Yokohama Mathematical
Journal Vol. 61, 2015

DISTINGUISHING COLORINGS OF 3-CONNECTED PLANAR GRAPHS WITH FIVE COLORS

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(Received October 29, 2014)

Abstract. A (proper) coloring of G with k colors is called a *distinguishing k-coloring* of G if there is no color-preserving automorphism of G other than the identity map. We shall prove that every 3-connected planar graph, with the exception of $K_{2,2,2}$ and $C_6+\overline{K}_2$, admits a distinguishing 5-coloring which uses color 5 only for one vertex. By contrast, we shall present examples of 3-connected planar graphs that have distinguishing 4-colorings but no distinguishing 4-coloring with one color used only once.

1. Introduction

Our graphs are simple, without loops or multiple edges. We denote the set of vertices and edges of a graph G by $V(G)$ and $E(G)$, respectively. An assignment of colors to vertices $c : V(G) \to \{1, 2, \ldots, k\}$ is called a (*proper*) *coloring* or a *k-coloring* of G if any adjacent vertices receive different colors. A k-coloring of G is said to be *distinguishing* or is called a *distinguishing k-coloring* of G if there is no color-preserving automorphism of G other than the identity map. If G admits a distinguishing k-coloring, then G is said to be *distinguishing k-colorable.* The *distinguishing chromatic number* of G is defined as the minimum number k such that G is distinguishing k-colorable and is denoted by $\chi_D(G)$.

These notions have been introduced by Collins and Trenk [3], who have determined the distinguishing chromatic numbers of many kinds of abstract graphs. The distinguishing chromatic number has also received attention in case of embedded graphs in [4, 6, 8], and in particular Negami [6] and Tucker [9] have independently shown that every 3-connected planar graph is distinguishing 6-colorable. The authors have recently proven a strengthening of this result. Here, we denote the *double wheel* $C_n + \overline{K}_2$ with rim C_n by DW_n.

Theorem 1. (Fijavž, Negami and Sano [4]) *Every 3-connected planar graph is distinguishing 5-colorable unless it is isomorphic to either $K_{2,2,2}$ or DW_6.*

2010 Mathematics Subject Classification: 05C10, 05C15
Key words and phrases: distinguishing colorings, planar graphs, topological graph theory

It should be noted that there are 2-connected planar graphs with arbitrarily large distinguishing chromatic number. For example, $\chi_D(K_{2,n}) = n+2$ for any $n \geq 1$.

In fact, the distinguishing 5-colorings constructed in the proof in [4] assign color 5 to very few vertices and the authors suspected that it might suffice to use color 5 only once. We shall show the affirmative answer to this:

Theorem 2. *Every 3-connected planar graph, except $K_{2,2,2}$ and DW_6, has a distinguishing 5-coloring which uses color 5 only for one vertex.*

The Four Color Theorem [1, 7] states that every planar graph is 4-colorable. The above theorem seems to suggest that we need only one extra color to modify a 4-coloring of a 3-connected planar graph G into a distinguishing one and use the additional color only once. However, when G is 3-colorable, we can use two more colors, 4 and 5, and use color 4 as many times as we want.

One might ask for a stronger result. Let G be a 3-connected planar graph, whose chromatic number does not exceed 3. Does there exist a distinguishing $(\chi(G)+1)$-coloring which uses the additional color for only one vertex?

We shall prove that the above speculation is false even if we allow a finite list of exceptional graphs. Namely we shall exhibit families of k-chromatic graphs for $k = 2, 3$ whose distinguishing chromatic number is equal to $k+1$ and such that every distinguishing $(k+1)$-coloring assigns each color to at least a pair of vertices.

The above mentioned examples follow in Section 3, and we shall devote Section 2 to a proof of Theorem 2.

2. Proof – using color 5 only once

Let G be a 3-connected planar graph. It is pointed out in [5] that the uniqueness of its dual, proved by Whitney [10], implies that G can be embedded on the sphere so that every automorphism of G extends to a transformation over the sphere. Such an embedding is said to be *faithful*. If G is faithfully embedded on the sphere, then its automorphism group $\mathrm{Aut}(G)$ acts on the sphere and the image of one specified face of G determines the automorphism of the whole G. In particular, if an automorphism of G fixes each vertex lying along the boundary cycle of some face, then it must be the identity map. We shall use this logic implicitly in our arguments below. See [4] for the details.

A cycle of length 3 in a 3-connected graph G is called a *separating 3-cycle* if the removal of its vertices disconnects G. It is not difficult to see that if G is embedded on the plane or the sphere, then a separating 3-cycle divides it into two regions so that each region contains at least one vertex in its interior.

Lemma 3. *If a* 3*-connected planar graph contains a separating* 3*-cycle, then it has a distinguishing* 5*-coloring with color* 5 *used only once.*

Proof. Let G be a 3-connected graph embedded on the sphere and suppose that G contains a separating 3-cycle. Let us choose an innermost one, say uvw. That is, the cycle uvw is one of the two corresponding regions bounded by uvw, say R, contains no other separating 3-cycles of G. Since uvw is a separating 3-cycle, R contains at least one vertex x.

Consider a 4-coloring of G with colors 1, 2, 3 and 4, and change the color of x to color 5 to obtain a 5-coloring of G. Then it is clear that any color-preserving automorphism σ of G maps the region R onto itself since the outside of R contains no vertex with color 5 and since the separating 3-cycle uvw is innermost. Furthermore, σ fixes each of u, v and w since they have three different colors. This implies that σ fixes the faces incident to uv, vw and uw, both inside and outside R. By the connectivity of G, we conclude that σ is the identity map. Therefore, the 5-coloring of G with color 5 used only for x is distinguishing. ■

The following lemma gives us a key fact to prove Theorem 2.

Lemma 4. *If a* 3*-connected planar graph has a vertex of odd degree, then it has a distinguishing* 5*-coloring with color* 5 *used only once.*

Proof. Let G be a 3-connected planar graph and v a vertex of degree $2k+1$ in G. Let $u_0, u_1, \dots, u_{2k}$ be the neighbors of v lying around v in this cyclic order and let A be the face incident to v with *corner* $u_k v u_{k+1}$. That is, u_k, v, u_{k+1} are three consecutive vertices along A. Consider the graph G' obtained from G by contracting the edge u_0v to a vertex x and by adding an edge joining u_k and u_{k+1} inside A unless already present. Then G' is a planar graph with no loops and admits a 4-coloring by the Four Color Theorem. This 4-coloring induces naturally a 4-coloring of $G - v$ in which u_0 gets the same color as x and u_k and u_{k+1} get two additional different colors, since x, u_k and u_{k+1} form a triangle in G'. Without loss of generality, we may assume that u_0, u_k and u_{k+1} get colors 1, 2 and 3, respectively.

Assign color 5 to v to form a 5-coloring of G. This choice implies that every color-preserving automorphism of G will fix the set of neighbors of v. By construction, u_0 is the only vertex adjacent to v which gets color 1. This implies that u_0v is a unique edge with colors 1 and 5 at its ends and hence every color-preserving automorphism σ of G fixes this edge. If σ is not the identity map, then it acts on the sphere as a reflexion and flips A, swapping u_k and u_{k+1}. However, this is impossible since u_k and u_{k+1} have colors 2 and 3. Therefore, σ is the identity map and the 5-coloring of G with color 5 only for v is distinguishing. ■

Proof of Theorem 2. Let G be a 3-connected planar graph and suppose that it is faithfully embedded on the sphere. By Lemmas 3 and 4, we may assume that G has no separating 3-cycle and that every vertex of G has even degree. First we shall show that there is a triangular face of G under these assumptions.

Let n, m and f denote the number of vertices, edges and faces of G. Assuming that G does not contain a triangular face, we have $2m \geq 4f$. Substituting this to Euler's formula $n - m + f = 2$, we conclude that $2m \leq 4n - 8$, and consequently the average vertex degree in G is strictly less than 4. This implies that G has a vertex of degree at most 3 and hence there is a vertex of degree 2 in G since every vertex of G has even degree. This is contrary to G being 3-connected. Therefore, there is a triangular face in G.

Let uvw be the boundary cycle of a triangular face, say A, and let B be the other face sharing the edge uv with A. Consider the planar graph G' obtained from G by contracting the edge uv to a vertex x. Since G' has no loops, G' admits a 4-coloring by the Four Color Theorem, which induces a 4-coloring of $G - v$. We may assume that u and w get colors 1 and 2 without loss of generality.

Assigning color 5 to v in addition, we obtain a 5-coloring of G so that uv is a unique edge with colors 1 and 5 at its ends. Then every color-preserving automorphism σ of G fixes the edge uv. If B is not a triangular face, then σ cannot swap the two faces A and B, and hence it is the identity map. Thus, the 5-coloring of G is distinguishing and uses color 5 only for v, as we want. Otherwise, we conclude that every triangular face is incident only to triangular faces. This implies that G is a triangulation of the sphere.

It is well-known that a triangulation G of the sphere is 3-colorable if and only if every vertex of G has even degree. Thus, G has a 3-coloring with colors 1, 2 and 3 by our assumption on the degrees of vertices in G. We shall modify this 3-coloring to a 5-coloring of G, adding two colors 4 and 5, as follows.

First suppose that G has a vertex v of degree at least 8. Let $u_0, u_1, \ldots, u_{d-1}$ be the neighbors of v with $d = \deg v$, lying around v in this cyclic order. We may assume that v has color 3, $u_0, u_2, \ldots$ have color 1 and $u_1, u_3, \ldots$ have color 2 in the 3-coloring of G. Since there is no separating 3-cycle in G, there is no edge between u_0 and u_3. Thus, we can assign color 4 to both u_0 and u_3, and assign color 5 to v to obtain a 5-coloring of G.

Choose a color-preserving automorphism σ of G. Since v is a unique vertex with color 5, σ fixes v and maps the cycle $u_0 u_1 \cdots u_{d-1}$ onto itself. The pair of vertices u_0 and u_3 divides the cycle into two segments, one of which has length 3 and with the other being longer since $d \geq 8$. Thus, σ cannot swap these two segments. Furthermore, it cannot flip the segment $u_0 u_1 u_2 u_3$ since u_1 and u_2 are adjacent and consequently have different colors. Hence σ fixes each of vertices $v, u_0, u_1, \ldots, u_{d-1}$ and hence it is the identity map. Therefore, the 5-coloring of

G is distinguishing with color 5 used once.

Now we suppose that there is no vertex of degree at least 8 in G. That is, every vertex in G has degree 4 or 6. Let us first treat the two regular cases.

Since G is a triangulation, if G is 4-regular then G is isomorphic to $K_{2,2,2}$ and this case is excluded as one of the exceptions. On the other hand, a routine application of Euler's formula shows that there exists no planar 6-regular graphs.

Hence we may assume that G is not a regular graph and there exists a pair of adjacent vertices v and u_0 which have degrees 6 and 4, respectively. Let $u_0, u_1, \ldots, u_5$ be the six neighbors of v, which form a cycle surrounding v. Let us note that if u_i and u_j are not consecutive around v, then u_i and u_j are not adjacent, since G is assumed to have no separating 3-cycle.

Assume first that there exist vertices u_i, u_{i+2} (for some $i = 0, \ldots, 5$ with addition modulo 6) which have different degrees. In this case, let us assign color 4 to both u_i and u_{i+2} and assign color 5 to v. Now a color-preserving automorphism σ of G fixes v and u_{i+1}, since u_{i+1} is the only vertex adjacent to three vertices of colors 4, 4 and 5; if so were another vertex, then we could find a separating 3-cycle passing through v and it. Similarly σ fixes both u_i and u_{i+2} since they have different degrees. Hence σ is the identity map and the 5-coloring of G is distinguishing in this case.

Finally we may assume that $\deg u_0 = \deg u_2 = \deg u_4 = 4$ and $\deg u_1 = \deg u_3 = \deg u_5$, which is equal to either 4 or 6. If $\deg u_1 = 4$, then G is isomorphic to DW_6 with rim $u_0 u_1 \cdots u_5$, which is also excluded as one of two exceptional graphs. Thus, we assume that $\deg u_1 = 6$. Let w be the fourth neighbor of u_2 other than v, u_1 and u_3. If w and u_0 were adjacent, then $w u_0 u_1$ would form a separating 3-cycle, which is absurd. Hence construct our final coloring by assigning color 4 to both u_0 and w, and color 5 to v to obtain a 5-coloring of G.

A color-preserving automorphism σ of G fixes v, u_0 and u_1, the latter being the only vertex adjacent to three vertices of colors 4, 4 and 5. Hence σ is the identity map and the constructed 5-coloring of G is distinguishing with color 5 used once, completing the proof. ∎

3. Planar graphs which are not 4-chromatic

In this section, we shall discuss distinguishing coloring of planar graphs which are not 4-*chromatic*, that is, ones that have 2- or 3-colorings. The authors [4] have already proved that every 3-connected bipartite planar graph is distinguishing 3-colorable with two exceptions, the 3-cube and its radial graph, which are distinguishing 4-colorable.

It is hard to grasp the class of 3-colorable planar graphs, and characterizing which 3-colorable planar graphs admit a distinguishing 4-coloring consequently feels out of reach. However we can focus on a more manageable subclass. A graph G is said to be *triangle-free* if G contains no cycle of length 3. Grötzsch's theorem [2] implies that every triangle-free planar graph is 3-colorable.

Theorem 5. *Every 3-connected triangle-free planar graph has a distinguishing 4-coloring with color 4 used for at most two vertices.*

Proof. Let G be a 3-connected triangle-free planar graph embedded on the sphere. First suppose that G is a bipartite graph, having a 2-coloring with colors 1 and 2. Choose an edge x_1y_2 and let A_1 and A_2 be two faces sharing x_1y_2. Let $y_1x_1y_2$ and $x_1y_2x_2$ be the corners of these faces A_1 and A_2, respectively.

Recolor x_1 and x_2 with color 3 and change colors of y_1 and y_2 to 4. Let σ be a color-preserving automorphism of G. Then we have $\sigma(\{x_1, x_2\}) = \{x_1, x_2\}$ and $\sigma(\{y_1, y_2\}) = \{y_1, y_2\}$. Since G is 3-connected, A_1 is the only face of G containing vertices y_1 and y_2. Hence, σ maps A_1 onto itself and similarly A_2 onto itself. Since x_1y_2 is the only edge shared by A_1 and A_2, we have both $\sigma(x_1) = x_1$ and $\sigma(y_2) = y_2$. Continuing along the facial walk of either A_1 or A_2 we infer that $\sigma(x) = x$ for every vertex in the union of A_1 and A_2. Now connectivity of G implies that σ is the identity map. Therefore, the 4-coloring of G is distinguishing and uses color 4 only for two vertices y_1 and y_2.

Now suppose that G is not bipartite. There exists a face A of G bounded by a cycle of odd length. Let $u_0u_1 \cdots u_{2k}$ be the boundary cycle of the face. Since G is triangle-free, we have $k \geq 2$. By Grötzsch's theorem, G has a 3-coloring with colors 1, 2 and 3. Let us recolor vertices u_1 and u_{2k} with color 4. Consider a color-preserving automorphism σ of G. Then σ maps A to itself with $\sigma(\{u_1, u_{2k}\}) = \{u_1, u_{2k}\}$, similarly to the previous case, as otherwise $\{u_1, u_{2k}\}$ would form a 2-cut of G. If σ flipped A, then it would swap u_k and u_{k+1}, but this is impossible since they have two different colors. Therefore, σ fixes each vertex of A and is the identity map. Hence the 4-coloring of G is distinguishing and uses color 4 only for u_1 and u_{2k}. ■

Let G be a 3-connected planar graph distinct from $K_{2,2,2}$ and DW_6. If $\chi(G) = 4$, then Theorem 2 implies that there exists a distinguishing 5-coloring of G which uses color 5 at most once. Does a similar property hold for 3-connected planar graphs with smaller chromatic number? Does there, possibly after excluding a finite list of exceptions, exists a distinguishing coloring in which color $\chi(G) + 1$ is used at most once? We shall in the final part of the paper argue that this does not hold in general.

Let $n \geq 6$ be an even number, and let $U_n = \{u_0, \ldots, u_{n-1}\}$, $W_n = \{w_0, \ldots, w_{n-1}\}$, and $Z_n = \{z_0, \ldots, z_{n-1}\}$ be three disjoint sets of n vertices. The graph T_n is defined as the graph with $U_n \cup W_n \cup Z_n$ as its vertex set such that for every $i \in \{0, \ldots, n-1\}$, z_i is adjacent to u_i, w_i, u_{i+1} and w_{i+1}, and that w_i is adjacent also to u_{i-1}, u_i and u_{i+1}, where the addition in indices is taken modulo n. Let L_n be the subgraph in T_n induced by $U_n \cup W_n$. Note that U_n, W_n, Z_n are independent sets in T_n, hence

$$\chi(L_n) = 2 \quad \text{and} \quad \chi(T_n) = 3,$$

and also note that their respective optimal colorings are unique.

Both T_n and L_n are 3-connected planar graphs. The ladder L_n consists of two disjoint cycles of length n and n edges $u_i w_i$ joining them and can be embedded on the plane like a cyclic ladder. We obtain T_n from this by adding one vertex z_i to each quadrilateral face of L_n. It is not difficult to determine their distinguishing chromatic numbers,

$$\chi_D(L_n) = 3 \qquad \text{and} \quad \chi_D(T_n) = 4.$$

Actually we can construct a distiguishing 4-coloring c of T_n by altering the unique 3-coloring — recoloring vertices z_1, z_2 and z_4 with color 4. Given a color-preserving automorphism σ, we have $\sigma(z_1) = z_1, \sigma(z_2) = z_2$, and $\sigma(z_4) = z_4$, as no two distances between these three vertices are the same. Similarly σ fixes both common neighbors of z_1 and z_2, and is consequently the identity map.

In an analogous construction, a distinguishing 3-coloring of L_n can be obtained by assigning color 3 to vertices u_1, u_2 and u_4. We leave the proof that a color-preserving automorphism is the identity map also in this case to the reader.

Now if x is an arbitrary vertex of T_n and if c' is a 3-coloring of $T_n - x$, then c' partitions $V(T_n - x)$ uniquely into color classes $\{U_n \setminus \{x\}, W_n \setminus \{x\}, Z_n \setminus \{x\}\}$. This last property follows from the fact that $T_n - x$ can be constructed by pasting triangles along edges and adding edges. Similarly, if y is a vertex of L_n, then the bipartition of the connected graph $L_n - y$ into $\{U_n \setminus \{y\}, W_n \setminus \{y\}\}$ is unique. These observations enable us to prove our final result.

Theorem 6. *Let G be isomorphic to either L_n or T_n, where $n \geq 6$ is an even number, and let c be an arbitrary distinguishing $\chi_D(G)$-coloring of G. Then c uses each color at least twice.*

Proof. Assume that c is a 4-coloring of $G = T_n$ using color 4 only on a single vertex x. We may assume that either $x = z_0$ or $x = u_0$, up to symmetry.

First suppose that $x = u_0$. Since the partition of $T_n - x$ by any 3-coloring is unique as we showed above, we may assume that each u_i ($\neq u_0$) has color 1, w_i

has color 2 and z_i has color 3 in c. Then we find an automorphism σ fixing u_0 and flipping each of two disjoint cycles of the ladder L_n, which preserves the colors of vertices. Since σ is not the identity map, c is not a distinguishing coloring.

Suppose that $x = z_0$. Then we may assume that $T_n - x$ is uniquely colored with colors 1, 2 and 3, as well as in the previous case. An automorphism σ fixing z_0 is uniquely determined by any choice among a neighbor of z_0 as the image of u_0. If $\sigma(u_0) = u_1$, then σ is not the identity map and preserves the colors of vertices. Therefore, c is not a distinguishing coloring, again.

The proof in case of $G = L_n$ follows along the same lines. ■

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