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THE DISTINGUISHING NUMBERS OF 4-REGULAR QUADRANGULATIONS ON THE TORUS

By

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Abstract. A graph G is said to be d-distinguishable if we can assign d distinct labels to its vertices so that the only automorphism of G preserving the labels of vertices is the identity map and its distinguishing number is defined as the minimum number d such that G is d-distinguishable. We shall determine the distinguishing number of 4-regular quadrangulations on the torus and show that they are 2-distinguishable with spesified exceptions, after classifying the 4-regular quadrangulations on the torus with standard forms.

Introduction

We deal with only finite simple graphs throughout this paper. The automorphism $\sigma : G \to G$ of G is a permutation over the vertex set $V(G)$ of G such that it preserves the adjacency of vertices and it extends to a permutation over the edge set $E(G)$ so that $\sigma(uv)$ becomes the edge between $\sigma(u)$ and $\sigma(v)$ for any edge $uv \in E(G)$ with ends u and v. The set of automorphisms of G forms a group with respect to thier compositions. We call it the *automorphism group* of G and denote it by $\mathrm{Aut}(G)$.

The automorphism group $\mathrm{Aut}(G)$ exhibits the symmetry over G. We shall consider how to break such symmetry, assigning labels to vertices. Given an assignment $c : V(G) \to \{1, 2, \ldots, d\}$ of labels, we define $\mathrm{Aut}(G, c)$ as the subgroup in $\mathrm{Aut}(G)$ consisting of automorphisms preserving the labels given by c:

$$\mathrm{Aut}(G, c) = \{\sigma \in \mathrm{Aut}(G) \ : \ c(\sigma(v)) = c(v) \text{ for any } v \in V(G)\}$$

We call c a *d-distinguishing labeling* of G if $\mathrm{Aut}(G, c)$ is trivial. In other words, the only automorphism that preserves the labeling of G is the identity map id_G over G if c is d-distinguishing. A graph G is said to be *d-distinguishable* if G admits a d-distinguishing labeling. Since G is $|V(G)|$-distinguishable clearly, we can consider the minimum number d such that G is d-distinguishable and define

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the *distinguising number* of G as it, which is denoted by $D(G)$. For example, it is easy to see that $D(K_n) = n$ and $D(K_{n,n,n}) = n + 1$ for $n \geq 2$. We can find several arguments on these notions in [1, 2, 5] for example.

Recently, the second author [4] has established a theory on the distinguishing number of graphs embedded on closed surfaces, using some methods in topological graph theory although the distinguishing number is defined for abstract graphs. In particular, he has discussed the distinguishing number of "polyhedral quadrangulations" on closed surfaces in a part of his theory. A *quadrangulation* on a closed surface is usually defined as a simple graph embedded on the surface so that each face is bounded by a cycle of length 4 and is said to be *polyhedral* if any two faces meet each other in at most one vertex or one edge. In this paper, we shall discuss the distinguishing number of 4-regular quadrangulations on the torus, not assuming that they are polyhedral.

There have been classified the 4-regular quadrangulations on the torus in [3]. According to the classification, any 4-regular quadrangulation on the torus is isomorphic to one of the *standard forms* $Q(p, q, r)$ with suitable parameters p, q and r described below. The standard form $Q(p, q, r)$ has a *geodesic* 2*-factor* which consists of r disjoint cycles of length p, say C_0, $C_1, \ldots, C_{r-1}$. Each cycle C_i goes straight at each vertex and each pair of C_i and C_{i+1} with indices modulo r bounds an annulus on the torus, which is divided into p quadrilaterals by edges between them. Let u_0, $u_1, \ldots, u_{p-1}$ be the vertices lying along C_0 in this order. Start at u_0 along the edge joining u_0 to C_1 and go straight. Then we will encounter C_0 again after crossing all of C_i orthogonally in order; the goal is u_q.

The following is our main theorem in this paper:

Theorem 1. *Every* 4*-regular quadrangulation on the torus is* 2*-distinguishable unless it is isomorphic to one of the followings:*

$$Q(5, 2, 1), \quad Q(10, 3, 1), \quad Q(3, 1, 2), \quad Q(3, 0, 3),$$
$$Q(2q + 2, q, 1) \ (q \geq 2), \quad Q(p, 2, 2) \ (p \geq 4)$$

The exceptions $Q(5, 2, 1)$, $Q(3, 1, 2)$ and $Q(3, 0, 3)$ are isomorphic to K_5, $K_{2,2,2}$ and $C_3 \times C_3$, respectively and they are not 2-distinguishable acutually; $D(Q(5, 2, 1)) = 5$, $D(Q(3, 1, 2)) = 3$ and $D(Q(3, 0, 3)) = 3$. Also the two infinite series of exceptions are not 2-distinguishable, which will be discussed later. The only polyhedral quadrangulation is $Q(3, 0, 3)$ among these exceptions in the theorem and hence we have the following corollary immediately:

Corollary 2. *Every polyhedral* 4*-regular quadrangulation on the torus is* 2*-distinguishable unless it is isomorphic to* $Q(3, 0, 3)$. ■

We shall describe the standard forms of 4-regular quadrangulations on the torus in detail in Section 1 and show that they are 2-distinguishable in easy cases with sufficiently large parameters in Section 2. In Section 3, we shall discuss the remaining cases, considering how to distinguish vertices. In Section 4, we shall show that the two series of exceptions in the theorem are actually exceptions and determine their distinguishing numbers. In Section 5, we shall investigate "the faithfulness of embeddings" of 4-regular quadrangulations on the torus and discuss the relation between our results and the general theory on polyhedral graphs on closed surfaces developed in [4].

1. Standard forms

Let Ω be the integral grid of the xy-plane $\boldsymbol{R}^2$ consisting of vertical and horizontal lines crossing at the points with integral coordinates. We regard Ω as an infinite 4-regular quadrangulation on the plane. Thus, we have $V(\Omega) = \{(x, y) \in \boldsymbol{R}^2 : x, y \in \boldsymbol{Z}\}$ and two vertices (x, y) and (x', y') are adjacent if and only if $|x - x'| + |y - y'| = 1$. This infinite 4-regular quadrangulation covers $Q(p, q, r)$ naturally and we can specify each vertex of $Q(p, q, r)$ with xy-coordinates as $u_{(x,y)}$, which corresponds to the point (x, y) in Ω. In particular, the set of vertices $\{u_{(i,j)} : \ j = 0, 1, \ldots, p-1\}$ may be assumed to form each cycle C_i in the geodesic 2-factor in $Q(p, q, r)$ and $u_{(0,j)}$ is identical with $u_{(r,j-q)}$.

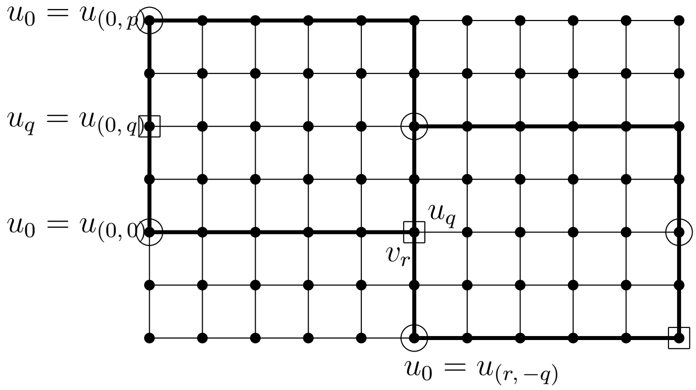

Figure 1 The plane grid covering $Q(p, q, r)$

It is easy to see that $u_{(x,y)} \in V(Q(p, q, r))$ is covered by the following set:

$$\{(x + \lambda r, y - \lambda q + \mu p) : \lambda, \mu \in \boldsymbol{Z}\}$$

Define a map $\tau_{(a,b)} : V(Q(p, q, r)) \to V(Q(p, q, r))$ by $\tau_{(a,b)}(u_{(x,y)}) = u_{(x+a,y+b)}$. It is easy to see that $\tau_{(a,b)}$ is well-defined and it becomes an automorphism of $Q(p, q, r)$. Since we can choose (a, b) arbitrarily, this implies that $\mathrm{Aut}(Q(p, q, r))$

is non-trivial and that $Q(p, q, r)$ is *vertex-transitive*, that is, there is an automorphism τ of $Q(p, q, r)$ with $\tau(u) = v$ for any pair of vertices u and v. In particular, $\tau_{(0,p)}$ and $\tau_{(r,-q)}$ induce the identity map over $Q(p, q, r)$.

To recognize the standard form presenting a given 4-regular quadrangulation G on the torus, we read the parameters p, q and r as follows. First we choose a vertex u_0 and one of the four edges incident to u_0, say u_0u_1, arbitrarily. Go straight toward the direction of u_0u_1 and counting the number of vertices until we meet u_0, again. Then we get the first parameters p with a cycle $C_0 = u_0u_1 \cdots u_{p-1}$ of length p. Secondly, choose another edge u_0v_1 incident to u_0 and not lying on C_0 and go straight toward the direction fo u_0v_1 until we encouter a vertex on C_0. Then we get a path $u_0v_1 \cdots v_r$ of length r with $v_r = u_q \in V(C)$.

Thus, the parameters p, q and r are uniquely determined by the choice of an ordered triple $\langle u_0, u_1, v_1\rangle$. There are 4×2 ways to choose u_1 and v_1 for a fixed u_0 but there are at most four distinct sets of parameters, as shown in the following theorem. The set of parameters are independent of the choice of u_0 since $Q(p, q, r)$ is vertex-transitive.

Theorem 3. *Two 4-regular quadrangulations $Q(p, q, r)$ and $Q(p', q', r')$ are isomorphic as maps if and only if one of the followings holds:*

(i) $p' = p, \;\; q' = q, \;\; r' = r.$
(ii) $p' = p, \;\; q' = p - q, \;\; r' = r.$
(iii) $p' = pr/(p, q), \;\; q' = nr, \;\; r' = (p, q),$
(iv) $p' = pr/(p, q), \;\; q' = pr/(p, q) - nr, \;\; r' = (p, q),$

In both (iii) *and* (iv), (p, q) *stands for the greatest common divisor of p and q and n can be obtained as the minimum non-negative integer satisfying $nq \equiv (p, q)$* $(\mathrm{mod}\ p)$.

Proof. Two maps are isomorphic if and only if there is a homeomorphism between the surfaces containing them that carries one onto the other, preserving their faces. If $Q(p, q, r)$ is isomorphic to $Q(p', q', r')$, then we can read the parameters p', q' and r' over $Q(p, q, r)$. Thus, it suffices to understand what parameters we can read, re-choosing $\langle u_0, u_1, v_1\rangle$ for a fixed u_0.

We use the same notations as given before the theorem. Let w_0, w_1, w_2 and w_3 be the four neighbors of u_0 lying around it in this cyclic order with $w_0 = u_1$, $w_1 = v_1$ and $w_2 = u_{p-1}$. Then we obtain a set of parameters (p', q', r') by choosing an ordered triple $\langle u_0, w_{\mathrm{s}}, w_{\mathrm{t}}\rangle$ for $s, t \in \{0, 1, 2, 3\}$ with $|s - t| \neq 2$. In particular, the original parameters (p, q, r) can be read with $\langle u_0, w_0, w_1\rangle$.

First, consider the parameters $(\bar{p}, \bar{q}, \bar{r})$ with $\langle u_0, w_2, w_3\rangle$. To read these, we go straight along C_0 backward and come back to u_0. Next we go straight toward the direction of u_0w_3. Then we reach a vertex on C_0, again after crossing $C_{r-1}, C_{r-2}, \ldots$. The vertex is u_{p-q} and is the qth vertex if we count the vertices along C_0 backward from u_0. These imply that $\bar{p} = p$, $\bar{q} = q$ and $\bar{r} = r$. This fact can be applied to each pair of ordered triples each of which can be obtained from the other by rotating it in 180°. Therefore, we oabtain only four sets of parameters from $\langle u_0, w_s, w_t\rangle$'s and they can be read with $\langle u_0, w_0, w_1\rangle$, $\langle u_0, w_2, w_1\rangle$, $\langle u_0, w_1, w_0\rangle$ and $\langle u_0, w_3, w_0\rangle$.

Read the parameters p', q' and r' with $\langle u_0, w_2, w_1\rangle$. Then the p vertices lying along C_0 are re-labeled backward and u_q labeled with $\langle u_0, w_0, w_1\rangle$ is re-labeled as the $(p-q)$th vertex on C_0. This implies that $p' = p$, $q' = p - q$ and $r' = r$, which corresponds to (ii) in the theorem.

Read the parameter p', q' and r' with $\langle u_0, w_1, w_0\rangle$ in turn. Go straight toward the direction of u_0w_1 until we come back to u_0; don't stop at u_q if $u_q \neq u_0$. Suppose that we encountered m vertices on C_0 finally. Then we obtain a cycle C_0' of length mr and $mq \equiv 0 \pmod p$. If $u_0 = u_{(0,0)}$, then the m vertices on C_0 we encountered are $u_{(q,0)}, u_{(2q,0)}, \ldots, u_{(mq,0)} = u_{(0,0)}$. These divide C_0 into m segments and the length of each segment is equal to r' and hence $p = mr'$.

Since $mq \equiv 0 \pmod p$, there is a poisitive integer α with $mq = \alpha p$ and hence $q = \alpha r'$. Since m is the minimum number satisfying these equalities, r' must be the greatest common divisor (p, q) of p and q. Since $pr = p'r'$ as the number of vertices, we have $p' = pr/r' = pr/(p,q)$. Go straight toward the direction of u_0w_0. Then we encounter $u_{(r',0)}$ and this is the nrth vertex along C_0' with $nq \equiv r' = (p,q)$. Thus, we have $q' = nr$. Now we have obtained the formulas corresponding to (iii).

The translation formula in (iv) can be obtained just as the composition of (iii) and (ii). So we need to discuss nothing more. ■

To eliminate duplications of $Q(p,q,r)$, we may assume that $0 \le q \le p/2$. For, $Q(p,q,r)$ with $p/2 < q \le p-1$ can be translated into $Q(p, p-q, r)$ with $0 \le p - q \le p/2$ by the formula (ii) in Theorem 3. Furthermore, we should exclude the parameters that present non-simple graphs. The following lemma shows those:

Lemma 4. *The standard form $Q(p,q,r)$ with $0 \le q \le p/2$ presents a simple graph if and only if none of the following holds:*

(i) $p \le 2$
(ii) $r = 1$ *and* $q \le 1$
(iii) $r = 1$ *and* $p = 2q$
(iv) $r = 2$ *and* $q = 0$.

Proof. It is easy to see that $Q(p,q,r)$ has a self-loop if and only if either $p=1$ or $(r,q)=(1,0)$ and that the other cases allows multiple edges. ■

2. Sufficiently large cases

In this section, we shall discuss the case when $r \geq 3$. In this case, we can prove easily that all $Q(p,q,r)$'s but one are 2-distinguishable. In general, a graph G is 2-distinguishable if and only if there is a subset $S \subset V(G)$ such that the only automorphism σ of G with $\sigma(S)=S$ is the identity map over G. We shall use this frequently below. Since $Q(p,q,r)$ is vertex-transitive, we have $D(Q(p,q,r)) \geq 2$. Thus, if $Q(p,q,r)$ is 2-distinguishable, then $D(Q(p,q,r))=2$. We use the same notation $u_{(x,y)}$, C_i and so on as in the previous section.

Lemma 5. *$D(Q(p,q,r))=2$ if $r \geq 4$.*

Proof. Put $S = V(C_0) \cup \{u_{(1,0)}, u_{(2,1)}\}$ and let H be the subgraph induced by S in $Q(p,q,r)$. Then H consists of the cycle C and two additional vertices $u_{(1,0)}$ and $u_{(2,1)}$. Since $r \geq 4$, the latter is isolated in H while the former has degree 1 in H and is adjacent to $u_{(0,0)}$, which has degree 3 in H. The other vertices has degree 2 in H. Thus, it is clear that any automorphism σ of $Q(p,q,r)$ with $\sigma(S)=S$ fixes each of $u_{(0,0)}$, $u_{(1,0)}$ and $u_{(2,1)}$ and that $\sigma(C_0)=C_0$.

The neighbors of p vertices lying along C_0 form two disjoint cycles C_1 and C_{r-1}. The former contains a vertex $u_{(1,0)}$ belonging to S but the latter does not. This implies that $\sigma(C_1)=C_1$ in particular. Similarly, the neighbors of vertices on C_i form two disjoint cycles C_{i-1} and C_{i+1}. If we have known that $\sigma(C_{i-1})=C_{i-1}$ and $\sigma(C_i)=C_i$, then we also conclude that $\sigma(C_{i+1})=C_{i+1}$. Therefore, σ sends any cycle C_i onto itself. In particular, either σ turns over C_1 with $\sigma(u_{(1,0)})=u_{(1,0)}$, or fixes C_1 pointwise. Similarly, σ turns over C_2 with $\sigma(u_{(2,1)})=u_{(2,1)}$, or fixes C_2 pointwise. However, the only compatible case is when $\sigma|_{C_1 \cup C_2}$ is the identify map over $C_1 \cup C_2$ since $u_{(1,0)}$ is adjacent to $u_{(2,0)}$, but not to $u_{(2,2)}$. Clearly, this extends to all of cycles C_i's and σ must be the identity map over $Q(p,q,r)$. Therefore, $Q(p,q,r)$ is 2-distinguishable if $r \geq 4$. ■

It should be noticed that the argument in the above proof does not work actually with S so defined if $r=3$. For, both $u_{(1,0)}$ and $u_{(2,1)}$ have degree 1 in H and σ may exchange C_1 and $C_2 = C_{r-1}$. For example, $Q(p,0,3) \cong C_p \times C_3$ admits an automorphism σ which rotates the face $u_{(1,0)}u_{(2,0)}u_{(2,1)}u_{(1,1)}$ in 180°. Such an automorphism σ exchanges $u_{(1,0)}$ and $u_{(2,1)}$ although we have $\sigma(S)=S$. Thus, we need to change the choice of S for the case of $r=3$, as shown in the proof below.

LEMMA 6. *$D(Q(p,q,3)) = 2$ if $p \geq 4$.*

Proof. Put $S = V(C_0) \cup \{u_{(1,0)}, u_{(2,1)}, u_{(2,2)}\}$ and let H be the subgraph induced by S in $Q(p,q,3)$. Then $u_{(1,0)}$ has degree 1 in H and both $u_{(2,1)}$ and $u_{(2,2)}$ have degree 2 in H. The subgraph H contains two cycles without chords, one of which is C_0 and the other is $u_{(2,1)}u_{(2,2)}u_{(0,q+2)}u_{(0,q+1)}$. However, the former contains a vertex adjacet to the unique vertex $u_{(1,0)}$ of degree 1 in H, but the latter does not unless $q = p-1$ or $p-2$. Thus, any automorphism σ of $Q(p,q,3)$ with $\sigma(S) = S$ fixes C_0 setwise in such a case. Here we may assume that $0 \leq q \leq p/2$ in addition. If $q = p-1 \leq p/2$, then we have $p \leq 2$, which forces $Q(p,q,r)$ to have multiple edges, a contradiction. If $q = p-2 \leq p/2$, then we have $p \leq 4$ and hence $(p,q) = (4,2)$ under the assumption in the lemma.

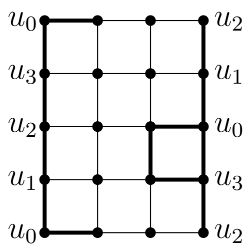

Figure 2 $Q(4,2,3)$

We need to discuss in more detail to conclude the same fact as above in this unique exceptional case, depicted in Figure 2. In this case, $u_{(1,0)}$ has degree 1, $u_0 = u_{(0,0)} = u_{(3,2)}$ has degree 4, $u_3 = u_{(0,3)} = u_{(3,1)}$ has degree 3 in H and the other vertices in S have degree 2 in H. These imply that σ fixes each of $u_{(1,0)}$, u_0 and u_3. On the other hand, five vertices not belonging to S form a path of length 4 and σ fixes each of $X = \{u_{(1,1)}, u_{(2,0)}\}$, $Y = \{u_{(1,2)}, u_{(2,3)}\}$ and $Z = \{u_{(1,3)}\}$ setwise. Then $u_{(2,1)}$ is adjacent to the two vertices in X and $u_{(2,2)}$ is adjacent to those in Y, but u_1 and u_2 have none of these properties. Therefore σ does not exchange u_1u_2 and $u_{(2,1)}u_{(2,2)}$ and hence σ fixes C_0 pointwise, rather than setwise as we want.

Similarly, σ fixes C_1 and C_2, too, since C_1 and C_2 can be recognized as two disjoint cycles in $Q(p,q,3) - C_0$ and since the number of vertices in C_1 belonging to S is different from that of C_2. Then σ flips C_1 with $\sigma(u_{(1,0)}) = u_{(1,0)}$, or fixes it pointwise, while σ fixes $\{u_{(2,1)}, u_{(2,2)}\}$. This implies that $\sigma|_{C_1 \cup C_2}$ must be the identity map over $C_1 \cup C_2$ if $p \geq 4$; σ may flip both C_1 and C_2 if $p = 3$. This extends to the whole of $Q(p,q,3)$ and σ must be the identity map over $Q(p,q,3)$. Therefore, $Q(p,q,3)$ is 2-distinguishable if $p \geq 4$. ■

The only remaining cases with $r \geq 3$ is $Q(3,q,3)$. Under the assumption of $q \leq p/2$, it suffices to treat the following two cases:

Lemma 7. *$D(Q(3,0,3)) = 3$ and $D(Q(3,1,3)) = 2$.*

Proof. Let G be the 4-regular quadrangulation $Q(3,0,3)$ on the torus, which is isomorphic to $C_3 \times C_3$. Then it has presicely six cycles of length 3, three of which are disjoint and correspond to the first factor of $C_3 \times C_3$ and the other three to the second factor. To distinguish these two type of cycles, we call them "vertical" and "horizontal", respectively. Suppose that G has a 2-distinguishing labeling $c : V(G) \to \{1,2\}$. We may assume that $|c^{-1}(1)| \leq 4 < |c^{-1}(2)|$, up to symmetry.

First, suppose that one of the six cycles, say C, gets only one kind of labels. Then it is clear that any automorphism $\bar{\sigma}$ of $G - C$ which preserves the labeling $c|_{G-C}$ extends to an automorphism of G preserving c. Also, it is not difficult to see that $G - C \cong C_3 \times K_2$ is not 2-distinguishable. This implies that there is an automorphism $\bar{\sigma}$ of $G - C$ which preserves $c|_{G-C}$ and is not the identity map over $G - C$. Then $\bar{\sigma}$ extends to an automorphism σ of G which preserves c and it is not the identity map over G. This is contrary to our assumption on c being 2-distinguishing.

Therefore, we may assume that any cycle of length 3 contains two knids of labels, "1" and "2". Then we have $c^{-1}(1) = 3$ or 4, and have three cases depicted as the first three in Figure 3, up to symmetry; the vertices encircled in the figure get label "1" while the others get "2". The first one in the figure presents the case when there are only three "1"'s. If $|c^{-1}(1)| = 4$, one of the three veritcal cycle gets two "1"'s and so does one of the three horizontal cycles. The second one presents the case when those two cycles have a common vertex with "1" and the third presents the case when they do not. Clearly, we can see the symmetry on the torus which induces naturally an automorphism of G of order 2 preserving the labeling in each case. However, this is contrary to c being 2-distinguihing, again and hence G is not 2-distinguishable with $D(G) \geq 3$.

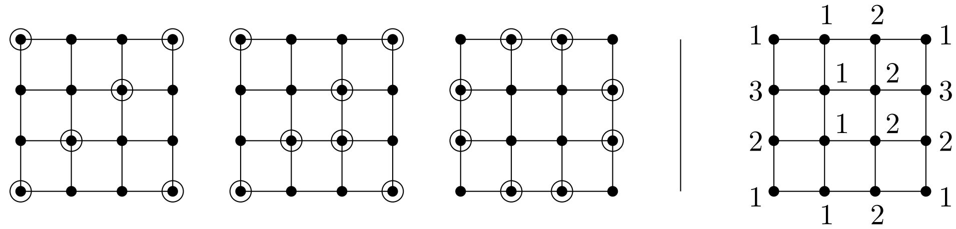

Figure 3 Labelings of $C_3 \times C_3$ with two and three labels

On the other hand, the fourth in Figure 3 presents a 3-distinguishing labeling of G. To see it, let σ be an automorphism of G preserving this labeling. There is a unique vertical cycle with three labels "1", "2" and "3", a unique cycle with only "1" and one with "2". Then σ must fix these three cycles setwise and must fix each of the vertices on the first one since they have different labels. This extends to the other two cycles and σ becomes the identity map over G. Therefore, G is 3-distinguishable and we have $D(G) = 3$.

Now let G be the 4-regular quadrangulation $Q(3, 1, 3)$ on the torus. It is not so difficult to see that G consists of three disjoint cycles of length 3 and one hamilton cycle and that this decomposition is unique. This implies that any automorphism σ of G fixes the hamilton cycle $v_0 v_1 \cdots v_8$ setwise. Define $c : V(G) \to \{1, 2\}$ by $c(v_0) = c(v_1) = c(v_3) = 1$ and $c(v) = 2$ for any other vertex v. Then σ must fix all vertices in addition if σ preserves this labeling c and hence c is 2-distinguishing. Therefore, G is 2-distinguishable with $D(G) = 2$. ■

3. Distinguishing vertices

In the previous section, we have determined the distinguishing number of all $Q(p, q, r)$ with $r \geq 3$. So it suffices to deal with the case when $r \leq 2$ here. Our argument below will work for all cases, but its main part concerns such a exceptional case.

Let G be a graph and S a subset of $V(G)$ in general. A vertex $v \in V(G)$ is said to be *distinguished* with S if $\sigma(v) = v$ for any automorphism σ of G with $\sigma(S) = S$. Thus, G is 2-distinguishable if and only if there is a subset $S \subset V(G)$ such that all vertices of G are distinguished with S. Furthermore, a set X or a subgraph H with vertex set X is said to be *distinguished* with S if each vertex in X is distinguished with S and it is *self-distinguished* if $S = X$. We try to find a self-distinguished set S and to extend the distinguishability of vertices.

Now consider any 4-regular quadrangulation G on the torus. Then it can be described with one of the standard forms $Q(p, q, r)$ with suitable parameters p, q and r. After translating parameters, we may assume that $0 \leq q \leq p/2$. Fix p, q and r for G with this condition and the labeling of vertices $u_{(x,y)}$, as given in Section 1, to realize this set of parameters. Furthermore, we should exclude the parameters which present non-simple graphs. Under these assumptions, let $H_{(x,y)}$ be the subgraph in $Q(p, q, r)$ with the following vertex set and edge set:

$$\begin{aligned} V(H_{(x,y)}) &= \{u_{(x,y)}, u_{(x,y+1)}, u_{(x,y+2)}, u_{(x+1,y)}, u_{(x+1,y+1)}\} \\ E(H_{(x,y)}) &= \{u_{(x,y)}u_{(x,y+1)}, u_{(x,y+1)}u_{(x,y+2)}, u_{(x+1,y)}u_{(x+1,y+1)}, \\ &\qquad\qquad u_{(x,y)}u_{(x+1,y)}, u_{(x,y+1)}u_{(x+1,y+1)}\} \end{aligned}$$

This subgraph $H_{(x,\,y)}$ consists of one square $u_{(x,\,y)}u_{(x,\,y+1)}u_{(x+1,\,y+1)}u_{(x+1,y)}$ and one edge $u_{(x,\,y+1)}u_{(x,\,y+2)}$ joining the square and a unique vertex $u_{(x,\,y+2)}$ of degree 1. However, $H_{(x,\,y)}$ may have *chords*, which are edges joing two vertices in $H_{(x,\,y)}$ but not belonging to $H_{(x,\,y)}$, and the five vertices listed above may not be all distinct in general. It is clear that $H_{(x,\,y)}$ has no chord and have exactly five distinct vertices if the parameters p, q and r are sufficiently large. Here $H_{(x,\,y)}$ is said to be *good* in such a case. It should be noticed that it does not depend on the choice of $(x,\,y)$ whether $H_{(x,\,y)}$ is good or not.

Lemma 8. *The subgraph $H_{(x,y)}$ in $Q(p,q,r)$ is good except the following cases:*

(i) $Q(p,2,1)$ $(p \geq 5)$, $Q(p,3,1)$ $(p \geq 7)$
(ii) $Q(2q+1,q,1)$ $(q \geq 3)$
(iii) $Q(p,1,2)$, $Q(p,2,2)$ $(p \geq 3)$
(iv) $Q(3,q,r)$ $(r \geq 3)$

Proof. It is clear that if $p \geq 4$ and $r \geq 3$, then $H_{(x,\,y)}$ is good and hence we should exclude only the case of $p = 3$ if $r \geq 3$. This is listed as (iv) in the lemma. Assume below that $r = 1$ or 2.

First consider the case of $r = 1$. The two vertices $u_{(x+1,\,y)}$ and $u_{(x+1,\,y+1)}$ in $H_{(x,\,y)}$ appear as $u_{(0,\,q)}$ and $u_{(0,\,q+1)}$ in this case. It is clear that the five vertices in $V(H_{(x,\,y)})$ are all distinct with no vertical chord if and only if $p \geq 7$ and $4 \leq q \leq p-3$. Denying these conditions, we obtain that $p \leq 6$, $q \leq 3$, or $p-2 \leq q$. However, the third one should be excluded since $p-2 \leq q \leq p/2$ implies $p \leq 4$. The graph is not simple also in case of $q = 0, 1$. Thus, all cases except $Q(p,2,1)$ and $Q(p,3,1)$ listed in (i) admit no vertical chord.

Label $u_{(x,\,y)}$, $u_{(x,\,y+1)}, \ldots$ by u_0, $u_1, \ldots$, according to the way to read the parameters p, q and r. Then $u_{(x+2,\,y)}$ is identical with u_{2q}. To exclude horizontal chords of $H_{(x,\,y)}$, it suffices to consider the position of u_0 in the column containing $u_{(x+2,\,y)}$. We conclude that there are horizontal chords in $H_{(x,\,y)}$ only when $u_{2q} = u_{p-1}, u_0, u_1$ or u_2, as Figure 4 suggests. However, there would be multiple egdes if $u_{2q} = u_0$ and $q \leq p/2$ does not hold if $u_{2q} = u_1, u_2$. Thus, we obtain only $Q(2q+1,q,1)$, which is (ii) in the lemma.

Now suppose that $r = 2$. Then neither $u_{(x+1,\,y)}$ nor $u_{(x+1,\,y+1)}$ appear the column containing $u_{(x,\,y)}$ and $H_{(x,\,y)}$ has a vertical chord only when $p = 3$. To find horizontal chords of $H_{(x,\,y)}$, we consider the poision of u_0 in the column containing $u_{(x+2,\,y)}$, which is identical with u_q, and conclude that $q = p-1, 0, 1, 2$ as well as in the previous case. However, the first case implies that $p \leq 2$ since $p-1 \leq p/2$ while there are multiple edges in the second case. Thus, we obtain $Q(p,1,2)$ and $Q(p,2,2)$, which cover all cases of $p = 3$; $Q(3,0,2)$ is not simple. They are listed in (iii). ■

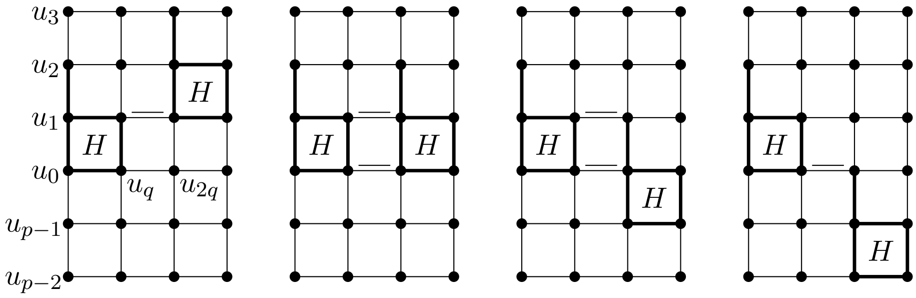

Figure 4 $H = H_{(x,y)}$ with horizontal chords in $Q(p, q, 1)$

Now we shall try our first purpose, finding a self-distingushed set. The subgraph $H_{(x,y)}$ defined above gives us an answer:

Lemma 9. *The set $V(H_{(x,y)})$ is self-distinguished except the following cases:*

(i) $Q(p, 2, 1)$ $(p \geq 5)$, $Q(p, 3, 1)$ $(p \geq 7)$
(ii) $Q(2q + 1, q, 1)$, $Q(2q + 2, q, 1)$ $(q \geq 3)$.
(iii) $Q(p, 1, 2)$, $Q(p, 2, 2)$ $(p \geq 3)$.
(iv) $Q(3, q, r)$ $(r \geq 3)$.
(v) $Q(4, 1, 3)$, $Q(4, 0, 4)$

Proof. Put $S = V(H_{(x,y)})$ and take any automorphism σ of G with $\sigma(S) = S$. Since the exceptional cases in the lemma include all exceptional cases in Lemma 8, $H_{(x,y)}$ is good. Then $\sigma|_{H_{(x,y)}}$ becomes an automorphism of $H_{(x,y)}$. This implies that σ fixes each of $v = u_{(x,y+2)}$, $u_{(x,y+1)}$ and $u_{(x+1,y)}$ and hence they are distinguished with S and encircled in Figure 5. Furthermore, the vertex $w = u_{(x-1,y+1)}$ does not belong to S since $H_{(x,y)}$ has no horizotal chord and it is a unique vertex in $V(G) - S$ incident to the distinguished vertex $u_{(x,y+1)}$. This implies that σ fixes w and hence it is distinguished with S, too. However, σ might exchange $u_{(x,y)}$ and $u_{(x+1,y+1)}$ yet although we deny it below.

Suppose that σ exchanges $u_{(x,y)}$ and $u_{(x+1,y+1)}$. Then σ exchanges their neighbors $\{u_{(x-1,y)}, u_{(x,y-1)}\}$ and $\{u_{(x+1,y+2)}, u_{(x+2,y+1)}\}$, too. Since the distinguished vertex v is adjacent to $u_{(x+1,y+2)}$, it must be identical with either (A) $u_{(x-2,y)}$, (B) $u_{(x-1,y-1)}$ or (C) $u_{(x,y-2)}$, each of which might be a neighbor of $\sigma(u_{(x+1,y+2)})$; also $u_{(x+1,y-1)}$ might be such a vertex, but $H_{(x,y)}$ would have a chord $u_{(x+1,y)}u_{(x+1,y-1)}$ in this case, a contradiction. We can recongize the parameters in each of the three cases as follows:

Case (A): We have either $Q(p, p-2, 2)$ or $Q(p, q, 1)$ with $2q \equiv -2 \pmod p$. Since $p - 2 \leq p/2$ implies $p \leq 4$, only $Q(3, 1, 2)$ and $Q(4, 2, 2)$ survive in the former case. On the other hand, the congruence in the latter case implies

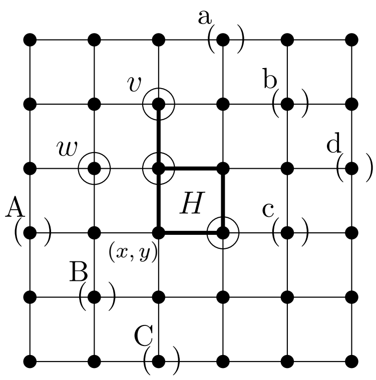

Figure 5 Candidates for distinguished vertices

that $2q = p - 2$ since $q \le p/2$, and hence we obtain $Q(2q+2, q, 1)$ with $q \ge 3$; the case of $q = 2$ is included in (i).

Case (B): We have $Q(p, p-3, 1)$ and $p - 3 \le p/2$. It follows that $p \le 6$ and $Q(p, p-3, 1)$ is simple only if $p = 5$. Thus, only $Q(5, 2, 1)$ survives.

Case (C): In this case, we can conclude only that $p = 4$.

As well as in the previous, the distinguished vertex w should be identical with either (a) $u_{(x+1,y+3)}$, (b) $u_{(x+2,y+2)}$, (c) $u_{(x+2,y)}$ or (d) $u_{(x+3,y+1)}$. These restrict the values of parameters as follows:

Case (a): We have $Q(p, p-2, 2)$ or $Q(p, q, 1)$ with $2q \equiv -2 \pmod p$. This is completely the same as Case (A).

Cases (b) and (c): We have $Q(p, p-1, 3)$, $Q(p, 1, 3)$ or $Q(p, q, 1)$ with $3q \equiv \pm 1 \pmod p$. However, the first case does not happen since $p - 1 \le p/2$ implies $p \le 2$. The congruence in the third case implies that $3q = p \pm 1$ since $q \le p/2$ and we obtain $Q(3q \pm 1, q, 1)$.

Case (d): We have $Q(p, 0, 4)$, $Q(p, q, 2)$ with $2q \equiv 0 \pmod p$ or $Q(p, q, 1)$ with $4q \equiv 0 \pmod p$. The congruence in the second case implies either $q = 0$ or $q = p/2$, but the former case yields multiple edges. Thus, only $Q(2q, q, 2)$ survives. The congruence in the third case implies either $q = 0$, $q = p/2$ or $q = p/4$. However, $Q(p, 0, 1)$ and $Q(2q, q, 1)$ are not simple and hence only $Q(4q, q, 1)$ survives.

One of Cases (A) to (C) and one of cases (a) to (d) must happen together. Clearly Cases (A) and (a) are one of such compatible pairs and $Q(2q+2, q, 1)$ appears in (ii). The quadrangulation $Q(4, 2, 2)$ in Case (A) survives also as $Q(2q, q, 2)$ in Case (d) while $Q(3, 1, 2)$ appears nowhere other than in Case (a). They are included in (iii). The unique surviver $Q(5, 2, 1)$ in Case (B) appears only as $Q(3q-1, q, 1)$ with $q = 2$ in Case (c) and is included as $Q(2q+1, q, 1)$ with $q = 2$ in (ii). Corresponding to Case (C) with $p = 4$, we find only $Q(4, 2, 2)$

in Cases (a) and (d), $Q(4,1,3)$ in Case (c) and $Q(4,0,4)$ in Case (d). The last two are listed in (v).

Now we have seen that all possible cases under the assumption that σ exchanges $u_{(x,y)}$ and $u_{(x+1,y+1)}$ are listed as the exceptional cases in the lemma. Therefore, σ fixes each of $u_{(x,y)}$ and $u_{(x+1,y+1)}$ except those cases. This implies that all of the five vertices in S are distinguished with S and hence $S = V(H_{(x,y)})$ is self-distinguihed. ■

Once we found a self-distinguished set, we can extend the set of distinguished vertices by the following two lemmas. The extending rule is depicted in Figure 6, where the vertices we have already known as distinguished are encircled and each arrow points a vertex which we find as another distinguished one. The first two, given by Lemma 10, work for any $Q(p,q,r)$, but the third depends on the parameters p, q and r.

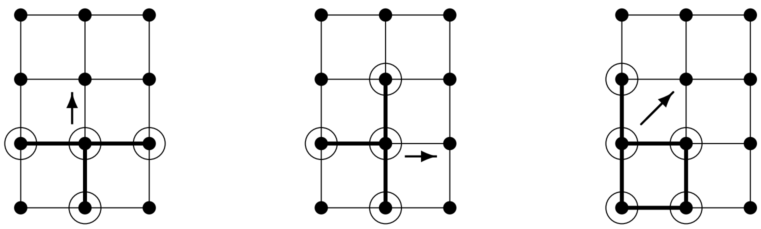

Figure 6 Extending the distinguishability of vertices

Lemma 10. *Let G be any 4-regular quadrangulation on the torus and v a vertex of G with four neighbors w_0, w_1, w_2 and w_3. If v, w_0, w_1 and w_2 are distinguished with a suitable set S, then so is w_3.*

Proof. Let σ be any automorphism of G with $\sigma(S) = S$. Then $\sigma(v) = v$ since v is distinguished with S and hence $\sigma(w_i)$ must be in the neighborhood of v for $i = 0,1,2,3$. Since $\sigma(w_i) = w_i$ for $i = 0,1,2$, we have $\sigma(w_3) = w_3$, too. ■

The following lemma corresponds to the third rule in Figure 6. We should refer to Lemma 8 for the condition to guarantee that $H_{(x,y)}$ is good.

Lemma 11. *If $H_{(x,y)}$ is good and if each vertex in $H_{(x,y)}$ is distinguished with a subset S, then so is $u_{(x+1,y+2)}$.*

Proof. Suppose that $u_{(x+1,y+2)}$ is not distinguished with S. Then there is an automorphism σ of G with $\sigma(S) = S$ and with $\sigma(u_{(x+1,y+2)}) \neq u_{(x+1,y+2)}$. Since the only neighbors of $u_{(x+1,y+1)}$ which are not distinguished are $u_{(x+1,y+2)}$ and $u_{(x+2,y+1)}$, σ exchanges $u_{(x+1,y+2)}$ and $u_{(x+2,y+1)}$, and the latter must be adjacent to the distinguished vertex $u_{(x,\,y+2)}$ as well as $u_{(x+1,\,y+2)}$ is. This implies that $u_{(x,\,y+2)}$ is identical with either $u_{(x+2,\,y)}$ or $u_{(x+3,\,y+1)}$. However, $H_{(x,\,y)}$ would have a chord $u_{(x+1,\,y)}u_{(x+2,\,y)}$ in the former case, contrary to $H_{(x,\,y)}$ being good.

On the other hand, if the latter happens, then $H_{(x,\,y)}$ is identical with $H_{(x+3,\,y-1)}$. Applying Lemma 10 to $u_{(x+3,\,y)}$, we conclude that $u_{(x+2,\,y)}$ is distinguished with S and it should be adjacent to $u_{(x+1,\,y+2)}$ as well as to $u_{(x+2,\,y+1)}$ since σ exchanges them. This implies that the distinguished vertex $u_{(x+2,\,y)}$ is identical wtih $u_{(x+1,\,y+3)}$ and that $H_{(x,\,y)}$ is identical with $H_{(x+2,\,y+2)}$. However, $H_{(x+2,\,y+2)}$ would have a chord $u_{(x+3,\,y+2)}u_{(x+3,\,y+1)}$, contrary to $H_{(x,\,y)}$ being good, again.

Therefore, any automorphism σ of G with $\sigma(S) = S$ fixes $u_{(x+1,\,y+2)}$ as we want and hence $u_{(x+1,\,y+2)}$ is distinguished with S. ■

Combining Lemmas 10 and 11, we can prepare another extending rule of the distinguishability of vertices, which is more useful to prove our main theorem.

Lemma 12. *Suppose that $H_{(x,\,y)}$ is good and exclude $Q(2q+2, q, 1)$. If $H_{(x,\,y)}$ is distinguished with a subset S, then so is $H_{(x+1,\,y)}$.*

Proof. It suffices to show that $u_{(x+1,\,y+2)}$, $u_{(x+2,\,y)}$ and $u_{(x+2,\,y+1)}$ are distinguished with S. We have already shown it for the first one as Lemma 11 and this implies immediately that $u_{(x+2,\,y+1)}$ is distinguished with S by Lemma 10.

Suppose that $u_{(x+2,\,y)}$ is not distinguished with S. Then there is an automorphism σ of G with $\sigma(S) = S$ such that $\sigma(u_{(x+2,\,y)}) \neq u_{(x+2,\,y)}$. Since $u_{(x+2,\,y)}$ is adjacent to the distinguished vertex $u_{(x+1,\,y)}$, $\sigma(u_{(x+2,\,y)})$ must be identical with $u_{(x+1,\,y-1)}$. Since $u_{(x+2,\,y)}$ is adjacent also to the distinguished vertex $u_{(x+2,\,y+1)}$, $\sigma(u_{(x+2,\,y)})$ must be identical with either $u_{(x+2,\,y+2)}$ or $u_{(x+3,\,y+1)}$. If the first case happens, then we have $\sigma(u_{(x+2,\,y)}) = u_{(x+1,\,y-1)} = u_{(x+2,\,y+2)}$ and it follows that $q = p-3 \leq p/2$ and $r = 1$. That is, we have $Q(p, p-3, 1)$ with $p \leq 6$, but $H_{(x,\,y)}$ is not good in this, which is contrary to our assumption. Thus, it must hold that $\sigma(u_{(x+2,\,y)}) = u_{(x+1,\,y-1)} = u_{(x+3,\,y+1)}$ and hence we have $Q(p, p-2, 2)$ or $Q(p, q, 1)$ with $2q \equiv -2 \pmod{p}$. The former case happens only in case of $Q(3, 1, 2)$ and $Q(4, 2, 2)$ since $p - 2 \leq p/2$, but this is not the case since $H_{(x,\,y)}$ is not good in $Q(3, 1, 2)$ and $Q(4, 2, 2)$. In the latter case, we have $Q(2q+2, q, 1)$ since $2q = p-2$, but this is excluded by the second assumption in the lemma. Therefore, $u_{(x+2,\,y)}$ is distinguished with S. ■

The following lemma covers partially the case when $H_{(x,\,y)}$ is not good:

Lemma 13. $D(Q(p, 2, 1)) = 2$ *for* $p \geq 7$*, and* $D(Q(p, 3, 1)) = 2$ *for* $p \geq 11$*.*

Proof. Each of $Q(p, 2, 1)$ and $Q(p, 3, 1)$ has a hamilton cycle $u_0u_1 \cdots u_{p-1}$. Put

$S = \{u_0, u_1, u_3\}$ and let H_S be the subgraph induced by the vertices of S and their neighbors in each graph. The isomorphism type of H_S depends of the value of p, but our arguments below will work in common.

Suppose that $p \geq 7$ for $Q(p, 2, 1)$. Then S induces a path $u_0u_1u_3$ of length 2 in $Q(p, 2, 1)$ and u_1 is distinguished with S since any automorphism σ with $\sigma(S) = S$ fixes the middle point u_1 of the path. Since $p \geq 7$, u_2 is a unique vertex in H_S which is adjacent to all of u_0, u_1 and u_3. The vertex u_0 has a common neighbor with u_1 other than u_2, namely u_{p-1}, while u_3 does not. This distinguishes u_0 from u_1. Thus, we have four consecutive distinguished vertices u_0 to u_3 in $Q(p, 2, 1)$.

Suppose that $p \geq 11$ for $Q(p, 3, 1)$. Then S induces a path $u_1u_0u_3$ of length 2 and hence its middle point u_0 is distinguished with S. There are only two cycles of length 4 containing the path $u_1u_0u_3$; namely $u_1u_0u_3u_2$ and $u_1u_0u_3u_4$. This implies that any automorphism σ with $\sigma(S) = S$ fixes the set $\{u_2, u_4\}$. Since $p \geq 11$, u_2 has a common neighbor with u_0 other than u_1 and u_3, namely u_{p-1}, but u_4 does not. These imply that each of u_2, u_4 and u_{p-1} is distinguished with S. The vertices u_{p-2} and u_6 have different degree in H_S and they are the neighbors of u_1 and of u_3 other than u_0, u_2 and u_4. This distinguishes u_1 from u_3. Thus, we have six consecutive distinguished vertices $u_{p-1} = u_{-1}$, u_0 to u_4.

Assume that four consecutive vertices u_{i-3} to u_i are distinguished with S in $Q(p, 2, 1)$ and that so are u_{i-4} and u_{u-5} in addition for $Q(p, 3, 1)$. Then u_{i+1} is one of the four neighbors of u_{i-1} (or u_{i-2}) and the other three u_i, u_{i-2} and u_{i-3} (or u_{i-1}, u_{i-3} and u_{i-5}) in $Q(p, 2, 1)$ (or in $Q(p, 3, 1)$). Since the vertices with indices smaller than $i + 1$ are all distinguished with S by the assumption, u_{i+1} becomes distinguished with S, too by Lemma 10. Therefore, we can conclude inductively that all vertices are distinguished with S and that $Q(p, 2, 1)$ and $Q(p, 3, 1)$ are 2-distinguishable under the assumption on p in the lemman. ■

All we need to prove our main theorem has been prepared. The fact that $H_{(x,y)}$ is self-distinguished will guarantee that $Q(p, q, r)$ is 2-distinguishable in most of cases. We shall show that the two infinite series listed in the theorem are acutally exceptions in the next section.

Proof of Theorem 1. First suppose that G is a 4-regular quadrangulation $Q(p, q, r)$ on the torus except ones listed in Lemma 9. Then $H_{(0,0)}$ is good and is self-distinguished. Furthermore, $H_{(1,0)}, H_{(2,0)}, H_{(3,0)}, \ldots$ are distinguished with $S = V(H_{(0,0)})$ by Lemma 12. By (iii) in Theorem 3, $H_{(p',0)}$ is identical with $H_{(0,0)}$ where $p' = pr/(p, q)$. If this sequence $\{H_{(x,0)}\}$ covers all vertices of G, then they are all distinguished with S. Otherwise, consider the sequence $\{H_{(x,1)}\}$. Since $u_{(0,3)}$ is distinguished with S by Lemma 10, $H_{(0,1)}$ is distinguished with S, too and hence all vertices covered by the second sequence also are distinguished with S. Continue this argument as far as we need. Finally we can conclude that all vertices are distinguished with S and hence G is 2-distinguishable with $D(G) = 2$.

Now consider the exceptional cases listed in Lemma 9. We have discussed the distinguishability of vertices in $Q(p, q, r)$, fixing the parameters p, q and r so far, but we shall allow to translate them suitably up to isomorphism below.

(i) $Q(p, 2, 1)$ $(p \geq 5)$, $Q(p, 3, 1)$ $(p \geq 7)$: We have already known that $Q(p, 2, 1)$ is 2-distinguishable except $Q(5, 2, 1)$ and $Q(6, 2, 1)$ by Lemma 13. The former exception is isomorphic to K_5 while the latter is $K_{2,2,2}$ and is isomorphic to $Q(3, 1, 2)$. They are excluded the first and the third exceptions in the theorem. Similarly, $Q(p, 3, 1)$ is 2-distinguishable if $p \geq 11$. by Lemma 13. The complement of $Q(7, 3, 1)$ is 2-regular and forms a hamilton cycle $u_0u_2u_4u_6u_1u_3u_5$. It follows that all vertices of $Q(7, 3, 1)$ are distinguished with $\{u_0, u_2, u_6\}$ and hence $Q(7, 3, 1)$ is 2-distinguishable. Also, it is easy to see that $Q(9, 3, 1)$ is 2-distinguishable since it forms a unique hamilton cycle with three triangles. On the other hand, $Q(8, 3, 1)$ and $Q(10, 3, 1)$ are listed as two of the exceptions in the theorem; the former can be regarded as $Q(2q+2, q, 1)$ with $q = 3$ The complement of $Q(10, 3, 1)$ is isomorphic to $K_5 \times K_2$. It is not so difficult to see that $D(Q(10, 3, 1)) = D(K_5 \times K_2) = 3$.

(ii) $Q(2q + 1, q, 1)$, $Q(2q + 2, q, 1)$ $(q \geq 3)$: The latter is excluded in the theorem while $Q(2q + 1, q, 1)$ is isomorphic to $Q(2q + 1, 2, 1)$, as shown below.

$$(2q - 1)q = (2q + 1)q - 2q \equiv -2q \equiv 1 \pmod{2q + 1}$$

Thus, $n = 2q - 1$ is the solution of $nq \equiv 1 \pmod{2q + 1}$ and hence we can translate the parameters of $Q(2q+1, q, 1)$ into the following parameters p', q' and r' by (iv) in Theorem 3 with this n:

$$\begin{aligned}
p' &= (2q + 1) \cdot 1/(2q + 1,\ q) = 2q + 1 \\
q' &= p' - n \cdot 1 = 2q + 1 - (2q - 1) = 2 \\
r' &= (2q + 1,\ q) = 1
\end{aligned}$$

Thus, we can omit $Q(2q + 1, 2, 1)$ as a special type of $Q(p, 2, 1)$ in (i).

(iii) $Q(p, 1, 2)$, $Q(p, 2, 2)$ $(p \geq 3)$: The latter appears as one of the exceptions in the theorem while $Q(p, 1, 2)$ can be translated into $Q(2p, 2, 1)$ by (iii) in Theorem 3. As well as in the previous case, we can omit $Q(2p, 2, 1)$ as a special type of $Q(p, 2, 1)$.

(iv) and (v) $Q(3, q, r)$ $(r \geq 3)$, $Q(4, 1, 3)$, $Q(4, 0, 4)$: By Lemmas 5, 6 and 7, they are 2-distinguishable except $Q(3, 0, 3)$. Thus, we do not exclude them as exceptions in the theorem.

We have just confirmed that all cases are either 2-distinguishable or listed as exceptions in the theorem. This completes the proof. ■

4. Exceptional cases

In this section, we shall discuss the two infinite series of exceptions $Q(2q+2,q,1)$ and $Q(p,2,2)$ in detail. They are different maps on the torus, but are isomorphic to each other as graphs, as we shall show below.

LEMMA 14. *The 4-regular quadrangulations $Q(2q+2,q,1)$ on the torus is isomorphic to $Q(q+1,2,2)$ as graphs for $q \geq 2$.*

Proof. The first kind $Q(2q+2,q,1)$ has a hamilton cycle $u_0u_1\cdots u_{2q+1}$ and each vertex u_i has four neighbors u_{i-1}, u_{i+1}, u_{i+q} and u_{i-q} with indices taken modulo $2q+2$. Let $N(u_i)$ be the set of these four vertices. Then we can find that $N(u_i) = N(u_{q+1+i})$. Put $U_i = \{u_i, u_{q+1+i}\}$ to clear the structure of $Q(2q+2,q,1)$. Then we have $V(Q(2q+2,q,1)) = U_0 \cup U_1 \cup \cdots \cup U_q$ and all pairs between U_i and U_{i+1} are adjacent with indices taken modulo $q+1$.

The second kind $Q(p,2,2)$ consists of two cycles $u'_0u'_1\cdots u'_{p-1}$, $u''_0u''_1\cdots u''_{p-1}$ and edges $u'_iu''_i$, $u''_iu'_{i+2}$ for $i \equiv 0,1,\ldots,p-1 \pmod p$. Then u'_i and u''_{i-1} have four common neighbors u'_{i-1}, u'_{i+1}, u''_i and u''_{i-2}. Put $U'_i = \{u'_i, u''_{i-1}\}$ for $i \equiv 0,1,\ldots,p-1 \pmod p$. Then all pairs between U'_i and U'_{i+1} are adjacent.

Comparing these two structures, we find an isomorphism Φ between $Q(2q+2,q,1)$ and $Q(q+1,2,2)$ so that $\Phi(u_i) = u'_i$ and $\Phi(u_{q+1+i}) = u''_{i-1}$ for $i = 0,1,\ldots,q$. ■

LEMMA 15. *The 4-regular quadrangulation $Q(2q+2,q,1)$ on the torus can be presented only by two standard forms $Q(2q+2,q,1)$ and $Q(2q+2,q+2,1)$ if q is odd and by two more standard forms $Q(q+1,q-1,2)$ and $Q(q+1,2,2)$ if q is even.*

Proof. First suppose that q is odd. Then we have $(2q+2,q) = (2,q) = 1$. Solve $nq \equiv (2,q) = 1 \pmod{2q+2}$.

$$q^2 - 1 = (q-1)(q+1) = \{(q-1)/2\}(2q+2) \equiv 0 \pmod{2q+2}$$

Thus we have $n = q$ and $Q(2q+2,q,1)$ can be translated into itself by (iii) in Theorem 3. Therefore, $Q(2q+2,q,1)$ admits only one more standard form $Q(2q+2,q+2,1)$ given by (ii).

Suppose that q is even. Then we have $(2q+2,q) = (2,q) = 2$. Solve $nq \equiv$

$(2, q) = 2 \pmod{2q+2}$.

$$(q-1)q = (q+1)q - 2q = (2q+2)(q/2) - 2q \equiv 2 \pmod{2q+2}$$

Thus, we have $n = q-1$ and $Q(2q+2, q, 1)$ can be translated into $Q(q+1, q-1, 2)$ by (iii) and into $Q(q+1, 2, 2)$ by (iv) in Theorem 3. ■

LEMMA 16. *The 4-regular quadrangulation $Q(p, 2, 2)$ can be presented only by two standard forms $Q(p, 2, 2)$ and $Q(p, p-2, 2)$ if p is even and by two more standard forms $Q(2p, p-1, 1)$ and $Q(2p, p+1, 1)$ if p is odd.*

Proof. First suppose that p is even and slove $n \cdot 2 \equiv (p, 2) = 2 \pmod{p}$. Clearly we have $n = 1$ and hence $Q(p, 2, 2)$ can be translated into itself by (iii) in Theorem 3. There is another standard form $Q(p, p-2, 2)$ given by (ii).

Suppose that p is odd and solve $n \cdot 2 \equiv (p, 2) = 1 \pmod{p}$. Then we have $n = (p+1)/2$ and hence $Q(p, 2, 2)$ can be translated into $Q(2p, p+1, 1)$ by (iii) and into $Q(2p, p-1, 1)$ by (iv) in Theorem 3. ■

By the above two lemmas, $Q(2q+2, q, 1)$ and $Q(p, 2, 2)$ present the same map on the torus only if $q = p-1$ is even. Otherwise, they exhibit two different embeddings of one graph on the torus.

LEMMA 17. *$D(Q(2q+2, q, 1)) = 3$ $(q \geq 2)$ and $D(Q(p, 2, 2)) = 3$ $(p \geq 3)$.*

Proof. If suffices to show the first equality since $Q(2q+2, q, 1)$ and $Q(p, 2, 2)$ are isomorphic as graphs with $q = p-1$. We use the same notations as in our proof of Lemma 14 to express the structure of $Q(2q+2, q, 1)$.

First suppose that there were a 2-distinguishing labeling $c : V(Q(2q+2, q, 1)) \to \{1, 2\}$. Since there is an automorphism σ_i of $Q(2q+2, q, 1)$ which exchanges u_i and u_{q+1+i} in U_i, fixing all other vertices, we have $c(u_i) \neq c(u_{q+1+i})$ for $i = 0, 1, \ldots, q$. Then we can define an automorphism σ of $Q(2q+2, q, 1)$ so that σ carries each vertex in U_i to one in U_{i+1} with the same label as it has. That is, σ preserves the labels given by c and is not the identity map. This is contrary to c being 2-distinguishing. Therefore, there is no 2-distinguishing labeling of $Q(2q+2, q, 1)$ and $D(Q(2q+2, q, 1)) \geq 3$.

Define a labeling $c : V(Q(2q+2, q, 1)) \to \{1, 2, 3\}$ by $\sigma(u_0) = \sigma(u_1) = \sigma(u_3) = 1$, $\sigma(u_i) = 2$ for $i = 2, 4, \ldots, q$ and $\sigma(u_j) = 3$ for $j = q+1, \ldots, 2q+1$. Take any automorphism σ of $Q(2q+2, q, 1)$ preserving this labeling. Then $C = u_{q+1} \cdots u_{2q+1}$ forms a unique cycle with only label "3". This implies that $\sigma(C) = C$ and hence σ preserves another cycle $C' = u_0 u_1 \cdots u_q$. Since C' has length at least six, the occurences of "1" force σ to fix all vertices lying along C' and this forces it to fix all vertices along C, too. Therefore, σ must be the identity map and c is a 3-distinguishing labeling. Now we have $D(Q(2q+2, q, 1)) \leq 3$ and hence $= 3$. ■

5. Faithful embeddings

Let G be a graph embedded on a closed surface F^2. Then G is said to be *faithfully embedded* on F^2 if any automorphism σ of G extends to an auto-homeormophism over F^2. In other words, the symmetry of G can be realized as that over F^2 in such a case. As is mentioned in introduction, the second author has already established a general theorem on the distinguishing number of polyhedral graphs on closed surfaces in [4]. As an easy corollary of his theorem, we can conclude that:

Theorem 18. (Negami [4]) *Any polyhedarl quadrangulation faithfully embedded on the torus is* 2*-distinguishable unless it is isomorphic to* $C_3 \times C_3$. ■

A natural question arises; does this theorem include our main theorem? Of course, it does not since there exist infinitely many 4-regular quadrangulations on the torus which are not polyhedral. So, what can we say about the faithfulness of embedding? For example, $Q(4, 0, 4)$ is isomorphic to the 4-cube $Q_4 \cong C_4 \times C_4$ and is not faithfully embedded on the torus. We shall describe it in more detail in the proof of Lemma 21.

In most of cases, we can decide the faithfulness of embedding of $Q(p, q, r)$ by general arguments, as we shall do later. Unfortunately, we need to deal with the following case separately:

Lemma 19. *The* 4*-reglar quadrangulation* $Q(4, 1, 3)$ *is faithfully embedded on the torus.*

Proof. It is clear that $Q(4, 1, 3)$ decomposes into the geodesic 2-factor consisting of three disjoint cycles C_0, C_1, C_2 of length 4 which run in parallel to one another and one hamilton cycle C which crosses the 2-factor orthogonally. Rename the vertices as $v_0, v_1, \ldots, v_{11}$ so that they form the hamilton cycle $C = v_0 v_1 \cdots v_{11}$. Then $u_i = v_{3i}$ and each vertex v_i is incident to four edges $v_i v_{i-3}$, $v_i v_{i-1}$, $v_i v_{i+1}$ and $v_i v_{i+3}$. Furthermore, there are two cycles $v_i v_{i+1} v_{i+2} v_{i+3}$, $v_i v_{i-1} v_{i-2} v_{i-3}$ meeting only at v_i and joined with two edges $v_{i-2} v_{i+1}$ and $v_{i-1} v_{i+2}$. It is just a routine to confirm that any automorphism τ of $Q(4, 1, 3)$ preserves this decomposition $(C_0, C_1, C_2; C)$ with the structures described above and it follows that τ extends an auto-homeomorphism over the torus. Therefore, $Q(4, 1, 3)$ is faithfully embedded on the torus. ■

Here we shall show an easy criterion for a graph not to be faithfully embedded on a closed surface in general:

Lemma 20. *Let G be a graph embedded on a closed surface and v a vertex of degree at least 4 with neighbors w_0, w_1, w_2, $w_3, \dots$ lying around v in this cyclic order. If there is an automorphism τ of G which exchanges w_1 and w_2, fixing v and w_0, then G is not faithfully embedded on the surface.*

Proof. By the assumption, $[\cdots w_0 v w_1 \cdots]$ is a segment of a boundary walk of a face. However, $[\cdots \tau(w_0)\tau(v)\tau(w_1)\cdots] = [\cdots w_0 v w_2 \cdots]$ cannot be the boundary walk of any face since it separates two edges vw_1 and vw_3 into its different sides. Therefore, τ does not extend to any auto-homeomorphism over the surface and hence G is not faithfully embedded on the surface. ■

Using the above, we shall show that $Q(p,q,r)$ is not faithfully embedded on the torus in the exceptional cases.

Lemma 21. *The 4-regular quadrangulations $Q(3,1,2)$, $Q(4,0,4)$, $Q(5,2,1)$, $Q(10,3,1)$, $Q(p,2,2)$ $(p \geq 4)$ and $Q(2q+2,q,1)$ $(q \geq 2)$ are not faithfully embedded on the torus.*

Proof. First find the isomorphisms $Q(3,1,2) \cong K_{2,2,2}$, $Q(4,0,4) \cong C_4 \times C_4 \cong Q_4$, $Q(5,2,1) \cong K_5$ and $Q(10,3,1) \cong K_{5,5}$ − a perfect matching to recognize their automorphisms. Then there is an automorphism τ which exchanges $u_{(0,0)}$ and $u_{(1,1)}$, fixing all the others for $Q(3,1,2)$, $Q(5,2,1)$, $Q(2q+2,q,1)$; recall that $U_0 = \{u_0, u_{q+1}\} = \{u_{(0,0)}, u_{(1,1)}\}$ for the last in Lemma 14. In this case, we can take $u_{(0,1)}$, $u_{(0,0)}$ and $u_{(1,1)}$ as v, w_1 and w_2 in Lemma 20 in order and hence they are not faithfully embedded on the torus.

Similarly, we find an automorphism τ of $Q(10,3,1)$ which exchanges $u_{(0,0)}$ and $u_{(1,1)}$, fixing $u_{(0,1)}$ and $u_{(-1,1)}$, but it exchanges $u_{(0,5)}$ and $u_{(0,9)}$, which are joined to $u_{(0,0)}$ and $u_{(1,1)}$ by the perfect matching in the complement of $Q(10,3,1)$. Nevertheless, this τ also works as τ in Lemma 20 for $Q(10,3,1)$.

For $Q(p,2,2)$, we take $u_{(0,0)}$, $u_{(0,-1)}$, $u_{(1,0)}$, $u_{(0,1)}$ and $u_{(-1,0)}$ as v, w_0, w_1, w_2 and w_3 in Lemma 20 in order. Since $U_1' = \{u_1', u_0''\} = \{u_{(0,1)}, u_{(1,0)}\}$ in Lemma 14, there is an automorphism τ which exchanges w_1 and w_2, fixing all the others. Therefore, $Q(p,2,2)$ is not faithfully embedded on the torus, too.

For the remaining one $Q(4,0,4)$, we need to discuss it more globally. Each pair of C_i and C_{i+1} induces a 3-cube in the 4-cube Q_4, say $[C_i, C_{i+1}]$, and the four 3-cubes form a solid torus. Our torus where $Q(4,0,4)$ is embedded is the boundary of this solid torus. The 4-cube Q_4 contains eight 3-cubes and each pair of them can be exchanged by its automorphism. For example, $[C_0, C_1]$ can be exchanged to the 3-cube induced by $\{u_{(i,j)} : i = 0, 1, 2, 3,\ j = 0, 1\}$, which lies in the hole of the solid torus. This can be realized by an automorphism τ which fixes the two squares $u_{(0,0)}u_{(1,0)}u_{(1,1)}u_{(0,1)}$ and $u_{(2,2)}u_{(3,2)}u_{(3,3)}u_{(2,3)}$ pointwise, and exchanges $u_{(0,2)}u_{(1,2)}u_{(1,3)}u_{(0,3)}$ and $u_{(3,1)}u_{(2,1)}u_{(2,0)}u_{(3,0)}$. In particular, $C_1 = u_{(0,0)}u_{(0,1)}u_{(0,2)}u_{(0,3)}$ does not bound any face while $\tau(C_1) = u_{(0,0)}u_{(0,1)}u_{(3,1)}u_{(3,0)}$ bounds a face. Therefore, τ does not extend to any auto-homeomorphism over the torus and hence $Q(4, 0, 4)$ is not faithfully embedded on the torus. ■

Now we shall prove that $Q(p, q, r)$ is faithfully embedded on the torus with specified exceptions, which we have disussed in the above lemma.

Theorem 22. *A 4-regular quadrangulation on the torus is faithfully embedded on the torus unless it is isomorphic to* $Q(3, 1, 2)$*,* $Q(4, 0, 4)$*,* $Q(5, 2, 1)$*,* $Q(10, 3, 1)$*,* $Q(p, 2, 2)$ $(p \geq 4)$ *or* $Q(2q + 2, q, 1)$ $(q \geq 2)$*.*

Proof. Suppose that $Q(p, q, r)$ with $0 \leq q \leq p/2$ is not faithfully embedded on the torus, that is, there is an automorphism σ of $Q(p, q, r)$ which does not extend to any auto-homeomorphism over the torus. Then there is a vertex $u_{(x,y)}$ such that σ does not preserve the rotation around $u_{(x,y)}$. Since the automorphism $\tau_{(a,b)}$ defined in Section 1 extends an auto-homeomorphism over the torus, we may assume that $(x,\ y) = (0,\ 0)$ and σ fixes $u_{(0,0)}$; consider $\tau_{(-x',-y')}\sigma\tau_{(x,y)}$ if $\sigma(u_{(x,y)}) = u_{(x',y')}$.

The rotation around $u_{(0,0)}$ reads $u_{(0,1)}u_{(1,0)}u_{(0,-1)}u_{(-1,0)}$ and should include two segments $\sigma(u_{(0,1)})\sigma(u_{(0,-1)})$ and $\sigma(u_{(1,0)})\sigma(u_{(-1,0)})$ since σ shuffles the rotation. Then there are two non-adjacent faces $u_{(0,0)}\sigma(u_{(0,1)})\sigma(w)\sigma(u_{(0,-1)})$ and $u_{(0,0)}\sigma(u_{(1,0)})\sigma(w')\sigma(u_{(-1,0)})$ for some vertex w and w'. If $u_{(1,1)} = u_{(-1,-1)}$ or $u_{(-1,1)} = u_{(1,-1)}$, then we conclude that $Q(p, q, r)$ is isomorphic to either $Q(p, 2, 2)$, $Q(p, p-2, 2)$ or $Q(p, q, 1)$ with $2q \equiv \pm 2 \pmod p$. The first one appears as an exception in the theorem. The second one includes only $Q(3, 1, 2)$ and $Q(4, 2, 2)$ since $p-2 \leq p/2$ implies $p \leq 4$. The former is one of the exceptions and the latter also appears as $Q(p, 2, 2)$ with $p = 4$. The third one is isomorphic to $Q(2q \pm 2, q, 1)$. However, only $Q(2q+2, q, 1)$ survives since $q \leq p/2$ and it also is an exception in the theorem. Thus, we may assume that $u_{(1,1)}$, $u_{(-1,-1)}$, $u_{(-1,1)}$ and $u_{(1,-1)}$ are all distinct. It is easy to see that $w \neq w'$ under this assumption.

Figure 7 presents all cases of the possible positions of w, up to symmetry. Since we fix the parameters p, q and r so that $0 \leq q \leq p/2$, the only symmetry we should consider is the rotation around $u_{(0,0)}$ in $180°$.

Case (I): We have $Q(4, q, r)$. Consider the position of w' and first suppose that $w' = u_{(-2,0)} = u_{(2,0)}$. Then we have $(4/r)q \equiv 0 \pmod 4$ for $r = 1, 2, 4$ and obtain $Q(4, 2, 2)$ and $Q(4, 0, 4)$, which appear as the exceptions in the

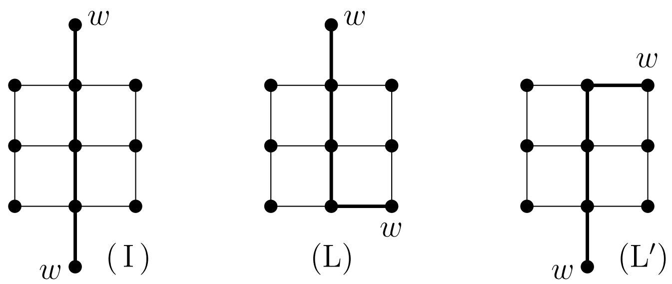

Figure 7 Shuffling the rotation around $u_{(0,0)}$

theorem. Otherwise, w' lies at one of $u_{(-2,0)}$ and $u_{(2,0)}$ and also at one of $u_{(\pm1,\pm1)}$. In either case, $\tau_{(3,\pm1)}$ induces the identity map over $Q(4,q,r)$ and we have $Q(4,q,3)$ with $q \equiv \pm1 \pmod 4$ since $4r \geq 5$, and hence only $Q(4,1,3)$ survives since $q \leq p/2$. This is not an exception in the theomre by Lemma 19.

Case (L): We have $Q(p,3,1)$. If $w' = u_{(-2,0)} = u_{(2,0)}$, then we must have $4 \times 3 \equiv 0 \pmod p$ and hence $p = 6$ or 12. However, $Q(6,3,1)$ is not simple and $Q(12,3,1)$ is isomorphic to $Q(4,1,3)$, which appears in the previous case. Otherwise, $\tau_{(3,\pm1)}$ induces the identity map over $Q(p,3,1)$ as well as in Case I, and $Q(p,3,1)$ is isomorphic to $Q(p,q,1)$ with $3q \equiv \pm1 \pmod p$, depending on the posision of w'. Thus, we have $3 \times 3 \pm 1 \equiv 0 \pmod p$ and hence p divides either 8 or 10. Since $3 \leq p/2$, we conclude that $p = 8$ or 10. However, $Q(8,3,1)$ is isomorphic to $Q(2q+2,q,1)$ with $q = 3$. On the other hand, $Q(10,3,1)$ is one of the exceptions in the theorem.

Case (L′): We have $Q(p,p-3,1)$ and $p \leq 6$ since $p-3 \leq p/2$. Thus, $Q(p,q,r)$ is isomorphic to either $Q(5,2,1)$ or $Q(6,3,1)$. The former appears as an exception in the theorem while the latter should be omitted since it is not simple.

Now we have found all exceptions listed in the theorem and they are not faithfully embedded on the torus actually by Lemma 21. ■

To know how much Theorem 18 covers our main theorem, we need to recognize the 4-regular quadrangulations on the torus that are not polyhedral.

Lemma 23. *A 4-regular quadrangulation $Q(p,q,r)$ is polyhedral unless it is isomorphic to one of $Q(p,2,1)$, $Q(p,1,2)$, $Q(p,2,2)$, $Q(2q+1,q,1)$, $Q(2q+2,q,1)$ with suitable p and q which make them simple.*

Proof. Suppose that $Q(p,q,r)$ with $0 \leq q \leq p/2$ is not polyhedral. Then there are two faces which meet each other at two vertices that are not joined by an edge which the two faces share. Since $Q(p, q, r)$ is vertex-transitive, we may assume that one of the two faces is $u_{(0,0)}u_{(1,0)}u_{(1,1)}u_{(0,1)}$, say A, and we have the four cases B_1 to B_4 for the other, up to symmetry, as depicted in Figure 8. Thus, one vertex incident to A and another incident to B_i should be identical and the difference between their coordinates is one of $(1, \pm 2)$, $(2, \pm 1)$ and $(2, \pm 2)$.

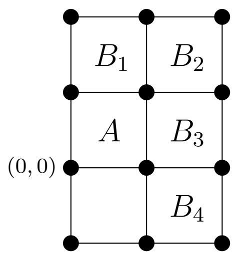

Figure 8 Non-polyhedral cases

In case of $(1, \pm 2)$, we have $Q(p, 2, 1)$ and $Q(p, p-2, 1)$, but the latter should be omitted since $q \leq p/2$. In case of $(2, \pm 1)$, we have $Q(p, q, 2)$ with $q \equiv \pm 1 \pmod p$ and $Q(p, q, 1)$ with $2q \equiv \pm 1 \pmod p$. However, only $Q(p, 2, 1)$ and $Q(2q + 1, q, 1)$ survive in this case. In case of $(2, \pm 2)$, we have $Q(p, q, 2)$ with $q \equiv \pm 2 \pmod p$ and $Q(p, q, 1)$ with $2q \equiv \pm 2 \pmod p$, and only $Q(p, 2, 2)$ and $Q(2q + 2, q, 1)$ survive under the assumption of $q \leq p/2$. Now we have found all exceptions in the lemma and they are not polyhedral actually. ■

If we restrict the 4-regular quadrangulations on the tous to polyhedral ones, then most of them are faithfully embedded on the torus.

Corollary 24. *Every polyhedral 4-regular quadrangulation on the torus is faithfully embedded on the torus except $Q(4, 0, 4)$ and $Q(10, 3, 1)$.*

Proof. It suffices to exclude ones not polyhedral from the exceptions in Theorem 22. ■

Unifying the exceptions in Theorem 22 and Lemma 23, we can complete the list of all 4-regular quadrangulations on the torus that Theorem 18 does not cover:

$$Q(4, 0, 4),\ Q(10, 3, 1),\ Q(p, 2, 1),\ Q(p, 1, 2),\ Q(p, 2, 2),$$
$$Q(2q + 1, q, 1),\ Q(2q + 2, q, 1)$$

Thus, we can prove Theorem 1 as a corollary of Theorem 18 if we decide the distinguishing number of these exceptions individually.

However, the proof of Theomre 18 in [4] proceeds under the strong assumption that there exsits an auto-homeomorphism over the surface as the extension of a given automorphism of a graph. On the other hand, our proof is purely combinatorial except that 4-regular quadrangulations are defined as graphs embedded on the torus.

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