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LIMITING DISTRIBUTIONS OF QUANTUM WALKS ON THE SQUARE LATTICE

By

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Abstract. In the study of quantum walks, determining their limit distributions is one of the important issues. In this paper, we propose a model of discrete-time quantum walks on the square lattice without localization and give its limit distribution. Based on the argument on G.Grimmett-S.Janson-P.F.Scudo [9]. We also discuss the relationship between our quantum walks and alternate quantum walks. In the last section, we give consideration on the positive-operator-valued measure (POVM) and express our main theorem in the context.

1. Introduction

The notion of quantum walks was introduced by Y.Aharonov et al. [2] as a quantum counterpart of the classical one-dimensional random walks. It was re-discovered in computer science by several authors, for instance, [1], [5], [18] around 2000. Recently quantum walks have been intensively studied in connection with quantum computing [4], [10], [20], [21] and quantum physics [3], [12]. Quantum walks is now studied intensively in mathematics and analysis long-time behavior is one of the main topics there. In this paper, we give some consideration of quantum walk on the square lattice.

In 2004, N.Inui-Y.Konishi-N.Konno [11] analyzed the two-dimensional Grover walk model and discovered an interesting phenomenon called a *localization*. Grover walk is a quantum walk that is given by the following $d \times d$ unitary matrix $G = (g_{i,j})_{i,j=1,2,\cdots,d}$,

$$g_{i,j} = \frac{2}{d} - \delta_{i,j}.$$

In two-dimensional discrete-time quantum walks, there are examples whose limit distributions are computed [8], [15] and [22]. In this paper we give an example of non-localization quantum walks on the two-dimensional lattice given by a 4×4 unitary matrix without assumption any initial condition.

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Let (X_n, Y_n) be our quantum walk at time n. We have the following limit theorem.

Main theorem (Theorem 2.1) *We start the walk at the origin. Let α, β be non-negative integers. For any initial state $\varphi = {}^{\mathrm{T}}(\varphi_1, \varphi_2, \varphi_3, \varphi_4) \in \mathbb{C}^4$ with $|\varphi|^2_{\mathbb{C}^4} = 1$, where T is the transpose operator. Then we have*

$$\lim_{n\to\infty} \mathbb{E}\left[\left(\frac{X_n}{n}\right)^{\alpha}\left(\frac{Y_n}{n}\right)^{\beta}\right] = \int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_y\ v_x^{\alpha}\ v_y^{\beta}\ \frac{4\ \chi_{\Omega}(v_x, v_y)}{\pi^2(1-4v_x^2)(1-4v_y^2)}\ m(v_x, v_y),$$

where $\chi_\Omega(v_x, v_y)$ is the characteristic function on the region $\Omega = \{(v_x, v_y) : v_x^2 + v_y^2 < (\frac{1}{2})^2\}$ and the weight function $m(v_x, v_y)$ is given by

$$m(v_x, v_y) = 1 - 2\Bigg((|\varphi_2|^2 - |\varphi_1|^2)v_x + 2\mathfrak{Re}(\varphi_2\overline{\varphi_1})v_y\Bigg) - 2\Bigg((|\varphi_4|^2 - |\varphi_3|^2)v_y + 2\mathfrak{Re}(\varphi_3\overline{\varphi_4})v_x\Bigg).$$

Remark 1.1. Our model has a relation to an alternate quantum walk on the square lattice [8]. They assumed initial state to be two-state, where we treat initial condition in four-state. Our result implies the alternate quantum walk introduced by [8] in the sense that we obtain their result by taking $\varphi_1 = \varphi_2 = 0$.

Let us explain the background of our result in [8], [22]. We consider the Hilbert space

$$\ell^2(\mathbb{Z}^2, \mathbb{C}^4) = \{f : \mathbb{Z}^2 \longrightarrow \mathbb{C}^4;\ \|f\|^2 = \sum_{x\in\mathbb{Z}^2} |f(x)|^2_{\mathbb{C}^4} < \infty\}$$

with the inner product defined by

$$\langle f, g\rangle = \sum_{x\in\mathbb{Z}^2} \langle f(x), g(x)\rangle_{\mathbb{C}^4}, \quad f,\ g \in \ell^2(\mathbb{Z}^2, \mathbb{C}^4),$$

where $|\cdot|_{\mathbb{C}^4}$ and $\langle\cdot,\cdot\rangle_{\mathbb{C}^4}$ are the standard norm and inner product on $\mathbb{C}^4$. For $x \in \mathbb{Z}^2$ and $\varphi \in \mathbb{C}^4$, define $\delta_x \otimes \varphi \in \ell^2(\mathbb{Z}^2, \mathbb{C}^4)$ by

$$(\delta_x \otimes \varphi)(y) = \begin{cases} \varphi & x = y, \\ 0 & \textit{otherwise}. \end{cases}$$

For $f \in \ell^2(\mathbb{Z}^2, \mathbb{C}^4)$ and $(x, y) \in \mathbb{Z}^2$, define the shift operators τ_1, τ_2 on $\ell^2(\mathbb{Z}^2, \mathbb{C}^4)$ by

$$(\tau_1 f)(x, y) = f(x-1, y), \quad (\tau_2 f)(x, y) = f(x, y-1).$$

Let $A = (a_{i,j})_{i,j=1,2,3,4}$ be a four-by-four unitary matrix. Decompose the matrix A as

$$A = P_1 + P_2 + P_3 + P_4,$$

where P_i is defined by

$$P_i = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & a_{i3} & a_{i4} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (i = 1, 2, 3, 4).$$

A quantum walk is described by a unitary operator $U_A : \ell^2(\mathbb{Z}^2, \mathbb{C}^4) \to \ell^2(\mathbb{Z}^2, \mathbb{C}^4)$ defined by

$$U_A = P_1\tau_1 + P_2\tau_1^{-1} + P_3\tau_2 + P_4\tau_2^{-1}. \tag{1.1}$$

We found that the following relation is derived from definition of U_A.

$$(U_A f)(x, y) = P_1 f(x-1, y) + P_2 f(x+1, y) + P_3 f(x, y-1) + P_4 f(x, y+1). \tag{1.2}$$

Given an initial state $\varphi \in \mathbb{C}^4$ with $|\varphi|^2_{\mathbb{C}^4} = 1$, the transition probability for existence at $(x, y) \in \mathbb{Z}^2$ in n-step is given by $|\psi_n(x, y)|^2_{\mathbb{C}^4}$, where the n-th iteration $\psi_n = U_A^n(\delta_0 \otimes \varphi)$ and 0 is the origin in $\mathbb{Z}^2$. One interesting future of the discrete-time quantum walk on the square lattice is a localization, the first example of which was shown by Grover walk [11]. K.Watabe-N.Kobayashi-M.Katori-N.Konno [22] showed the density function of a generalization of Grover walk associated with matrix A_1;

$$A_1 = \begin{pmatrix} -p & q & \sqrt{pq} & \sqrt{pq} \\ q & -p & \sqrt{pq} & \sqrt{pq} \\ \sqrt{pq} & \sqrt{pq} & -q & p \\ \sqrt{pq} & \sqrt{pq} & p & -q \end{pmatrix},$$

where $p + q = 1$ and $p, q \in (0, 1)$. They showed the following theorem.

Theorem 1.2. (Watabe-Kobayashi-Katori-Konno (2008) [22]) *For any initial state* $\varphi = {}^{\mathrm{T}}(\varphi_1, \varphi_2, \varphi_3, \varphi_4) \in \mathbb{C}^4$,

$$\lim_{n\to\infty} \mathbb{E}\left[\left(\frac{X_n}{n}\right)^\alpha \left(\frac{Y_n}{n}\right)^\beta\right] = \int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_y \; v_x^\alpha \; v_y^\beta \; \nu(v_x, v_y).$$

Here the limiting distribution ν is explicit given by $\mu_p(v_x, v_y)\ m(v_x, v_y) + \Delta\delta_0(v_x)\delta_0(v_y)$, where $\mu_p(x, y)$ is the density function, $m(v_x, v_y)$ and Δ are the weight functions. Detailed definitions are formed in [22]. Since ν contains the Dirac's delta function, this model has also localization.
It is interesting to see that the density function of our quantum walk appears in [8] and [15]. Franco et al. construct a model of two-state quantum walk without localization. They call the model *alternate quantum walk* and determine limit distribution as in the following.

Theorem 1.3. (Franco-Gettrick-Machida-Busch (2011) [8]) *For any initial state* $\varphi = {}^{\mathrm{T}}(\varphi_1, \varphi_2) \in \mathbb{C}^2$,

$$\lim_{n\to\infty} \mathbb{E}\left[\left(\frac{X_n}{n}\right)^{\alpha}\left(\frac{Y_n}{n}\right)^{\beta}\right] = \int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_y\ v_x^{\alpha}\ v_y^{\beta}\ \frac{\chi_\Omega(v_x, v_y)}{\pi^2(1-v_x^2)(1-v_y^2)} m(v_x, v_y),$$

where

$$m(x, y) = 1 - (|\varphi_1|^2 - |\varphi_2|^2)y - \frac{\mathfrak{Re}(\varphi_1\varphi_2^*)}{cs}\left[c^2(x-y) + s^2(x+y)\right],$$

and

$$\Omega = \left\{(x, y) : \frac{(x+y)^2}{4c^2} + \frac{(x-y)^2}{4s^2} < 1\right\},$$

$$c = \cos\gamma,\ \ s = \sin\gamma,\ \ \gamma \in (0.2\pi),\ \ \gamma \neq \frac{\pi}{2}, \pi, \frac{3\pi}{2}.$$

This paper organized as follows. In this section 1, we define the notion of a discrete-time quantum walk on the square lattice. By calculating the eigenvalues of a time evolution matrix of the quantum walk in the wave number space, the long-time behavior of the joint moments of X_n and Y_n is analyzed and we see that the Konno function appears as the density function with respect to the radial direction in our quantum walk in section 2. In section 3, we discuss a relation of their model with ours. Finally, we give another expression of our result from the view point of the quantum information, the POVM in particular.

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2. Main result

Now we consider the quantum walk associated with matrix A_2, where

$$A_2 = \begin{pmatrix} 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix}.$$

THEOREM 2.1. (Main Theorem) *We start the walk at the origin. Let α, β be non-negative integers. For any initial state $\varphi = {}^{\mathrm{T}}(\varphi_1, \varphi_2, \varphi_3, \varphi_4) \in \mathbb{C}^4$ with $|\varphi|^2_{\mathbb{C}^4} = 1$, where T is the transpose operator. Then we have*

$$\lim_{n\to\infty} \mathbb{E}\left[\left(\frac{X_n}{n}\right)^{\alpha}\left(\frac{Y_n}{n}\right)^{\beta}\right] = \int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_y\, v_x^{\alpha}\, v_y^{\beta}\, \frac{4\ \chi_{\Omega}(v_x, v_y)}{\pi^2(1-4v_x^2)(1-4v_y^2)}\, m(v_x, v_y),$$

where $\chi_{\Omega}(v_x, v_y)$ is the characteristic function on the region $\Omega = \{(v_x, v_y) : v_x^2 + v_y^2 < (\frac{1}{2})^2\}$ and the weight function $m(v_x, v_y)$ is given by

$$m(v_x, v_y) = 1 - 2\Big((|\varphi_2|^2 - |\varphi_1|^2)v_x + 2\mathfrak{Re}(\varphi_2\overline{\varphi_1})v_y\Big) - 2\Big((|\varphi_4|^2 - |\varphi_3|^2)v_y + 2\mathfrak{Re}(\varphi_3\overline{\varphi_4})v_x\Big).$$

From now on, we prepare lemmas to prove our main theorem following the method by [9]. For $\psi_n \in \ell^2(\mathbb{Z}^2, \mathbb{C}^4)$, we define the Fourier transformation $\hat{\psi}_n$ by

$$\hat{\psi}_n(k_x, k_y) = \sum_{(x,y)\in\mathbb{Z}^2} e^{-i(k_x x + k_y y)}\psi_n(x, y). \tag{2.3}$$

Then we get

$$\psi_n(x, y) = \int_{-\pi}^{\pi} \frac{dk_x}{2\pi} \int_{-\pi}^{\pi} \frac{dk_y}{2\pi} e^{i(k_x x + k_y y)}\hat{\psi}_n(k_x, k_y).$$

LEMMA 2.2. *For $\psi_n \in \ell^2(\mathbb{Z}^2, \mathbb{C}^4)$ and $\varphi \in \mathbb{C}^4$ with $|\varphi|_{\mathbb{C}^4} = 1$, we get the following relation*

$$\hat{\psi}_n(k_x, k_y) = \Big(V(k_x, k_y)\Big)^n \varphi,$$

where

$$V(k_x, k_y) = \begin{pmatrix} e^{-ik_x} & 0 & 0 & 0 \\ 0 & e^{ik_x} & 0 & 0 \\ 0 & 0 & e^{-ik_y} & 0 \\ 0 & 0 & 0 & e^{ik_y} \end{pmatrix} A.$$

Proof. From (1.2) and (2.3), we have

$$\begin{aligned} \hat{\psi}_{n+1}(k_x, k_y) &= \sum_{(x,y)\in\mathbb{Z}^2} \psi_{n+1}(x, y)\ e^{-i(k_x x + k_y y)} \\ &= (e^{-ik_x}P_1 + e^{ik_x}P_2 + e^{-ik_y}P_3 + e^{ik_y}P_4) \sum_{(x,y)\in\mathbb{Z}^2} \psi_n(x, y)\ e^{-i(k_x x + k_y y)} \\ &= V(k_x, k_y)\ \hat{\psi}_n(k_x, k_y). \end{aligned}$$

□

Using the Fourier transformation of ψ_n, we get the following formula of the joint moments $\mathbb{E}[X_n^\alpha Y_n^\beta]$.

$$\mathbb{E}[X_n^\alpha Y_n^\beta] = \int_{-\pi}^{\pi} \frac{dk_x}{2\pi} \int_{-\pi}^{\pi} \frac{dk_y}{2\pi} \hat{\psi}_n^*(k_x, k_y) \left(i \frac{\partial}{\partial k_x} \right)^\alpha \left(i \frac{\partial}{\partial k_y} \right)^\beta \hat{\psi}_n(k_x, k_y), \quad (2.4)$$

where $\hat{\psi}_n^*$ is an adjoint operator of ψ_n and X_n and Y_n are x and y coordinates of the position of the walker at time n.

$$\begin{aligned} \hat{\psi}_n(k_x, k_y) &= \Big(V(k_x, k_y) \Big)^n \varphi \\ &= \sum_{j=1}^{4} (\lambda_j)^n \, \boldsymbol{v}_j(k_x, k_y) \, C_j(k_x, k_y). \end{aligned}$$

Here, λ_j is the eigenvalue of $V(k_x, k_y)$ and $\boldsymbol{v}_j(k_x, k_y)$ is the normalized eigenvector corresponding to the eigenvalue λ_j, $1 \le j \le 4$ and

$$C_j(k_x, k_y) := \boldsymbol{v}_j^*(k_x, k_y) \varphi, \quad (2.5)$$

where φ is an initial state. From (2.4) and (2.5), we need to analysis eigenvalues of $V(k_x, k_y)$. The eigenvalues of $V(k_x, k_y)$ are given by

$$\lambda_1 = e^{-i\{\frac{\omega(k_x,k_y)+\pi}{2} + m\pi\}}, \ \lambda_2 = -e^{-i\{\frac{\omega(k_x,k_y)+\pi}{2} + m\pi\}},$$

$$\lambda_3 = e^{i\{\frac{\omega(k_x,k_y)}{2} + m\pi\}}, \ \lambda_4 = -e^{i\{\frac{\omega(k_x,k_y)}{2} + m\pi\}},$$

where $m \in \mathbb{Z}$ and

$$\cos \omega(k_x, k_y) = \sqrt{1 - \sin^2 k_y \cos^2 k_x}, \quad \sin \omega(k_x, k_y) = \sin k_y \cos k_x. \quad (2.6)$$

Then we have

$$\begin{aligned} &\hat{\psi}_n^*(k_x, k_y) \left(i \frac{\partial}{\partial k_x} \right)^\alpha \left(i \frac{\partial}{\partial k_y} \right)^\beta \hat{\psi}_n(k_x, k_y) \\ &= M(k_x, k_y) \left(\frac{1}{2} \frac{\partial \omega(k_x, k_y)}{\partial k_x} \right)^\alpha \left(\frac{1}{2} \frac{\partial \omega(k_x, k_y)}{\partial k_y} \right)^\beta \times (n)_{\alpha+\beta} + \mathcal{O}\Big(n^{\alpha+\beta-1} \Big). \end{aligned} \quad (2.7)$$

Here, $(n)_k = n(n-1) \cdots (n-k+1)$ and

$$M(k_x, k_y) = |C_1(k_x, k_y)|^2 + |C_2(k_x, k_y)|^2 + (-1)^{\alpha+\beta} \Big(|C_3(k_x, k_y)|^2 + |C_4(k_x, k_y)|^2 \Big). \quad (2.8)$$

By using (2.4) and (2.7), we get the following lemma.

LEMMA 2.3.

$$\lim_{n\to\infty} \mathbb{E}\left[\left(\frac{X_n}{n}\right)^{\alpha}\left(\frac{Y_n}{n}\right)^{\beta}\right]$$
$$= \int_{-\pi}^{\pi}\frac{dk_x}{2\pi}\int_{-\pi}^{\pi}\frac{dk_y}{2\pi}M(k_x,k_y)\left(\frac{1}{2}\frac{\partial\omega(k_x,k_y)}{\partial k_x}\right)^{\alpha}\left(\frac{1}{2}\frac{\partial\omega(k_x,k_y)}{\partial k_y}\right)^{\beta}.$$

To obtain the formula like Theorem 1.2, we put $v_x := -\frac{1}{2}\frac{\partial\omega(k_x,k_y)}{\partial k_x}$ and $v_y := \frac{1}{2}\frac{\partial\omega(k_x,k_y)}{\partial k_y}$. Then by simple computation apply to (2.6), v_x and v_y explicit computable as

$$v_x = \frac{1}{2}\frac{\sin k_x \sin k_y}{\sqrt{1-\cos^2 k_x \sin^2 k_y}}, \quad v_y = \frac{1}{2}\frac{\cos k_x \cos k_y}{\sqrt{1-\cos^2 k_x \sin^2 k_y}}. \tag{2.9}$$

Then the domain of integration is given by

$$\Omega = \left\{(v_x,v_y) : v_x^2 + v_y^2 < \left(\frac{1}{2}\right)^2\right\}.$$

Let K be $\{(\frac{1}{2},0),(0,-\frac{1}{2}),(-\frac{1}{2},0),(0,\frac{1}{2})\}$. We give the Jacobian of the map

$$\begin{array}{cccc} \Phi : [-\pi,\pi)^2 & \longrightarrow & \Omega\cup K \\ \Cup & & \Cup \\ (k_x,k_y) & \longmapsto & (v_x,v_y). \end{array}$$

By using (2.9), we have

$$det\left(\frac{\partial v_i}{\partial k_j}\right)_{\substack{i=x,y\\ j=x,y}} = \frac{1}{4}\left(\frac{\sin k_x \cos k_y}{1-\cos^2 k_x \sin^2 k_y}\right)^2. \tag{2.10}$$

Next we compute the density function.

LEMMA 2.4.

(1) *The map* Φ *is one-to-four.*
(2) *Suppose that* $\cos^2 k_x \neq 1$ $(\sin k_x \neq 0)$ *and* $\sin^2 k_y \neq 1$ $(\cos k_y \neq 0)$. *Then we get the following relation.*

$$\left(\frac{\sin k_x \cos k_y}{1-\cos^2 k_x \sin^2 k_y}\right)^2 = (1-4v_x^2)(1-4v_y^2).$$

Proof. Firstly, we prove Lemma 2.4 (1). If $(k'_x,k'_y) = (k_x-\pi,k_y-\pi)$, $(k'_x,k'_y) = (\pi-k_x,\pi-k_y)$ and $(k'_x,k'_y) = (-k_x,-k_y)$, then we have $(v_x,v_y) = (v'_x,v'_y)$. When $\sin k_x = 0$ and $\cos k_y = 0$, the Jacobian has singular points $(v_x,v_y) = (0,\pm 1/2)$

and $(v_x, v_y) = (\pm 1/2, 0)$ given by (2.9). This map $\Phi : [-\pi,\pi)^2 \longrightarrow \Omega \cup K$ is one-to-four.

Next, we prove Lemma 2.4 (2). Suppose that $\sin^2 k_y \neq 0$. From (2.9), we get

$$\sin^2 k_y = \frac{4v_x^2}{1-4v_y^2}, \quad \cos^2 k_y = \frac{1-4v_y^2-4v_x^2}{1-4v_y^2} \tag{2.11}$$

$$\sin^2 k_x = \frac{1-4v_y^2-4v_x^2}{1-4v_x^2}, \quad \cos^2 k_x = \frac{4v_y^2}{1-4v_x^2}. \tag{2.12}$$

By (2.11) and (2.12), we have

$$\left(\frac{\sin k_x \cos k_y}{1-\cos^2 k_x \sin^2 k_y}\right)^2 = (1-4v_x^2)(1-4v_y^2).$$

Suppose that $\sin^2 k_y = 0$ ($\cos^2 k_y = 1$),

$$\left(\frac{\sin k_x \cos k_y}{1-\cos^2 k_x \sin^2 k_y}\right)^2 = \sin^2 k_x.$$

Using $4v_x^2 = 0$ and $4v_y^2 = \cos^2 k_x$,

$$\left(\frac{\sin k_x \cos k_y}{1-\cos^2 k_x \sin^2 k_y}\right)^2 = \sin^2 k_x = 1-\cos^2 k_x = 1 \times (1-4v_y^2) = (1-4v_x^2)(1-4v_y^2).$$

Then we have

$$\left(\frac{\sin k_x \cos k_y}{1-\cos^2 k_x \sin^2 k_y}\right)^2 = (1-4v_x^2)(1-4v_y^2).$$

□

Next we compute the weight function $m(v_x, v_y) := M\big(k_x(v_x,v_y), k_y(v_x,v_y)\big)$. Let us recall (2.8). To compute (2.8) explicit by using (2.5), we see that the normalized eigenvectors $\boldsymbol{v}_j$ corresponding to the eigenvalues λ_j of $V(k_x,k_y)$, $1 \leq j \leq 4$, are given by

$$\boldsymbol{v}_1 = \frac{1}{N_1(k_x,k_y)}\begin{pmatrix} 1 \\ \frac{e^{ik_x}\cos k_y}{S_1} \\ \frac{e^{-i\frac{\omega+\pi}{2}}\left(e^{-ik_y}+e^{i(\omega+k_x)}\right)}{\sqrt{2}S_1} \\ \frac{e^{-i\frac{\omega+\pi}{2}}\left(e^{ik_y}-e^{i(\omega+k_x)}\right)}{\sqrt{2}S_1}\end{pmatrix}, \quad \boldsymbol{v}_2 = \frac{1}{N_1(k_x,k_y)}\begin{pmatrix} 1 \\ \frac{e^{ik_x}\cos k_y}{S_1} \\ \frac{-e^{-i\frac{\omega+\pi}{2}}\left(e^{-ik_y}+e^{i(\omega+k_x)}\right)}{\sqrt{2}S_1} \\ \frac{-e^{-i\frac{\omega+\pi}{2}}\left(e^{ik_y}-e^{i(\omega+k_x)}\right)}{\sqrt{2}S_1}\end{pmatrix},$$

$$\boldsymbol{v}_3 = \frac{1}{N_2(k_x,k_y)}\begin{pmatrix} 1 \\ \frac{e^{ik_x}\cos k_y}{S_2} \\ \frac{e^{i\frac{\omega}{2}}\left(e^{-ik_y}-e^{-i(\omega-k_x)}\right)}{\sqrt{2}S_2} \\ \frac{e^{i\frac{\omega}{2}}\left(e^{ik_y}+e^{-i(\omega-k_x)}\right)}{\sqrt{2}S_2}\end{pmatrix}, \quad \boldsymbol{v}_4 = \frac{1}{N_2(k_x,k_y)}\begin{pmatrix} 1 \\ \frac{e^{ik_x}\cos k_y}{S_2} \\ \frac{-e^{i\frac{\omega}{2}}\left(e^{-ik_y}-e^{-i(\omega-k_x)}\right)}{\sqrt{2}S_2} \\ \frac{-e^{i\frac{\omega}{2}}\left(e^{ik_y}+e^{-i(\omega-k_x)}\right)}{\sqrt{2}S_2}\end{pmatrix}.$$

Here $S_1 = -e^{i\omega} + ie^{-ik_x}\sin k_y,\ S_2 = e^{-i\omega} + ie^{-ik_x}\sin k_y$ and the normalization factors $N_1(k_x,k_y)$ and $N_2(k_x,k_y)$ are given by

$$\frac{1}{N_1(k_x,k_y)^2}=\frac{S_1^2}{4(1-\sin k_y\sin(\omega+k_x))},\ \frac{1}{N_2(k_x,k_y)^2}=\frac{S_2^2}{4(1-\sin k_y\sin(\omega-k_x))}.$$

We denote ω is $\omega(k_x,k_y)$ for simplicity. Let us denote a_i^j are coefficients of each eigenvector, i.e. j is the index of an eigenvalue and i is the i-th component of an eigenvector. There are following relations;

$$a_1^1=a_1^2,\ a_2^1=a_2^2,\ -a_3^1=a_3^2,\ -a_4^1=a_4^2,$$

$$a_1^3=a_1^4,\ a_2^3=a_2^4,\ -a_3^3=a_3^4,\ -a_4^3=a_4^4.$$

Lemma 2.5. *The weight function $m(v_x,v_y)=M\big(k_x(v_x,v_y),k_y(v_x,v_y)\big)$ of the density function is given by*

$$(-1)^\alpha m(v_x,v_y)=(-1)^\alpha m_1(v_x,v_y)+(-1)^\beta m_2(v_x,v_y),$$

where

$$\begin{aligned}m_1(v_x,v_y)&=\frac{1}{2}-v_x(|\varphi_1|^2-|\varphi_2|^2)-v_y(|\varphi_4|^2-|\varphi_3|^2)\\&+\sqrt{1-4v_x^2-4v_y^2}\Big(\mathfrak{Im}(\varphi_3\overline{\varphi_4})-\mathfrak{Im}(\varphi_2\overline{\varphi_1})\Big)+2\Big(v_x\mathfrak{Re}(\varphi_3\overline{\varphi_4})-v_y\mathfrak{Re}(\varphi_2\overline{\varphi_1})\Big),\end{aligned}$$

$$\begin{aligned}m_2(v_x,v_y)&=\frac{1}{2}-v_x(|\varphi_2|^2-|\varphi_1|^2)-v_y(|\varphi_3|^2-|\varphi_4|^2)\\&+\sqrt{1-4v_x^2-4v_y^2}\Big(\mathfrak{Im}(\varphi_2\overline{\varphi_1})-\mathfrak{Im}(\varphi_3\overline{\varphi_4})\Big)+2\Big(v_y\mathfrak{Re}(\varphi_2\overline{\varphi_1})-v_x\mathfrak{Re}(\varphi_3\overline{\varphi_4})\Big).\end{aligned}$$

Proof.

$$\begin{aligned}&M(k_x,k_y)\\&=|C_1(k_x,k_y)|^2+|C_2(k_x,k_y)|^2+(-1)^{\alpha+\beta}\Big(|C_3(k_x,k_y)|^2+|C_4(k_x,k_y)|^2\Big)\\&=\Big(2\sum_{i=1}^{4}|a_i^1|^2|\varphi_i|^2+4\mathfrak{Re}(a_1^1\overline{a_2^1}\varphi_2\overline{\varphi_1})+4\mathfrak{Re}(a_4^1\overline{a_3^1}\varphi_3\overline{\varphi_4})\Big)\\&\qquad+(-1)^{\alpha+\beta}\Big(2\sum_{i=1}^{4}|a_i^3|^2|\varphi_i|^2+4\mathfrak{Re}(a_1^3\overline{a_2^3}\varphi_2\overline{\varphi_1})+4\mathfrak{Re}(a_4^3\overline{a_3^3}\varphi_3\overline{\varphi_4})\Big).\end{aligned}\tag{2.13}$$

It should be noted that

$$
\begin{aligned}
&2\sum_{i=1}^{4}|a_i^1|^2|\varphi_i|^2\\
&\quad=\frac{1}{2}\Big\{(1-2v_x)|\varphi_1|^2+(1+2v_x)|\varphi_2|^2+(1+2v_y)|\varphi_3|^2+(1-2v_y)|\varphi_4|^2\Big\},\\
&2\sum_{i=1}^{4}|a_i^3|^2|\varphi_i|^2\\
&\quad=\frac{1}{2}\Big\{(1+2v_x)|\varphi_1|^2+(1-2v_x)|\varphi_2|^2+(1-2v_y)|\varphi_3|^2+(1+2v_y)|\varphi_4|^2\Big\},
\end{aligned}
\tag{2.14}
$$

$$
\begin{aligned}
&4\mathfrak{Re}(a_1^1\overline{a_2^1}\varphi_2\overline{\varphi_1})+4\mathfrak{Re}(a_4^1\overline{a_3^1}\varphi_3\overline{\varphi_4})\\
&=\sqrt{1-4v_x^2-4v_y^2}\Big(\mathfrak{Im}(\varphi_3\overline{\varphi_4})-\mathfrak{Im}(\varphi_2\overline{\varphi_1})\Big)+2\Big(v_x\mathfrak{Re}(\varphi_3\overline{\varphi_4})-v_y\mathfrak{Re}(\varphi_2\overline{\varphi_1})\Big),
\end{aligned}
\tag{2.15}
$$

$$
\begin{aligned}
&4\mathfrak{Re}(a_1^3\overline{a_2^3}\varphi_2\overline{\varphi_1})+4\mathfrak{Re}(a_4^3\overline{a_3^3}\varphi_3\overline{\varphi_4})\\
&=\sqrt{1-4v_x^2-4v_y^2}\Big(\mathfrak{Im}(\varphi_2\overline{\varphi_1})-\mathfrak{Im}(\varphi_3\overline{\varphi_4})\Big)+2\Big(v_y\mathfrak{Re}(\varphi_2\overline{\varphi_1})-v_x\mathfrak{Re}(\varphi_3\overline{\varphi_4})\Big).
\end{aligned}
\tag{2.16}
$$

From (2.13), (2.14), (2.15) and (2.16), we have

$$
\begin{aligned}
&(-1)^\alpha m(v_x,v_y)=(-1)^\alpha M\big(k_x(v_x,v_y),k_y(v_x,v_y)\big)\\
&=(-1)^\alpha\Bigg(\frac{1}{2}-v_x(|\varphi_1|^2-|\varphi_2|^2)-v_y(|\varphi_4|^2-|\varphi_3|^2)\\
&\qquad+\sqrt{1-4v_x^2-4v_y^2}\big(\mathfrak{Im}(\varphi_3\overline{\varphi_4})-\mathfrak{Im}(\varphi_2\overline{\varphi_1})\big)+2\big(v_x\mathfrak{Re}(\varphi_3\overline{\varphi_4})-v_y\mathfrak{Re}(\varphi_2\overline{\varphi_1})\big)\Bigg)\\
&\quad+(-1)^\beta\Bigg(\frac{1}{2}-v_x(|\varphi_2|^2-|\varphi_1|^2)-v_y(|\varphi_3|^2-|\varphi_4|^2)\\
&\qquad+\sqrt{1-4v_x^2-4v_y^2}\big(\mathfrak{Im}(\varphi_2\overline{\varphi_1})-\mathfrak{Im}(\varphi_3\overline{\varphi_4})\big)+2\big(v_y\mathfrak{Re}(\varphi_2\overline{\varphi_1})-v_x\mathfrak{Re}(\varphi_3\overline{\varphi_4})\big)\Bigg)\\
&=(-1)^\alpha m_1(v_x,v_y)+(-1)^\beta m_2(v_x,v_y).
\end{aligned}
$$

□

Finally, we prove the Theorem 2.1.

Proof of Theorem 2.1. Using Lemma 2.3, (2.10), Lemma 2.4 and Lemma 2.5, we get the following formula.

$$\begin{aligned}
&\int_{-\pi}^{\pi} \frac{dk_x}{2\pi} \int_{-\pi}^{\pi} \frac{dk_y}{2\pi} M(k_x,k_y) \left(\frac{1}{2}\frac{\partial\omega(k_x,k_y)}{\partial k_x}\right)^{\alpha} \left(\frac{1}{2}\frac{\partial\omega(k_x,k_y)}{\partial k_y}\right)^{\beta} \\
&= \int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_y \ (-v_x)^{\alpha} \ v_y^{\beta} \ \mu(v_x,v_y) \ m_1(v_x,v_y) \\
&\qquad\qquad + \int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_y \ v_x^{\alpha} \ (-v_y)^{\beta} \ \mu(v_x,v_y) \ m_2(v_x,v_y) \\
&= \int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_y \ v_x^{\alpha} \ v_y^{\beta} \ \mu(v_x,v_y) \ m(v_x,v_y),
\end{aligned}$$

where the density function $\mu(v_x, v_y)$ is given by

$$\mu(v_x,v_y) = \frac{4}{\pi^2}\frac{1}{(1-4v_x^2)(1-4v_y^2)}\chi_{\Omega}(v_x,v_y),$$

and the weight function $m(v_x, v_y)$ is given by

$$\begin{aligned}
&m(v_x,v_y) \\
&= 1 - 2\Big((|\varphi_2|^2 - |\varphi_1|^2)v_x + 2\mathfrak{Re}(\varphi_2\overline{\varphi_1})v_y\Big) - 2\Big((|\varphi_4|^2 - |\varphi_3|^2)v_y + 2\mathfrak{Re}(\varphi_3\overline{\varphi_4})v_x\Big).
\end{aligned}$$

Then we have

$$\lim_{n\to\infty} \mathbb{E}\left[\left(\frac{X_n}{n}\right)^{\alpha}\left(\frac{Y_n}{n}\right)^{\beta}\right] = \int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_y \ v_x^{\alpha} \ v_y^{\beta} \ \mu(v_x,v_y) \ m(v_x,v_y).$$

□

REMARK 2.6. We see that the Konno function [13] appears as the density function with respect to the radial direction in our quantum walk. It is interesting to see the following formula,

$$\frac{1}{4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu(v_x,v_y) \ dv_x \ dv_y = \int_0^{\infty} r \ \underbrace{f_K(r;\frac{1}{\sqrt{2}})}_{Konno\ function} \ dr.$$

Proof. We put

$$v_x = r\cos\theta, \quad v_y = r\sin\theta, \quad z = e^{i\theta}.$$

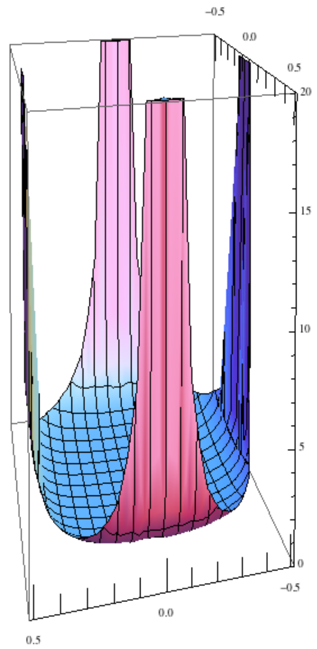

Figure 1 the density function of our quantum walk

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{4}{\pi^2}\frac{1}{(1-4v_x^2)(1-4v_y^2)}\chi_{\{v_x^2+v_y^2<\frac{1}{4}\}}(v_x,v_y)\ dv_x\ dv_y$$
$$=\int_0^{\frac{1}{2}} r\left\{\int_0^{2\pi}\left(\frac{4}{\pi^2}\frac{1}{(1-4r^2\cos^2\theta)(1-4r^2\sin^2\theta)}\right)d\theta\right\}dr$$
$$=\int_0^{\frac{1}{2}} r\left\{\frac{4i}{\pi^2 r^4}\int_{|z|=1}\frac{z^3}{(z^2-\frac{1}{r}z+1)(z^2+\frac{1}{r}z+1)(z^2-\frac{i}{r}z-1)(z^2+\frac{i}{r}z-1)}dz\right\}dr$$
$$=\int_0^{\frac{1}{2}} r\left\{\frac{4i}{\pi^2 r^4}\int_{|z|=1} f(z)\ dz\right\}dr$$

where,

$$f(z)=\frac{z^3}{(z-z_+)(z-z_-)(z+z_+)(z+z_-)(z-iz_+)(z-iz_-)(z+iz_+)(z+iz_-)}$$

with

$$z_+=\frac{1}{2r}(1+\sqrt{1-4r^2}),\quad z_-=\frac{1}{2r}(1-\sqrt{1-4r^2}).$$

There are four singular points at $z=z_-$, $z=-z_-$, $z=iz_-$, $z=-iz_-$ inside of the unit circle centered at the origin $\{z \in \mathbb{C} : |z| = 1\}$. We get

$$Res(f, z_-) = \frac{-r^4}{4\sqrt{1-4r^2}(1-2r^2)}, \tag{2.17}$$

and $Res(f, z_-) = Res(f, -z_-) = Res(f, iz_-) = Res(f, -iz_-)$. By using the residue theorem and (2.17), we have

$$\begin{aligned}
&\int_0^{\frac{1}{2}} r\left\{\frac{4i}{\pi^2 r^4}\int_{|z|=1} f(z)\ dz\right\}dr \\
&= \int_0^{\frac{1}{2}} r\left\{\frac{4i}{\pi^2 r^4}2\pi i\Big(Res(f,z_-)+Res(f,-z_-)+Res(f,iz_-)+Res(f,-iz_-)\Big)\right\}dr \\
&= \int_0^{\frac{1}{2}} r\left\{\frac{8}{\pi}\frac{1}{\sqrt{1-4r^2}(1-2r^2)}\right\}dr \\
&= 4\int_0^{\frac{1}{\sqrt{2}}} r\left\{\frac{1}{\pi\sqrt{2}}\frac{1}{\sqrt{\frac{1}{2}-r^2}(1-r^2)}\right\}dr \\
&= 4\int_0^{\infty} r\ f_K(r;\frac{1}{\sqrt{2}})\ dr.
\end{aligned}$$

The integral in the fourth line is exactly the Konno function

$$f_K(r;a) = \frac{\sqrt{1-a^2}}{\pi(1-r^2)\sqrt{a^2-r^2}}\chi_{\{0<r<a\}}(r)$$

as we remarked earlier. □

3. Alternate quantum walk

In this section, we discuss the relationship between an alternate quantum walk [8] and our quantum walk. An alternate quantum walk on the square lattice is defined as a unitary operator on the Hilbert space $\ell^2(\mathbb{Z}^2, \mathbb{C}^2)$. To explain the set-up of an alternate quantum walk, we consider the Hilbert space $\ell^2(\mathbb{Z}^2, \mathbb{C}^2)$ with the inner product defined by

$$\langle f, g\rangle = \sum_{x\in\mathbb{Z}^2}\langle f(x), g(x)\rangle_{\mathbb{C}^2}, \quad f,\ g \in \ell^2(\mathbb{Z}^2, \mathbb{C}^2),$$

where $|\cdot|_{\mathbb{C}^2}$ and $\langle\cdot,\cdot\rangle_{\mathbb{C}^2}$ are the standard norm and inner product on $\mathbb{C}^2$. For $f \in \ell^2(\mathbb{Z}^2, \mathbb{C}^2)$ and $(x, y) \in \mathbb{Z}^2$, define the shift operators σ_1, σ_2 on $\ell^2(\mathbb{Z}^2, \mathbb{C}^2)$ (see Fig 2, Fig 3) by

$$(\sigma_1 f)(x, y) = f(x-1, y), \quad (\sigma_2 f)(x, y) = f(x, y-1).$$

Let $C = (c_{i,j})_{i,j=1,2}$ be a two-by-two unitary matrix. Decompose the matrix C as

$$C = Q_1 + Q_2,$$

where Q_i is defined by

$$Q_1 = \begin{pmatrix} c_{11} & c_{12} \\ 0 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & 0 \\ c_{21} & c_{22} \end{pmatrix}.$$

An alternate quantum walk is described by a unitary operator $W_{A,i} : \ell^2(\mathbb{Z}^2, \mathbb{C}^2) \longrightarrow \ell^2(\mathbb{Z}^2, \mathbb{C}^2)$ defined by

$$W_{A,1} = (Q_1\sigma_2 + Q_2\sigma_2^{-1})(Q_1\sigma_1 + Q_2\sigma_1^{-1}), \quad W_{A,2} = (Q_1\sigma_1 + Q_2\sigma_1^{-1})(Q_1\sigma_2 + Q_2\sigma_2^{-1}).$$

We found that the following relations are derived from definition of $W_{A,i}$.

$$W_{A,1} = Q_1\ Q_1(\sigma_2 \circ \sigma_1) + Q_1\ Q_2(\sigma_2 \circ \sigma_1^{-1}) + Q_2\ Q_1(\sigma_2^{-1} \circ \sigma_1) + Q_2\ Q_2(\sigma_2^{-1} \circ \sigma_1^{-1}), \tag{3.18}$$

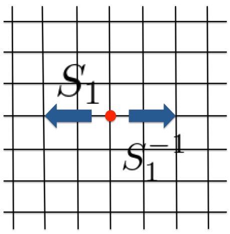

Figure 2

σ_1 (resp. σ_1^{-1}) is the shift operator induced by S_1 (resp. S_1^{-1}).

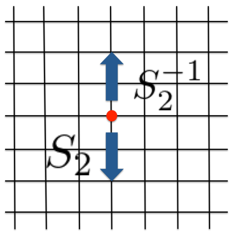

Figure 3

σ_2 (resp. σ_2^{-1}) is the shift operator induced by S_2 (resp. S_2^{-1}).

$$W_{A,2} = Q_1\ Q_1(\sigma_2 \circ \sigma_1) + Q_2\ Q_1(\sigma_2 \circ \sigma_1^{-1}) + Q_1\ Q_2(\sigma_2^{-1} \circ \sigma_1) + Q_2\ Q_2(\sigma_2^{-1} \circ \sigma_1^{-1}). \tag{3.19}$$

Let $\varphi_H \in \mathbb{C}^2$ be an initial state of an alternate quantum walk $W_{A,1}$ and $\varphi_V \in \mathbb{C}^2$ be an initial state of an alternate quantum walk $W_{A,2}$. From (3.18) and (3.19), we get

$$\begin{aligned} &W_{A,1}(\delta_{(0,0)} \otimes \varphi_H)(x,y) \\ &= \Big(\delta_{(1,1)} \otimes [Q_1\ Q_1\varphi_H]\Big)(x,y) + \Big(\delta_{(-1,1)} \otimes [Q_1\ Q_2\varphi_H]\Big)(x,y) \\ &\qquad + \Big(\delta_{(1,-1)} \otimes [Q_2\ Q_1\varphi_H]\Big)(x,y) + \Big(\delta_{(-1,-1)} \otimes [Q_2\ Q_2\varphi_H]\Big)(x,y), \end{aligned} \tag{3.20}$$

$$\begin{aligned} &W_{A,2}(\delta_{(0,0)} \otimes \varphi_V)(x,y) \\ &= \Big(\delta_{(1,1)} \otimes [Q_1\ Q_1\varphi_V]\Big)(x,y) + \Big(\delta_{(-1,1)} \otimes [Q_2\ Q_1\varphi_V]\Big)(x,y) \\ &\qquad + \Big(\delta_{(1,-1)} \otimes [Q_1\ Q_2\varphi_V]\Big)(x,y) + \Big(\delta_{(-1,-1)} \otimes [Q_2\ Q_2\varphi_V]\Big)(x,y). \end{aligned} \tag{3.21}$$

Now we rephrase our model so that we can compare it with the alternate quantum walks. We take a four-by-four unitary matrix;

$$A = \begin{pmatrix} 0 & C_1 \\ C_2 & 0 \end{pmatrix},$$

where 0 is a two-by-two zero matrix and C_i $(i = 1, 2)$ is a two-by-two unitary matrix. Suppose that the components, a_i, b_i, of the matrix C_i are non-zero. Decompose the matrix C_i as

$$C_1 = V_1 + V_2, \quad C_2 = R_1 + R_2,$$

where V_i and R_i are defined by

$$V_1 = \begin{pmatrix} a_1 & b_1 \\ 0 & 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & 0 \\ c_1 & d_1 \end{pmatrix}, \quad R_1 = \begin{pmatrix} a_2 & b_2 \\ 0 & 0 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 & 0 \\ c_2 & d_2 \end{pmatrix}.$$

By using V_1, V_2, R_1, R_2, we re-write P_1, P_2, P_3, P_4;

$$P_1 = \begin{pmatrix} 0 & V_1 \\ 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & V_2 \\ 0 & 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 & 0 \\ R_1 & 0 \end{pmatrix}, \quad P_4 = \begin{pmatrix} 0 & 0 \\ R_2 & 0 \end{pmatrix}.$$

We notice that

$$P_1 \cdot P_2 = P_2 \cdot P_1 = 0, \quad P_3 \cdot P_4 = P_4 \cdot P_3 = 0, \quad P_i \cdot P_i = 0 \ (i = 1, 2, 3, 4). \tag{3.22}$$

From (1.1) and (3.22), we get the following formula.

$$\begin{aligned} &(U_A)^2 \\ &= (P_1\tau_1 + P_2\tau_1^{-1} + P_3\tau_2 + P_4\tau_2^{-1})(P_1\tau_1 + P_2\tau_1^{-1} + P_3\tau_2 + P_4\tau_2^{-1}) \\ &= \begin{pmatrix} V_1 \cdot R_1 & 0 \\ 0 & R_1 \cdot V_1 \end{pmatrix} \tau_2 \circ \tau_1 + \begin{pmatrix} V_2 \cdot R_1 & 0 \\ 0 & R_1 \cdot V_2 \end{pmatrix} \tau_2 \circ \tau_1^{-1} \\ &\qquad + \begin{pmatrix} V_1 \cdot R_2 & 0 \\ 0 & R_2 \cdot V_1 \end{pmatrix} \tau_2^{-1} \circ \tau_1 + \begin{pmatrix} V_2 \cdot R_2 & 0 \\ 0 & R_2 \cdot V_2 \end{pmatrix} \tau_2^{-1} \circ \tau_1^{-1}. \end{aligned} \tag{3.23}$$

We are ready to see that we obtain an alternate quantum walk if $C_1 = C_2 = C$. Actually, suppose that $C_1 = C_2 = C$. From (3.23), we get

$$
\begin{aligned}
&(U_A)^2(\delta_{(0,0)} \otimes \varphi)(x,y) \\
&= \left(\delta_{(1,1)} \otimes \left[\begin{pmatrix} Q1 \cdot Q_1 & 0 \\ 0 & Q_1 \cdot Q_1 \end{pmatrix} \varphi\right]\right)(x,y) + \left(\delta_{(-1,1)} \otimes \left[\begin{pmatrix} Q_2 \cdot Q_1 & 0 \\ 0 & Q_1 \cdot Q_2 \end{pmatrix} \varphi\right]\right)(x,y) \\
&\quad + \left(\delta_{(1,-1)} \otimes \left[\begin{pmatrix} Q_1 \cdot Q_2 & 0 \\ 0 & Q_2 \cdot Q_1 \end{pmatrix} \varphi\right]\right)(x,y) + \left(\delta_{(-1,-1)} \otimes \left[\begin{pmatrix} Q_2 \cdot Q_2 & 0 \\ 0 & Q_2 \cdot Q_2 \end{pmatrix} \varphi\right]\right)(x,y) \\
&= \left(\delta_{(1,1)} \otimes \left[\begin{pmatrix} Q_1 \cdot Q_1 \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \\ Q_1 \cdot Q_1 \begin{pmatrix} \varphi_3 \\ \varphi_4 \end{pmatrix} \end{pmatrix}\right]\right)(x,y) + \left(\delta_{(-1,1)} \otimes \left[\begin{pmatrix} Q_2 \cdot Q_1 \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \\ Q_1 \cdot Q_2 \begin{pmatrix} \varphi_3 \\ \varphi_4 \end{pmatrix} \end{pmatrix}\right]\right)(x,y) \\
&\quad + \left(\delta_{(1,-1)} \otimes \left[\begin{pmatrix} Q_1 \cdot Q_2 \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \\ Q_2 \cdot Q_1 \begin{pmatrix} \varphi_3 \\ \varphi_4 \end{pmatrix} \end{pmatrix}\right]\right)(x,y) + \left(\delta_{(-1,-1)} \otimes \left[\begin{pmatrix} Q_2 \cdot Q_2 \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \\ Q_2 \cdot Q_2 \begin{pmatrix} \varphi_3 \\ \varphi_4 \end{pmatrix} \end{pmatrix}\right]\right)(x,y).
\end{aligned}
\tag{3.24}
$$

$\pi_1 : \mathbb{C}^4 \longrightarrow \mathbb{C}^2$ (*resp.* $\pi_2 : \mathbb{C}^4 \longrightarrow \mathbb{C}^2$) denotes the orthogonal projection onto the two-dimensional subspace $\mathbb{C}\boldsymbol{e}_1 + \mathbb{C}\boldsymbol{e}_2$ (*resp.* $\mathbb{C}\boldsymbol{e}_3 + \mathbb{C}\boldsymbol{e}_4$) in $\mathbb{C}^4$ where $\{\boldsymbol{e}_1, \cdots, \boldsymbol{e}_4\}$ denotes the standard basis on $\mathbb{C}^4$; namely

$$
\pi_1\left(\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}\right) = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \quad \pi_2\left(\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}\right) = \begin{pmatrix} z_3 \\ z_4 \end{pmatrix}.
$$

By using (3.20), (3.21) and (3.24), we have the following relationships between an alternate quantum walk and our quantum walk;

$$
\begin{aligned}
(\pi_1 U_A^2)(\delta_{(0,0)} \otimes \varphi)(x,y) &= W_{A,2}(\delta_{(0,0)} \otimes \varphi_V)(x,y), \\
(\pi_2 U_A^2)(\delta_{(0,0)} \otimes \varphi)(x,y) &= W_{A,1}(\delta_{(0,0)} \otimes \varphi_H)(x,y),
\end{aligned}
$$

where the orthogonal projection π_i acts the Hilbert space $\ell^2(\mathbb{Z}^2, \mathbb{C}^4)$. Then we get the following Corollary 3.1.

COROLLARY 3.1.

(1) *Suppose that* $\varphi_3 = \varphi_4 = 0$. *Let* $\varphi_V = {}^{\mathrm{T}}(\varphi_1, \varphi_2)$ *be a unit vector in* $\mathbb{C}^2$. *We have*

$$
\lim_{n\to\infty} |(\pi_1 U_A^{2n})(\delta_{(0,0)} \otimes \varphi)(x,y)|^2_{\mathbb{C}^2} = \lim_{n\to\infty} |W_{A,2}^n(\delta_{(0,0)} \otimes \varphi_V)(x,y)|^2_{\mathbb{C}^2}.
$$

(2) *Suppose that* $\varphi_1 = \varphi_2 = 0$. *Let* $\varphi_H = {}^{\mathrm{T}}(\varphi_3, \varphi_4)$ *be a unit vector in* $\mathbb{C}^2$. *We have*

$$
\lim_{n\to\infty} |(\pi_2 U_A^{2n})(\delta_{(0,0)} \otimes \varphi)(x,y)|^2_{\mathbb{C}^2} = \lim_{n\to\infty} |W_{A,1}^n(\delta_{(0,0)} \otimes \varphi_H)(x,y)|^2_{\mathbb{C}^2}.
$$

4. POVM (positive-operator-valued measure)

In this section, we give consideration on the positive-operator-valued measure (POVM) and express Theorem 2.1 in the context. POVMs are the generalized quantum measurements which appear in the quantum information theory. They play an important role to distinguish the quantum states [16], [17], [19]. Recently, experimentally realization of a generalized measuring device by using a quantum walk is studied in [6], [14]. We briefly see the definition of POVM and refer the interested readers to [7], [19].

Let $(X, \mathcal{F})$ be a measurable space and $\mathcal{H}$ be a separable Hilbert space. Denote by $\mathcal{L}_s(\mathcal{H})$ the set of self-adjoint linear operator on the Hilbert space $\mathcal{H}$. A positive-operator-valued measure (POVM) on X is defined to be a map $\Pi : \mathcal{F} \longrightarrow \mathcal{L}_s(\mathcal{H})$ such that

(1) $\Pi(E) \geq \Pi(\emptyset) = 0 \quad$ for all $E \in \mathcal{F}$.

(2) If $\{E_n\}$ is a countable collection of disjoint sets in $\mathcal{F}$ then

$$\Pi(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \Pi(E_n),$$

where the series convergence on the right hand side is in the weak operator topology.

(3) $\Pi(\Omega) = I_{\mathcal{H}}$,
where $I_{\mathcal{H}}$ is an identity operator on the Hilbert space $\mathcal{H}$.

We chose $(X, \mathcal{F}) = (\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$ and $\mathcal{H} = (L^2(\mathbb{R}^2, \mathbb{C}^4), \mu)$ with the inner product defined by

$$\langle f, g \rangle_\mu = \int_{\mathbb{R}^2} \langle f(s,t), g(s,t) \rangle_{\mathbb{C}^4} \mu(s,t) ds dt, \quad f,\ g \in L^2(\mathbb{R}^2, \mathbb{C}^4),$$

where

$$\mu(s,t) = \frac{4\chi_\Omega(s,t)}{\pi^2(1-4s^2)(1-4t^2)},$$

in order to put our model in the POVM-frame work. Then we take a map Π

$$\begin{array}{cccc} \Pi : \mathcal{B}(\mathbb{R}^2) & \longrightarrow & & M_4(\mathbb{C}) \\ \cup & & & \cup \\ \Delta & \longmapsto & \Pi(\Delta) := \int_\Delta E(s,t) d\mu(s,t), & \end{array}$$

where $E(s,t)$ is the following positive and self adjoint operator on $\mathbb{C}^4$;

$$E(s,t) = \chi_\Omega(s,t) \left\{ I - 2 \begin{pmatrix} -s & t & 0 & 0 \\ t & s & 0 & 0 \\ 0 & 0 & -t & s \\ 0 & 0 & s & t \end{pmatrix} \right\}$$

and $\Omega = \{(x,y) \in \mathbb{R}^2; x^2+y^2 < (\frac{1}{2})^2\}$.

Proposition 4.1. *$\Pi : \mathcal{B}(\mathbb{R}^2) \longrightarrow M_4(\mathbb{C})$ is a POVM, that is,*

(1) $\Pi(\Delta) \geq \Pi(\emptyset) = 0$ *for all* $\Delta \in \mathcal{B}(\mathbb{R}^2)$.

(2) *If $\{\Delta_n\}$ is a countable collection of disjoint sets in $\mathcal{B}(\mathbb{R}^2)$ then*

$$\Pi(\bigcup_{n=1}^{\infty} \Delta_n) = \sum_{n=1}^{\infty} \Pi(\Delta_n),$$

where the series convergence on the right hand side is in the weak operator topology.

(3) $\Pi(\mathbb{R}^2) = I$.

Proof. The above conditions (1) and (2) are shown from the definition of the integral and $E(s,t)$. So we show that the map Π is satisfied with the condition (3). Define the functions $F(\theta,r)$ and $G(\theta,r)$ by

$$F(\theta,r) = \frac{\cos\theta}{(1-4r^2\cos^2\theta)(1-4r^2\sin^2\theta)},$$
$$G(\theta,r) = \frac{\sin\theta}{(1-4r^2\cos^2\theta)(1-4r^2\sin^2\theta)}.$$

Since $G(\theta,r)$ is an odd function and

$$\begin{aligned}
&\int_0^{2\pi} F(\theta,r)d\theta \\
&= \int_0^{\pi} F(\theta,r)d\theta + \int_{\pi}^{2\pi} F(\theta,r)d\theta \\
&= \int_0^{\pi} F(\theta,r)d\theta + \int_0^{\pi} \frac{\cos(\theta'+\pi)}{(1-4r^2\cos^2(\theta'+\pi))(1-4r^2\sin^2(\theta'+\pi))}d\theta' \\
&= \int_0^{\pi} F(\theta,r)d\theta + \int_0^{\pi} -\frac{\cos\theta'}{(1-4r^2\cos^2\theta')(1-4r^2\sin^2\theta')}d\theta' \\
&= 0,
\end{aligned}$$

We get

$$\int_{\mathbb{R}^2} \frac{4s\ \chi_\Omega(s,t)}{\pi^2(1-4s^2)(1-4t^2)}dsdt = 0, \quad \int_{\mathbb{R}^2} \frac{4t\ \chi_\Omega(s,t)}{\pi^2(1-4s^2)(1-4t^2)}dsdt = 0. \quad (4.25)$$

From (4.25), it holds that

$$\Pi(\mathbb{R}^2) = \int_{\mathbb{R}^2} E(s,t)\mu(s,t)dsdt = I.$$

□

Thus $\{E(s,t)\}_{(s,t)\in\mathbb{R}^2}$ is a POVM. For any initial state $\varphi = {}^{\mathrm{T}}(\varphi_1, \varphi_2, \varphi_3, \varphi_4) \in \mathbb{C}^4$ with $|\varphi|^2_{\mathbb{C}^4} = 1$, we have

$$\begin{aligned}
&\langle \varphi, E(s,t)\varphi\rangle_{\mathbb{C}^4} \\
&= 1 - 2\Big((|\varphi_2|^2 - |\varphi_1|^2)s + 2\mathfrak{Re}(\varphi_2\overline{\varphi_1})t \Big) \\
&\qquad\qquad\qquad - 2\Big((|\varphi_4|^2 - |\varphi_3|^2)t + 2\mathfrak{Re}(\varphi_3\overline{\varphi_4})s \Big).
\end{aligned}$$

Then we have

$$\begin{aligned}
\lim_{n\to\infty} P\Big(\Big(\frac{X_n}{n}, \frac{Y_n}{n}\Big) \in \Delta\Big) &= \int_{-\infty}^{x}\int_{-\infty}^{y} \mu(s,t)\ m(s,t)dsdt \\
&= \int_{\Delta} \langle \varphi, E(s,t)\varphi\rangle_{\mathbb{C}^4}\mu(s,t)dsdt \\
&= \langle \varphi, \Pi(\Delta)\varphi\rangle_{\mu},
\end{aligned} \tag{4.26}$$

where $\Delta = \{(s,t) \in \mathbb{R}^2; s \le x,\ t \le y\}$. The formula in the left side of (4.26) is the probability that a quantum walk exists in the region $\{(i,j) \in \mathbb{Z}^2; i \le nx, j \le ny\}$ for the large time step n while the right hand side of (4.26) turns a POVM upon the initial state $\varphi \in \mathbb{C}^4$ with $|\varphi|_{\mathbb{C}^4} = 1$.

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