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SELF-NORMALIZED U-STATISTICS DEFINED BY ABSOLUTELY REGULAR SEQUENCES

By

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Abstract. In this paper, we establish some limit theorems for self-normalized U-statistics defined by absolutely regular sequences.

1. Introduction and results

Let $\{\xi_n; -\infty < n < \infty\}$ be a strictly stationary stochastic sequence with values in some countably generated measurable space $(\mathsf{X}, \mathbf{X})$. Denote by F the distribution of ξ_1 and put $\mathbf{M}_b^a = \sigma\{\xi_a, \ldots, \xi_b\}$ $(a \le b)$. We say that $\{\xi_n\}$ absolutely regular if it satisfices the condition

$$\beta(n) = E\left\{\sup_{B\in M_n^\infty} \left|P(B|\mathbf{M}_{\infty}^0) - P(B)\right|\right\} \downarrow 0 \quad (n \to \infty).$$

A measurable function $h : \mathsf{X}^m \to \mathsf{R}$ is called a kernel for

$$\theta = \int \cdots \int h(x_1, \ldots, x_m) \prod_{i=1}^{m} dF(x_i)$$

if it is symmetric in its m arguments. A U-statistic U_n is then given by

$$U_n = U_n(h) = \binom{n}{m}^{-1} \sum_{1\le i_1<i_2<\cdots<i_m\le n} h(\xi_{i_1}, \ldots, \xi_{i_m}).$$

A kernel h is called degenerate for the distribution F if for all choices of $a_i \in \mathsf{X}$ $(1 \le i \le m)$ and all $j \in \{1, \ldots, m\}$

$$Eh(a_1, \ldots a_{j-1}, \xi_j, a_{j+1}, \ldots, a_m) = 0.$$

A U-statistic is called degenerate if the corresponding kernel has this property.

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It is known that by the Hoeffding projection method every U-statistic can be written as a finite weighted sum of degerate ones, namely,

$$U_n = \sum_{l=0}^{m} \binom{m}{l} U_n^{(l)}$$

where U_n^l denotes the U-statistic obtained from the degenerate kernel

$$\hat{h}_l(x_1, \ldots, x_l) = \sum_{j=0}^{l} \binom{l}{j} (-1)^{l-j} h_l(x_1, \ldots, x_l)$$

with

$$\hat{h}_l(x_1, \ldots, x_l) = \int \cdots \int h(x_1, \ldots, x_m) \prod_{i=l+1}^{m} dF(x_i).$$

Define

$$\sigma_n^2 = E\left(\sum_{i=1}^{n} (\hat{h}_1(\xi_i) - \theta)\right)^2 \tag{1}$$

and

$$\sigma^2 = E(\hat{h}_1(\xi_i) - \theta)^2 + \sum_{i=1}^{\infty} E(\hat{h}_1(\xi_1) - \theta)(\hat{h}_1(\xi_{1+i}) - \theta) \tag{2}$$

if the sum converges absolutely.

Let $0 < \gamma < 1/8$. Throughout this paper, we fix this γ. Let r and k be integer-valued functions of n such that

$$r = r(n) = o(n^{(1/4)-\gamma}) \quad \text{and} \quad k = k(n) = \left[\frac{n}{r}\right]. \tag{3}$$

where $[u]$ denotes the largest integer j such that $j \le u$. Put

$$S_n = \sum_{i=1}^{n} \hat{h}_1(\xi_i)$$

and consider the new normalizer

$$C_n^2 = \sum_{j=1}^{k} \left(\sum_{i=1}^{r} (\hat{h}_1(\xi_{(j-1)r+i}) - n^{-1} S_n)\right)^2. \tag{4}$$

In Yoshihara (1976) it was shown that if (6) and (7) (below) hold, then

$$E(U_n(h))^2 = O(n^{-1-\epsilon}) \quad (2 \le l \le m)$$

for some $\epsilon > 0$, which implies

$$\frac{U_n(h) - \theta}{\sqrt{n}\sigma} - \frac{S_n - \theta}{\sqrt{n}\sigma} \to 0 \quad \text{(in probability)}$$

as $n \to 1$, provided $\sigma > 0$. Thus, to study the limiting behavior of $(U_n(h) - \theta)/(\sqrt{n}\sigma)$ it is enough to consider the limiting behavior of $(S_n - \theta)/(\sqrt{n}\sigma)$, which is defined by a strictly stationary sequence $\{\hat{h}_1(\xi_i) - \theta; i \geq 1\}$ of random variables satisfying the absolutely regular condition. Using this fact many important results have been established by many authors. But, unfortunately, we can not use those results, directly, in practice, since they are depend on σ and we can not caliculate σ explicitely. Hence, to solve the real problems using the theoretical results, it is needed to consider new self-normalizers which may be used without knowing the concrete value of σ. The new normalizer C_n^2 is one of the normalizers which generalizes the self-normalizer in the independent case since in that case we may take $r = 1$ and $k = n$.

The object of this paper is to study the limiting behavior of

$$\frac{n}{mC_n}(U_n(h) - \theta).$$

Here, the self-normalizer C_n is used instead of σ_n.

Denker and Keller (1983) showed that the limiting behavior of

$$\frac{n}{m\sigma_n}(U_n(h) - \theta).$$

depends on that of S_n. More specifically, if $(n/m\sigma_n)(S_n - \theta)$ converges in probability or $a.s.$ under some conditions, then $(n/m\sigma_n)(U_n(h) - \theta)$ converges in the same manner under the same conditions (See Yoshihara (1993)). Thus, to know the limiting behavior of $(n/mC_n)(U_n(h) - \theta)$ it is enough only to consider that of

$$\left| \frac{1}{\sigma_n} - \frac{1}{C_n} \right|.$$

We prove the following theorems (Theorems 1-5 (below)).

To formulate the central limit theorem let ρ be the sup-norm metric for functions in $D = D[0, 1]$. Let $\mathcal{G}$ be a set of all functions $g : D \to R$ that are $(D, \mathbf{D})$-measurable and ρ-continuous, or ρ-continuous except at points forming a set of Wiener measure zero on $(D, \mathbf{D})$, where $\mathbf{D}$ denotes the σ-algebra of subsets of D generated by the finite-dimensional subset of D.

Define the $D[0, 1]$-valued random function $X_n = \{X_n(t); 0 \leq t \leq 1\}$ by

$$X_n(t) = \frac{n}{mC_n}(U_{[nt]} - \theta) \quad (0 \leq t \leq 1) \tag{5}$$

Theorem 1. *Let $g \in \mathcal{G}$ be arbitrary. Suppose there exists a $0 < \delta \le 1$ such that*

$$\sup_{1 \le i_1 < \cdots < i_m} E\,|h(\xi_{i_1}, \ldots, \xi_{i_m})|^{2+\delta} < \infty \tag{6}$$

and

$$\sum_{n=1}^{\infty} \beta^{\frac{\delta}{16+\delta}}(n) < \infty. \tag{7}$$

If $\sigma > 0$, then

$$P(g(X_n(\cdot)) < x) \to P(g(W(\cdot)) < x) \tag{8}$$

for any $x \in \mathsf{R}$, where $\{W(t) : 0 \le t \le 1\}$ denotes a standard Wiener process.

Theorem 1 may be generalized as Theorem 2 (below). Let $\mathcal{Q}$ be the class of positive functions $q(t)$ on $(0, 1]$ which are nondecreasing near zero and let

$$I(q, a) = \int_{0+}^{1} \frac{1}{t} \exp\left(-\frac{aq^2(t)}{t}\right) dt, \quad 0 < a < \infty.$$

We define the weighted sup-norm metric $\| \cdot /q\|$ by

$$\left\| \frac{x - y}{q} \right\| = \sup_{0<t<1} \left| \frac{x(t) - y(t)}{q(t)} \right|.$$

Theorem 2. *Let $q \in \mathcal{Q}$ be arbitrary. Suppose the conditions of Theorem 1 hold. Then, for any $g \in \mathcal{G}$ we have that as $n \to \infty$*

$$g\left(\frac{X_n([n\cdot])}{q(\cdot)}\right) \xrightarrow{D} g\left(\frac{W(\cdot)}{q(\cdot)}\right) \quad \text{on } (D[0,1], \| \cdot /q\|), \tag{9}$$

if $I(g, a) < \infty$ for any $a > 0$.

Let

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt.$$

We prove the following Berry-Esseen theorem.

Theorem 3. *Let $h : \mathsf{X}_m \to \mathsf{R}$ be a non-degenerate kernel. Let $\{\xi_n\}$ be absolutely regular with mixing coefficient $\beta(\cdot)$. Suppose there exists a $0 < \delta \le 1$ such that (6) and (7) hold. Then*

$$\Delta_n = \sup_{x \in \mathsf{R}} |P(X_n(1) \le x) - \Phi(x)| = O(n^{-\lambda}) \tag{10}$$

where

$$\lambda = \frac{(1 - \epsilon)\delta}{144} \quad \text{with } \epsilon = \frac{2 - 2\delta - 2\delta^2}{2 + \delta}. \tag{11}$$

Denote by

$$\pi(P,Q) = \inf\{\epsilon > 0 | P(A) \le Q(A^\epsilon) + \epsilon \text{ for all closed } A \subset \mathsf{R}\}$$

the Prohorov distance of two distributions P and Q on R, where

$$A^\epsilon = \{x \in \mathsf{R} | \mathrm{dist}(x,A) < \epsilon\}.$$

THEOREM 4. *Let $h : \mathsf{X}^m \to \mathsf{R}$ be a non-degenerate kernel. Let $\{\xi_n\}$ be absolutely regular with mixing coefficient $\beta(\cdot)$. Suppose there exists a $0 < \delta \le 1/2$ such that (6) and (7) hold. Then*

$$\pi(\mathcal{L}(X_n), N(0,1)) = O(n^{-\lambda}) \tag{12}$$

where λ is the one defined by (11).

THEOREM 5. *Suppose $h : \mathsf{X}^m \to \mathsf{R}$ is a nondegenerate kernel. Suppose furthermore $\{\eta_i\}$ is absolutely regular with mixing coefficient $\beta(\cdot)$ such that*

$$\sup_{1 \le i_1 < \cdots < i_m} E\left|h(\zeta_{i_1}, \ldots, \zeta_{i_m})\right|^8 < \infty \tag{13}$$

and for some $0 < \delta \le 1$

$$\sum_{n=1}^{\infty} (n+1)^{\frac{56-9\delta}{\delta}} \beta(n) < \infty. \tag{14}$$

if $\sigma > 0$.

Then, we can redefine $\{\xi_i\}$ without changing its distribution on a richer space together with a Wiener process $\{W(t); 0 \le t < \infty\}$ such that

$$\frac{n^{\frac{3}{2}}}{mC_n}(U_n - \theta) - W(n) = O\left(n^{\frac{1}{2}-\lambda'}\right) \quad a.s. \tag{15}$$

for some $\lambda' > 0$.

2. Preparatory materials

In this section, we always denote by $\{\zeta_n\}$ a strictly stationary absolutely regular sequence of zero mean random variables.

We use often the following inequalities.

LEMMA A. (I) *Suppose that there exist positive numbers p and κ such that $0 < p + \kappa \le 1$, $E|\zeta_1|^{1+p+\kappa} < \infty$ and*

$$\sum_{i=1}^{\infty} \beta^{\frac{\kappa}{2+\kappa}}(i) < \infty.$$

Then,

$$E\left|\sum_{i=1}^{m} \zeta_i\right|^{1+p} \le cm \left\{E|\zeta_1|^{1+p+\kappa}\right\}^{\frac{1}{1+p+\kappa}}. \tag{16}$$

(See, Utev (1984).)
(II) *If for some $\delta > 0$*

$$E|\zeta_1|^{2+\delta} < \infty,$$

then

$$E\left(\sum_{i=1}^{m} \zeta_i\right)^2 \le cm \left\{E|\zeta_1|^2 + \left\{E|\zeta_1|^{2+\delta}\right\}^{\frac{2}{2+\delta}} \sum_{i=1}^{m} \beta^{\frac{\delta}{2+\delta}}(i)\right\}. \tag{17}$$

(III) *If for $p \ge 2$ and for some $\delta > 0$*

$$E|\zeta_1|^{p+\delta} < \infty \quad \text{and} \quad \sum_{i=0}^{\infty}(i+1)^{\frac{(p-2)(p+\delta)+p}{\delta}}\beta(i) < \infty \tag{18}$$

then

$$E\left|\sum_{i=1}^{m} \zeta_i\right|^{p} \le cm^{\frac{p}{2}} \left(E|\zeta_1|^{p+\delta}\right)^{\frac{p}{p+\delta}}. \tag{19}$$

Let r and k be the ones defined by (3) and put

$$\vartheta_j = \sum_{i=(j-1)r+1}^{jr} \zeta_i \quad \text{and} \quad B_n - 2 = \sum_{j=1}^{k} \vartheta_j^2.$$

Furthermore, put

$$\sigma_m^2(\zeta) = E\left(\sum_{i=1}^{m} \zeta_i\right)^2 \quad \text{and} \quad \sigma^2(\zeta) = E\zeta_1^2 + 2\sum_{i=1}^{\infty} E\zeta_1\zeta_{1+i}.$$

If $\sigma^2(\zeta)$ exists, then, by Lemma A we have

$$\sigma_m^2(\zeta) = m\sigma^2(\zeta)(1 + o(1)). \tag{20}$$

We prove the following lemma.

Lemma 1. (I) *Suppose there exists a* $0 < \delta \leq 1$ *such that*

$$E|\zeta_1|^{2+\delta} < \infty \quad \textit{and} \quad \sum_{i=1}^{\infty} \beta^{\frac{\delta}{8+\delta}}(i) < \infty. \tag{21}$$

If $\sigma(\zeta) > 0$, *then*

$$P\left(B_n^2 \leq \frac{1}{5}\sigma_n^2(\zeta)\right) = o(n^{-\gamma}). \tag{22}$$

(II) *If for some* $0 < \delta < \min\{16\gamma/(1+2\gamma), 1\}$

$$E|\zeta_1|^{4+\delta} < \infty \quad \textit{and} \quad \sum_{i=0}^{\infty} (i+1)^{\frac{12+2\delta}{\delta}} \beta(i) < \infty \tag{23}$$

and $\sigma(\zeta) > 0$, *then*

$$B_n^2 \geq \frac{1}{6} kr\sigma^2(\zeta) \quad a.s. \tag{24}$$

Proof. By (20) to prove (22) it suffices to show that

$$P\left(B_n^2 \leq \frac{1}{5}k\sigma_r^2(\zeta)\right) = o(n^{-\gamma}). \tag{25}$$

By the Jensen inequality

$$\frac{1}{k}\sum_{j=1}^{k}\left(\frac{\vartheta_j}{\sigma_r(\zeta)}\right)^2 \geq \left(\frac{1}{k}\sum_{j=1}^{k}\frac{|\vartheta_j|}{\sigma_r(\zeta)}\right)^2,$$

we have that for all n sufficiently large and for any $M > 0$

$$P\left(\frac{1}{k}\sum_{j=1}^{k}\left(\frac{\vartheta_j}{\sigma_r(\zeta)}\right)^2 < M\right) \leq P\left(\frac{1}{k}\sum_{j=1}^{k}\frac{|\vartheta_j|}{\sigma_r(\zeta)} < \sqrt{M}\right)$$

Thus, to prove (25) it suffices to show that

$$P\left(\frac{1}{k}\sum_{j=1}^{k}\frac{|\vartheta_j|}{\sigma_r(\zeta)} < \frac{1}{2}\right) = o(n^{-\gamma}) \quad (n \to \infty). \tag{26}$$

We note that by Melevède and Peligrad (2000)

$$\lim_{n\to\infty} E\frac{|\vartheta_j|}{\sigma_r(\zeta)} = \lim_{n\to\infty} E\left|\frac{1}{\sigma_r(\zeta)}\sum_{i=1}^{r}\zeta_i\right| = \sqrt{\frac{2}{\pi}}$$

if (21) holds and hence for all n sufficiently large

$$E\frac{|\vartheta_j|}{\sigma_r(\zeta)} > \sqrt{\frac{1}{2}}.$$

Let $\tilde{\kappa} = \sqrt{1/2} - 1/2 > 0$. Since $\{\vartheta_j\}$ is absolutely regular with mixing coefficient $\beta(\cdot)$ and $E\vartheta_1^2 = \sigma_r(\zeta)$, using the Chebyshev inequality, (16) with $p = (2-\delta)/(2+\delta)$ and (19) with $p = 2$

$$\begin{aligned}
&L.\ H.\ S.\ \text{of (26)} \\
&\le P\left(\frac{1}{k}\sum_{j=1}^{k}\left(\frac{|\vartheta_j|}{\sigma_r(\zeta)} - E\frac{|\vartheta_j|}{\sigma_r(\zeta)}\right) \le \frac{1}{2} - E\frac{|\vartheta_j|}{\sigma_r(\zeta)}\right) \\
&\le P\left(\frac{1}{k}\left|\sum_{j=1}^{k}\left(\frac{|\vartheta_j}{\sigma_r(\zeta)} - E\frac{|\vartheta_j|}{\sigma_r(\zeta)}\right)\right| \ge \tilde{\kappa}\right) \\
&\le \frac{1}{k^{\frac{4}{2+\delta}}\tilde{\kappa}^{\frac{4}{2+\delta}}} E\left|\sum_{j=1}^{k}\left(\frac{|\vartheta_j|}{\sigma_r(\zeta)} - E\frac{|\vartheta_j|}{\sigma_r(\zeta)}\right)\right|^{\frac{4}{2+\delta}} \\
&\le \frac{c}{k^{\frac{4}{2+\delta}}} k\left(E\left|\frac{|\vartheta_j|}{\sigma_r(\zeta)} - E\frac{|\vartheta_j|}{\sigma_r(\zeta)}\right|^2\right)^{\frac{1}{2}} \\
&\le \frac{c}{k^{\frac{2-\delta}{2+\delta}}}\left(E\left|\frac{|\vartheta_1|}{\sigma_r(\zeta)}\right|^2\right)^{\frac{1}{2}} \le \frac{c}{k^{\frac{2-\delta}{2+\delta}}} = o(n^{-\gamma}),
\end{aligned} \tag{27}$$

which implies (26) and hence (22) is obtained.

To prove (24) we need to prove

$$P\left(B_n^2 < \frac{1}{5}k_n\sigma_{r_n}^2(\zeta)\ i.o.\right) = 0,$$

which is shown by the Borel-Cantelli lemma if we show

$$\sum_{n=1}^{\infty} P\left(B_n^2 < \frac{1}{5}k_n\sigma_{r_n}^2(\zeta)\right) < \infty. \tag{28}$$

Using the proof of (27) and (19) with $p = 3$ first and then (19) with $p = 3 + \delta$, we have

$$
\begin{aligned}
& P\left(B_n^2 < \frac{1}{5}k_n\sigma_{r_n}^2(\zeta)\right)\\
& \le P\left(\left|\frac{1}{k_n}\sum_{j=1}^{k_n}\left(\frac{|\vartheta_j|}{\sigma_r(\zeta)} - E\frac{|\vartheta_j|}{\sigma_r(\zeta)}\right)\right| \ge \tilde{\kappa}\right)\\
& \le \frac{1}{\tilde{\kappa}^3 k_n^3} E\left|\sum_{j=1}^{k_n}\left(\frac{|\vartheta_j|}{\sigma_{r_n}(\zeta)} - E\frac{|\vartheta_j|}{\sigma_{r_n}(\zeta)}\right)\right|^3\\
& \le \frac{c}{k_n^3}k_n^{\frac{3}{2}}\left(E\left|\frac{|\vartheta_1|}{\sigma_{r_n}(\zeta)} - E\frac{|\vartheta_1|}{\sigma_{r_n}(\zeta)}\right|^{3+\delta}\right)^{\frac{3}{3+\delta}}\\
& \le \frac{c}{k_n^{\frac{3}{2}}}\left(E\left|\frac{\vartheta_1}{\sigma_{r_n}(\zeta)}\right|^{3+\delta}\right)^{\frac{3}{3+\delta}} \le \frac{c}{k_n^{\frac{3}{2}}} \le \frac{c}{n^{1+\tau}}
\end{aligned}
$$

for some $\tau < 0$. Hence

$$
L.\ H.\ S.\ \text{of (28)} \ \le \sum_{n=1}^{\infty}\frac{c}{n^{1+\tau}} < \infty
$$

and (24) is obtained. □

Put

$$
T_n = \sum_{i=1}^{n}\eta_i
$$

and

$$
\sigma_m^2(\eta) = E\left(\sum_{i=1}^{m}\eta_i\right)^2 \quad \text{and} \quad \sigma^2(\eta) = E\eta_1^2 + 2\sum_{i=1}^{\infty}E\eta_1\eta_{1+i}.
$$

Lemma 2. *Let $\{\eta_n\}$ be a strictly stationary absolutely regular sequence of random variables with $E\eta_1 = \theta$ and mixing coefficient $\beta(\cdot)$. Let $0 < \epsilon < (1/2)\gamma$ be arbitrary.*

(I) *Suppose there exists a $0 < \delta < 1$ such that*

$$
E|\eta_1|^{2+\delta} < \infty \quad \textit{and} \quad \sum_{i=1}^{\infty}\beta^{\frac{\delta}{4+\delta}}(i) < \infty. \tag{29}
$$

If $\sigma(\eta) > 0$, then

$$
P\left(\left|\frac{T_n}{n} - \theta\right| \ge n^{-\frac{1}{4}+\epsilon}\right) = o(n^{-\frac{1}{2}}). \tag{30}
$$

(II) *If for some* $\delta > 0$

$$E|\eta_1|^{4+\delta} < \infty \quad \textit{and} \quad \sum_{i=0}^{\infty}(i+1)^{\frac{12+2\delta}{\delta}}\beta(i) < \infty \tag{31}$$

and $\sigma(\zeta) > 0$*, then*

$$\left|\frac{T_n}{n} - \theta\right| \leq n^{-\frac{1}{4}+\epsilon} \quad a.s. \tag{32}$$

Proof. (I) By the Chebyshev inequality and (19) with $p = 2$ we have

$$\begin{aligned} &P\left(\left|\frac{T_n}{n} - \theta\right| \geq n^{-\frac{1}{4}+\epsilon}\right) \\ &\leq P\left(\left|\sum_{i=1}^{n}(\eta_i - \theta)\right| \geq n^{\frac{3}{4}+\epsilon}\right) \leq \frac{1}{n^{\frac{3}{2}+2\epsilon}} E\left|\sum_{i=1}^{n}(\eta_i - \theta)\right|^2 \\ &\leq cn^{-\frac{3}{2}-2\epsilon} \cdot n = cn^{-\frac{1}{2}-2\epsilon} \end{aligned}$$

which implies (30).

(II) To prove (32), we need to show that

$$P\left(\left|\frac{T_n}{n} - \theta\right| \geq n^{-\frac{1}{4}+\epsilon} i.o.\right) = 0.$$

Thus, it suffices to show

$$\sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{n}(\eta_i - \theta)\right| \geq n^{\frac{3}{4}+\epsilon}\right) < \infty,$$

which is obtained from the Chebyshev inequality, since by (19) with $p = 4$

$$\begin{aligned} &P\left(\left|\sum_{i=1}^{n}(\eta_i - \theta)\right| \geq n^{\frac{3}{4}+\epsilon}\right) \\ &\quad \leq \frac{1}{n^{3+4\epsilon}} E\left|\sum_{i=1}^{n}(\eta_i - \theta)\right|^4 \leq cn^{-3-4\epsilon} \cdot n^2 = cn^{-1-4\epsilon} \end{aligned}$$

for all n. □

Lemma 3. *Let* $\{\eta_n\}$ *be a strictly stationary absolutely regular sequence of random variables with* $E\eta_1 = \theta$ *and mixing coefficient* $\beta(\cdot)$*.*

(I) *Suppose there exists a* $0 < \delta \leq 1$ *such that*

$$E|\eta_1|^{2+\delta} < \infty \quad \textit{and} \quad \sum_{i=1}^{\infty}\beta^{\frac{\delta}{16+\delta}}(i) < \infty. \tag{33}$$

If $\sigma(\eta) > 0$, then

$$P\left(\left|\frac{1}{\sigma_n(\eta)} - \frac{1}{C_n(\eta)}\right| \geq n^{-(1/2)-\rho}\right) = o(n^{-\frac{\delta}{16}}). \tag{34}$$

(II) *If for some $0 < \delta < 8\gamma/(1+2\gamma)$*

$$E|\eta_1|^8 < \infty \quad and \quad \sum_{i=0}^{\infty}(i+1)^{\frac{56-6\delta}{\delta}}\beta(i) < \infty \tag{35}$$

and $\sigma(\zeta) > 0$, then

$$\left|\frac{1}{\sigma_n(\eta)} - \frac{1}{C_n(\eta)}\right| \leq \frac{1}{6}n^{-\frac{1}{2}-\rho}\sigma(\eta) \quad a.s. \tag{36}$$

Here, $\rho = \delta/16(4+\delta)$ and

$$C_n^2(\eta) = \sum_{j=1}^{k}\left(\sum_{i=1}^{r}\left(\eta_{(j-1)r+i} - \frac{T_n}{n}\right)\right)^2.$$

Proof. Let

$$B_n^2(\eta) = \sum_{j=1}^{k}\left(\sum_{i=1}^{r}(\eta_{(j-1)r+i} - \theta)\right)^2.$$

We note first that

$$\begin{aligned}
&\left|\frac{1}{\sigma_n(\eta)} - \frac{1}{C_n(\eta)}\right| \\
&\leq \left|\frac{1}{\sigma_n(\eta)} - \frac{1}{B_n(\eta)}\right| + \left|\frac{1}{B_n(\eta)} - \frac{1}{C_n(\eta)}\right| \\
&= \left|\frac{1}{\sigma_n(\eta)} - \frac{1}{B_n(\eta)}\right| + \frac{|B_n^2(\eta) - C_n^2(\eta)|}{B_n(\eta)C_n(\eta)(B_n(\eta)+C_n(\eta))}
\end{aligned}$$

By the Cauchy-Schwarz inequality

$$
\begin{aligned}
&|B_n^2(\eta) - C_n^2(\eta)| \\
&\le \sum_{j=1}^{k} \left| \left(\sum_{i=1}^{r} (\eta_{(j-1)r+i} - \theta) \right)^2 - \left(\sum_{i=1}^{r} (\eta_{(j-1)r+i} - n^{-1}T_n) \right)^2 \right| \\
&= \sum_{j=1}^{k} \left| \sum_{i=1}^{r} (\eta_{(j-1)r+i} - \theta) - \sum_{i=1}^{r} (\eta_{(j-1)r+i} - n^{-1}T_n) \right| \\
&\qquad \times \left| \sum_{i=1}^{r} (\eta_{(j-1)r+i} - \theta) + \sum_{i=1}^{r} (\eta_{(j-1)r+i} - n^{-1}T_n) \right| \\
&= \sum_{j=1}^{k} r|n^{-1}T_n - \theta| \left| 2\sum_{i=1}^{r} (\eta_{(j-1)r+i} - \theta) - r(n^{-1}T_n + \theta) \right| \\
&\le 2r|n^{-1}T_n - \theta| \sum_{j=1}^{k} \left| \sum_{i=1}^{r} (\eta_{(j-1)r+i} - \theta) \right| \\
&\qquad + kr^2 |n^{-1}T_n - \theta|\, |n^{-1}T_n + \theta| \\
&\le 2r\sqrt{k}|n^{-1}T_n - \theta| B_n(\eta) + kr^2|n^{-1}T_n - \theta|\, |n^{-1}T_n + \theta|
\end{aligned}
$$

and hence

$$
\begin{aligned}
&\frac{|B_n^2(\eta) - C_n^2(\eta)|}{B_n(\eta)C_n(\eta)(B_n(\eta) + C_n(\eta))} \\
&\le \frac{2\sqrt{k}r|n^{-1}T_n - \theta|}{C_n^2(\eta)} + \frac{kr^2|n^{-1}T_n - \theta|\, |n^{-1}T_n + \theta|}{B_n(\eta)C_n^2(\eta)}.
\end{aligned}
$$

Combining these relations, we have

$$
\begin{aligned}
\left| \frac{1}{\sigma_n(\eta)} - \frac{1}{C_n(\eta)} \right| &\le \left| \frac{1}{\sigma_n(\eta)} - \frac{1}{B_n(\eta)} \right| \\
&\quad + \frac{2\sqrt{k}r|n^{-1}T_n - \theta|}{C_n^2(\eta)} + \frac{kr^2|n^{-1}T_n - \theta|\, |n^{-1}T_n + \theta|}{B_n(\eta)C_n^2(\eta)}.
\end{aligned} \tag{37}
$$

Furthermore, if $|n^{-1}T_n - \theta| \le n^{-(1/4)+\epsilon}$ and $B_n^2(\eta) \ge (1/5)kr\sigma^2(\eta)$, then $|n^{-1}T_n + \theta| \le c_0$ for some $c_0 > 0$ and

$$
C_n(\eta) \ge \sqrt{\frac{1}{5}kr\sigma^2(\eta) + o(n)} \ge \frac{1}{3}\sqrt{kr}\sigma(\eta),
$$

which implies

$$
\begin{aligned}
&\frac{2\sqrt{k}r|n^{-1}T_n - \theta|}{C_n^2(\eta)} + \frac{kr^2|n^{-1}T_n - \theta|\, |n^{-1}T_n + \theta|}{B_n(\eta)C_n^2(\eta)} \\
&\le c\left(\frac{1}{\sqrt{k}} \frac{1}{n^{\frac{1}{4}-\epsilon}} + \frac{\sqrt{r}}{\sqrt{k}} \frac{1}{n^{\frac{1}{4}-\epsilon}} \right) \le cn^{-\frac{1}{2}-\frac{\gamma}{2}+\epsilon}.
\end{aligned}
$$

Since we can prove

$$\sigma_n^2(\eta) - k\sigma_r^2(\eta) = O(r),$$

using the elementary inequality

$$|\sqrt{a} - \sqrt{b}| \le \sqrt{|a-b|} \quad (a, b \ge 0)$$

we have

$$\begin{aligned}
(38) \qquad & \left|\frac{1}{\sigma_n(\eta)} - \frac{1}{B_n(\eta)}\right| = \frac{|B_n(\eta) - \sigma_n(\eta)|}{\sigma_n(\eta)B_n(\eta)} \\
& \le \frac{|B_n(\eta) - \sqrt{k\sigma_r^2(\eta)}|}{\sigma_n(\eta)B_n(\eta)} + \frac{|\sigma_n(\eta) - \sqrt{k\sigma_r^2(\eta)}|}{\sigma_n(\eta)B_n(\eta)} \\
& \le c\left\{\frac{\sqrt{|B_n^2(\eta) - k\sigma_r^2(\eta)|}}{kr\sigma^2(\eta)} + \frac{|\sigma_n^2(\eta) - k\sigma_r^2(\eta)|}{\sigma_n(\eta)B_n(\eta)(\sigma_n(\eta) + \sqrt{k\sigma_r^2(\eta)})}\right\} \\
& \le c\left\{\frac{\sqrt{|B_n^2(\eta) - k\sigma_r^2(\eta)|}}{kr\sigma^2(\eta)} + \frac{1}{\sqrt{kn}}\right\}.
\end{aligned}$$

Assume that the conditions of (I) hold. Let $0 < \delta \le 1$. Then, by (16) with $p = \epsilon = \delta/8$ first and then by (19) with $p = 2 + (\delta/2)$ we have

$$\begin{aligned}
(39) \qquad & E\left\{\frac{\sqrt{|B_n^2(\eta) - k\sigma_r^2(\eta)|}}{\sqrt{kr}\sigma(\eta)}\right\}^{2+(\delta/4)} \\
& = \frac{1}{(\sqrt{kr}\sigma(\eta))^{2+(\delta/4)}} E\left|\sum_{j=1}^{k}\left\{\left(\sum_{i=1}^{r}(\eta_{(j-1)r+i} - \theta)\right)^2 - \sigma_r^2(\eta)\right\}\right|^{1+\frac{\delta}{8}} \\
& \le \frac{c}{(kr\sigma^2(\eta))^{1+\frac{\delta}{8}}} k\left\{E\left|\left(\sum_{i=1}^{r}(\eta_i - \theta)\right)^2 - \sigma_r^2(\eta)\right|^{1+\frac{\delta}{4}}\right\}^{\frac{1}{1+\frac{\delta}{4}}} \\
& \le \frac{c}{(kr\sigma^2(\eta))^{1+\frac{\delta}{8}}} k\left\{E\left|\sum_{i=1}^{r}(\eta_i - \theta)\right|^{2+\frac{\delta}{2}}\right\}^{\frac{4}{4+\delta}} \\
& = \frac{c}{k^{\frac{\delta}{8}} r^{1+\frac{\delta}{8}}} (r^{1+\frac{\delta}{4}})^{\frac{4}{4+\delta}} = \frac{c}{k^{\frac{\delta}{8}} r^{\frac{\delta}{8}}} = cn^{-\frac{\delta}{8}}.
\end{aligned}$$

From (37), (39) and the definition of ρ we have

$$\begin{aligned}
& P\left(\left|\frac{1}{\sigma_n(\eta)} - \frac{1}{C_n(\eta)}\right| \ge n^{-\frac{1}{2}-\rho}, \right. \\
& \qquad \left. \left|\frac{T_n}{n} - \theta\right| \le n^{-\frac{1}{4}+\epsilon}, B_n^2(\eta) \ge \frac{1}{5}kr\sigma^2(\eta)\right) \\
& \le P\left(\left|\frac{1}{\sigma_n(\eta)} - \frac{1}{B_n(\eta)}\right| \ge \frac{1}{2}n^{-\frac{1}{2}-\rho}, B_n^2(\eta) \ge \frac{1}{5}kr\sigma^2(\eta)\right) \\
& \le cn^{(\frac{1}{2}+\rho)(2+\frac{\delta}{2})} \frac{1}{(kr)^{\frac{1}{2}(2+\frac{\delta}{2})}} E\left\{\frac{\sqrt{|B_n^2(\eta) - k\sigma_r^2(\eta)|}}{\sqrt{kr}\sigma(\eta)}\right\}^{2+\frac{\delta}{2}} \le cn^{-\frac{\delta}{16}}.
\end{aligned}$$

Thus, by Lemmas 1(I) and 2(I)

$$\begin{aligned}
&P\Big(\Big|\frac{1}{\sigma_n(\eta)}-\frac{1}{C_n(\eta)}\Big|\geq 2n^{-\frac{1}{2}-\rho}\Big)\\
&\leq P\Big(\Big|\frac{1}{\sigma_n(\eta)}-\frac{1}{C_n(\eta)}\Big|\geq n^{-\frac{1}{2}-\rho},\\
&\qquad \Big|\frac{T_n}{n}-\theta\Big|\leq n^{-\frac{1}{4}+\epsilon}, B_n^2(\eta)\geq \frac{1}{5}kr\sigma^2(\eta)\Big)\\
&\quad +P\Big(\Big|\frac{T_n}{n}-\theta\Big|\geq n^{-\frac{1}{4}+\epsilon}\Big)+P\Big(B_n^2(\eta)\leq \frac{1}{5}kr\sigma^2(\eta)\Big)\\
&=o(n^{-\frac{\delta}{16}})+o(n^{-\frac{1}{2}})+o(n^{-\gamma})=o(n^{-\frac{\delta}{16}}),
\end{aligned}$$

which completes the proof of (34).

Next, assume that the conditions of (II) hold. Using (35), (19) with $p=4-\delta$ firstly and then (19) with $p=8-\delta$ we have that

$$\begin{aligned}
(40)\qquad &E\Big\{\frac{\sqrt{|B_n^2(\eta)-k\sigma_r^2(\eta)|}}{\sqrt{kr}\sigma(\eta)}\Big\}^{8-2\delta}\\
&=\frac{1}{(\sqrt{kr}\sigma(\eta))^{8-2\delta}}E\Big|\sum_{j=1}^{k}\Big\{\Big(\sum_{i=1}^{r}\eta_{(j-1)r+i}-\theta\Big)^2-\sigma_r^2(\eta)\Big\}\Big|^{4-\delta}\\
&\leq \frac{c}{(kr)^{4-\delta}}k^{2-\frac{\delta}{2}}\Big\{E\Big|\Big(\sum_{i=1}^{r}\eta_i-\theta\Big)^2-\sigma_r^2(\eta)\Big|^{4-\frac{\delta}{2}}\Big\}^{\frac{4-\delta}{4-(\delta/2)}}\\
&\leq \frac{c}{k^{2-\frac{\delta}{2}}r^{4-\delta}}\Big\{E\Big|\sum_{i=1}^{r}\eta_i-\theta\Big|^{8-\delta}\Big\}^{\frac{8-2\delta}{8-\delta}}\\
&\leq \frac{c}{k^{2-\frac{\delta}{2}}r^{4-\delta}}(r^{4-\frac{\delta}{2}})^{\frac{8-2\delta}{8-\delta}}=\frac{c}{k^{2-\frac{\delta}{2}}}=\frac{c}{(n^{\frac{3}{4}+\gamma})^{2-\frac{\delta}{2}}}=\frac{c}{n^{1+\tau}}
\end{aligned}$$

for some $\tau>0$. Hence, (36) follows from (37), (38), Lemmas 1(II), 2(II) and the Borel-Cantelli lemma. □

Lemma 4. (I) *Suppose conditions of Lemma 3(I) holds. Then,*

$$\frac{1}{\sigma_n(\eta)}\sum_{i=1}^{n}\eta_i-\frac{1}{C_n(\eta)}\sum_{i=1}^{n}\eta_i \xrightarrow{P} 0 \quad (n\to\infty).$$

(II) *Suppose conditions of Lemma 3(II) holds. Then,*

$$\frac{1}{\sigma_n(\eta)}\sum_{i=1}^{n}\eta_i-\frac{1}{C_n(\eta)}\sum_{i=1}^{n}\eta_i \;\to\; 0 \quad a.s. \quad (n\to\infty).$$

Proof. We note that if

$$\left|\frac{1}{\sigma_n(\eta)} - \frac{1}{C_n(\eta)}\right| \le c_0 n^{-\frac{1}{2}-\rho}$$

for some $c_0 > 0$, then

$$\begin{aligned}&\left|\frac{1}{\sigma_n(\eta)}\sum_{i=1}^{n}\eta_i - \frac{1}{C_n(\eta)}\sum_{i=1}^{n}\eta_i\right| \\ &\le \left|\frac{1}{\sigma_n(\eta)} - \frac{1}{C_n(\eta)}\right|\left|\sum_{i=1}^{n}\eta_i\right| \le \frac{c_0}{n^{\frac{1}{2}+\rho}}\left|\sum_{i=1}^{n}\eta_i\right|.\end{aligned}$$

Since for the sequence $\{\eta_i\}$ the law of the iterated logarithm holds under the conditions in Lemma 3(I) and consequently in Lamma 3(II), the last term in the above inequalities is equal to 0 almost surely. Hence, we have the desired conclusions. □

From Lemma 4(I) and the well-known weak convergence theorem for strictly stationary sequence we have the following theorem.

Theorem 6. *Let $\{\zeta_i\}$ be a strictly stationary absolutely regular sequence of zero mean random variables with mixing coefficient $\beta(\cdot)$. Suppose there exists a $0 < \delta \le 1$ such that (31) holds and $\sigma(\eta) > 0$. Then, for any $g \in \mathcal{G}$*

$$P(g(\bar{X}_n([n\cdot])) < u) \to P(g(W(\cdot)) < u)$$

for all $u \in \mathsf{R}$, where

$$\bar{X}_n(l) = \left(\sum_{j=1}^{k}\left(\sum_{i=1}^{r}\zeta_{(j-1)r+i}\right)^2\right)^{-\frac{1}{2}}\sum_{i=1}^{l}\zeta_i.$$

Remark. By the same method, we can deduce the same conclusion for the strictly stationary strong mixing sequence under the analogous conditions in Theorem 6.

3. Proof of Theorems

Proof of Theorem 1. Since $g \in \mathcal{G}$ and $E\hat{h}_1(\xi_1) = \theta$, putting $\zeta_i = \hat{h}_1(\xi_1) - \theta$ from Theorem 6, we have the theorem. □

Proof of Theorem 2. The proof is easily obtained from Theorem 1. □

To prove Theorems 3 and 4, we need the following theorem due to Denker and Keller (1983).

Theorem A. *Suppose $h : \mathsf{X}^m \to \mathsf{R}$ is a nondegenerate kernel. Suppose furthermore $\{\eta_i\}$ is absolutely regular with mixing coefficient $\beta(\cdot)$ satisfying $\beta^{\delta/(2+\delta)}(n) = O(n^{-2+\epsilon})$ for some $0 < \delta \le 1$, $0 \le \epsilon < 1$, $\sigma(\zeta) > 0$ and*

$$\sup_{1\le i_1<\cdots<i_m} E|h(\zeta_{i_1}, \cdots, \zeta_{i_m})|^{2+\delta} < \infty.$$

Then

$$\sup_{u\in\mathsf{R}} \left| P\left(\frac{n}{\sigma_n(\zeta)} (U_n - \theta) < u \right) - \Phi(u) \right| = O(n^{-\lambda}) \tag{41}$$

and

$$\pi\left(\mathcal{L}\left(\frac{n}{\sigma_n(\zeta)} (U_n - \theta) \right), N(0,1) \right) = O(n^{-\lambda}) \tag{42}$$

where λ is the one defined by (11).

Proof of Theorem 3. To prove Theorem 3 we note that

$$\begin{aligned}
&\sup_{u\in\mathsf{R}} \left| P\left(\frac{U_n - n\theta}{C_n(\eta)} < u \right) - \Phi(u) \right| \\
&\le P\left(\left| \frac{1}{\sigma_n(\zeta)} - \frac{1}{C_n(\eta)} \right| |U_n - n\theta| \ge n^{-\lambda} \right) \\
&\quad + \sup_{u\in\mathsf{R}} \max\Bigg\{ \left| P\left(\frac{1}{\sigma_n(\zeta)} (U_n - n\theta) < u + n^{-\lambda} \right) - \Phi(u + n^{-\lambda}) \right|, \\
&\qquad\qquad \left| P\left(\frac{1}{\sigma_n(\zeta)} (U_n - n\theta) < u - n^{-\lambda} \right) - \Phi(u - n^{-\lambda}) \right| \Bigg\} \\
&\quad + \max\{ \sup_{u\in\mathsf{R}} |\Phi(u + n^{-\lambda}) - \Phi(u)|, \sup_{u\in\mathsf{R}} |\Phi(u - n^{-\lambda}) - \Phi(u)| \} \\
&= I_1 + I_2 + I_3, \quad \text{(say)}.
\end{aligned}$$

By (41) $I_2 = O(n^{-\lambda})$. On the other hand, $I_3 = O(n^{-\lambda})$ follows from the elementary inequality

$$\sup_{u\in\mathsf{R}} |\Phi(u + q) - \Phi(u)| \le \frac{|q|}{\sqrt{2\pi}}.$$

Finally, by Lemma 3, the Chebyshev inequality and Lemma A we have

$$\begin{aligned}
I_1 &\le P\left(\left| \frac{1}{\sigma_n(\zeta)} - \frac{1}{C_n(\eta)} \right| \ge n^{-\frac{1}{2} - 2\lambda} \right) \\
&\quad + P\left(n^{-\frac{1}{2} - 2\lambda} |S_n - n\theta| \ge n^{-\lambda} \right) \\
&= O(n^{-2\lambda}) + O(n^{-2\lambda}) = O(n^{-2\lambda}).
\end{aligned}$$

Hence, Theorem 3 is obtained. □

Proof of Theorem 4. Using the same method in the proof of Theorem 3, we can prove Theorem 4 by (42), (cf. Dudley (1968)). □

To prove Theorem 5 we use the following theorem due to Denker and Keller (1983).

THEOREM B. *Suppose* $h : \mathsf{X}^m \to \mathsf{R}$ *is a nondegenerate kernel. Suppose furthermore* $\{\xi_i\}$ *is absolutely regular with mixing coefficient* $\beta(\cdot)$ *satisfying* $\beta^{\delta/(2+\delta)}(n) = O(n^{-2+\epsilon})$ *for some* $0 < \delta \le 1$, $0 \le \epsilon < 1/2$, $\sigma(\zeta) > 0$ *and*

$$\sup_{1\le i_1<\cdots<i_m} E|h(\zeta_{i_1}, \cdots, \zeta_{i_m})|^{2+\delta} < \infty.$$

Then, we can redefine $\{\xi_i\}$ *without changing its distribution on a richer space together with a Wiener process* $\{W(t); 0 \le t < \infty\}$ *such that*

$$\frac{n}{m\sigma}(U_n - \theta) - W(n) = O(n^{\frac{1}{2}-\lambda'}) \quad a.s. \quad \text{for some } \lambda' > 0. \tag{43}$$

Proof of Theorem 5. If the conditions of Theorem 5 are satisfied, then by Lemma 3(II)

$$\begin{aligned}
&\left|\frac{1}{\sigma_n(\zeta)} - \frac{1}{C_n(\eta)}\right| |U_n - n\theta| \\
&= n^{\frac{1}{2}(1+\rho)}\left|\frac{1}{\sigma_n(\zeta)} - \frac{1}{C_n(\eta)}\right| \cdot n^{-\frac{1}{2}(1+\rho)}|U_n - n\theta| \\
&= O(n^{-\frac{\rho}{2}})o(1) = o(n^{-\frac{\rho}{2}}) \quad a.s.,
\end{aligned}$$

since for the sequence $\{U_n - n\theta\}$ the law of the iterated logarithm holds. Hence, the desired conclusion follows. □

Remark. Define a von Mises' functional V_n by

$$V_n = \frac{1}{n^m}\sum_{i_1=1}^{n}\cdots\sum_{i_m=1}^{n} h(\xi_{i_1}, \cdots, \xi_{i_m}) \quad (n \ge 1).$$

The statements in Theorems 1-5 hold for von Mises' functionals under the analogous assumptions, when (6) and (13) are replaced by

$$\sup_{i_1,\cdots,i_m\ge 1} E|h(\xi_{i_1}, \cdots, \xi_{i_m})|^{2+\delta} < \infty$$

and

$$\sup_{i_1,\cdots,i_m\ge 1} E|h(\xi_{i_1}, \cdots, \xi_{i_m})|^{8} < \infty,$$

respectively.

4. Simulation examples

To demonstrate simulation examples, in this section, we consider strictly stationary sequences $\{X_i\}$ of l-dependent random variables. Hence, the mixing condition $\beta(n) = 0$ $(n > l)$ is satisfied. Suppose $E\exp(t|X_i|) < \infty$ $(t > 0)$. Then, from the proofs in the preceding section it is obvious that we may choose r as $r = [n^{(1/2)-\gamma}]$ $(0 < \gamma < (1/2))$ and hence $k = [n/r] = [n^{(1/2)+\gamma}]$.

Example 1. Let $Z_1, Z_2, \cdots, Z_n$ be independent $[0,1]$ uniform random variables. For each i, let

$$X_j = \sum_{i=j}^{j+15} Z_i - 8 \quad (j = 1, 2, \cdots).$$

Then, the sequence $\{X_j\}$ is a 16-dependent random variables with $X_j = 0$.

Let $n > 1$ be fixed and put

$$\bar{X}_n = \frac{1}{n}(X_1 + \cdots + X_n) = \bar{h}_1.$$

Furthermore, let $r = [n^{0.48}]$ and $k = [n/r]$ and put

$$A_j^2 = \left\{ \sum_{i=1}^{r} (X_{(j-1)r+i} - \bar{X}_n) \right\}^2 \quad (j = 1, \cdots, k).$$

Define

$$Y_n = \frac{X_1 + X_2 + \cdots + X_n}{\sqrt{A_1^2 + A_2^2 + \cdots + A_k^2}}.$$

We consider the case where $n = 200, r = 12, k = 16$, repeat 300 times the above procedure and obtain Figure 1.

Example 2. Let $h(x,y) = |x - y|$ and consider the U-statistics

$$U_n = \frac{2}{n(n-1)} \sum_{1 \le i < j \le n} |X_j - X_i| = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} |X_j - X_i|,$$

where $\{X_i\}$ is a sequence of 8-dependent $N(0,1)$ random variables. In this case, $\theta = 2/\sqrt{\pi}$.

To construct the sequence, let $\{Z_i\}$ be independent $N(0,1)$ random variables. Put

$$X_j = \frac{1}{\sqrt{8}} \sum_{i=j}^{j+7} Z_i \quad (j = 1, 2, \cdots).$$

Then, the sequence $\{X_j\}$ satisfies the required condition.

Let $\Phi(x)$ be the standard normal distribution function, $\phi(x) = \Phi'(x)$ and put

$$h_1(x) = x\{2\Phi(x) - 1\} + 2\phi(x).$$

Let

$$\bar{h}_1 = \frac{1}{n}\sum_{i=1}^{n} h_1(X_i)$$

and put

$$A_j^2 = \left\{\sum_{i=1}^{r}(h_1(X_{(j-1)r+i}) - \bar{h}_1)\right\}^2 \quad (j = 1, \cdots, k).$$

Compute

$$Y_n = \frac{n(U_n - \theta)}{2\sqrt{A_1^2 + A_2^2 + \cdots + A_k^2}}.$$

We consider the case where $n = 50, r = 4, k = 12$, repeat 200 times the above procedure and obtain Figure 2.

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