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UPPER BOUNDS ON THE NON-RANDOM FLUCTUATIONS IN FIRST PASSAGE PERCOLATION WITH LOW MOMENT CONDITIONS

By

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Abstract. We consider first passage percolation with i.i.d. weights on edges of the d-dimensional cubic lattice $\mathbb{Z}^d$. Under the assumptions that a weight is equal to zero with probability smaller than the critical probability of bond percolation in $\mathbb{Z}^d$, and has the α-th moment for some $\alpha > 1$, we investigate upper bounds on the so-called non-random fluctuations of the model. In addition, we give an application of our result to a lower bound for variance of the first passage percolation in the case where the limit shape has flat edges.

1. Introduction

1.1 The model and the main result

First passage percolation was originally introduced in 1965 by Hammersley and Welsh [5]. In this model, we place i.i.d. random weights on edges of the d-dimensional cubic lattice $\mathbb{Z}^d$, and consider the minimum (random) traveling time from a subset of $\mathbb{Z}^d$ to another one. Let $\mathcal{E}$ be the edge set of $\mathbb{Z}^d$ and consider the measurable space $\Omega := [0,\infty)^{\mathcal{E}}$ endowed with the canonical σ-field $\mathcal{G}$. Moreover, for a given probability measure ν on $[0,\infty)$, let $P := \nu^{\otimes \mathcal{E}}$ be the corresponding product measure on $(\Omega, \mathcal{G})$. For a nearest neighbor path $\gamma = (\gamma_0, \dots, \gamma_l)$ on $\mathbb{Z}^d$, we define the *passage time* of γ as

$$T(\gamma) := \sum_{i=0}^{l-1} \omega(\{\gamma_i, \gamma_{i+1}\})$$

with the convention $\sum_{i=0}^{-1} \omega(\{\gamma_i, \gamma_{i+1}\}) := 0$. Here we use the notation $\{x, y\}$ to denote the edge of $\mathbb{Z}^d$ with endpoints x and y. For any two subsets A and B of $\mathbb{Z}^d$ we define the *first passage time* from A to B as

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$$T(A,B) := \inf\left\{T(\gamma); \begin{matrix}\gamma \text{ is a nearest neighbor path on } \mathbb{Z}^d \\ \text{from some site in } A \text{ to some site in } B\end{matrix}\right\}.$$

In particular, write $T(x,y) = T(\{x\},\{y\})$ for $x,y \in \mathbb{Z}^d$. We may extend the first passage time over $\mathbb{R}^d$. For $x \in \mathbb{R}^d$, let $[x]$ be a lattice point such that

$$\|[x]-x\|_\infty = \min\{\|v-x\|_\infty; v \in \mathbb{Z}^d\} \leq \frac{1}{2},$$

where $\|\cdot\|_\infty$ is the ℓ_∞-norm. If x and y are in $\mathbb{R}^d$, we rewrite $T(x,y) := T([x],[y])$. To shorten notation, given a vector $\xi \in \mathbb{R}^d$, the first passage time from the origin 0 to $n\xi$ is denoted by

$$a_{0,n}(\xi) := T(0, n\xi).$$

It is well known from the standard subadditive ergodic theorem that if $E[\omega(e)] < \infty$, then for any $\xi \in \mathbb{Z}^d$, P-a.s. and in L^1,

$$\mu(\xi) = \lim_{n\to\infty}\frac{1}{n}a_{0,n}(\xi) = \lim_{n\to\infty}\frac{1}{n}E[a_{0,n}(\xi)] = \inf_{n\geq 1}\frac{1}{n}E[a_{0,n}(\xi)]. \tag{1.1}$$

From [6, pages 158–160], such a limit also exists for a general $\xi \in \mathbb{R}^d$, and we call $\mu(\xi)$ the *time constant* for $\xi \in \mathbb{R}^d$.

In this paper, we study rates of convergence to the time constant in the first passage percolation. Kesten [7, (3.2), page 317] derived a bound on the so-called *non-random fluctuations* in first passage percolation, i.e., there exists a constant $C > 0$ such that

$$E[a_{0,n}(\xi)] - n\mu(\xi) \leq Cn^{1-1/(2d+4)}(\log n)^{1/(d+2)}, \qquad \xi \in \mathbb{R}^d, \tag{1.2}$$

under the assumptions that

$$\nu(\{0\}) < p_c \tag{1.3}$$

where p_c is the critical probability of bond percolation in $\mathbb{Z}^d$, and

$$\mathbb{E}[e^{\alpha\omega(e)}] < \infty \text{ for some } \alpha > 0. \tag{1.4}$$

Alexander [1] improved (1.2) by a different method. On the other hand, Zhang [10] studied the same problem under a weaker moment condition than (1.4): If

$$m_{\nu,\alpha} := E[\omega(e)^\alpha] < \infty \text{ for some } \alpha > 1, \tag{1.5}$$

then there exists a constant $C > 0$ such that for each coordinate direction ξ' of $\mathbb{R}^d$,

$$E[a_{0,n}(\xi')] - n\mu(\xi') \le Cn^{1/2}(\log n)^7. \tag{1.6}$$

For the proof of (1.6), he used symmetry properties of $\mathbb{Z}^d$ with respect to the coordinate axis. Therefore, his approach does not work for any direction except coordinate axis, and we need a new method. The next theorem is our main result.

Theorem 1.1. *Assume* (1.3) *and* (1.5). *Then, there exists a constant* $C > 0$ *such that for all* ℓ_2*-unit vector* $\xi \in \mathbb{R}^d$,

$$E[a_{0,n}(\xi)] - n\mu(\xi) \le Cn^{1-1/(6d+12)}(\log n)^{1/3}. \tag{1.7}$$

1.2 Application of Theorem 1.1

In this subsection, we state an application of Theorem 1.1. Bound (1.7) may not be optimal, but it is very useful that for all direction ξ we can uniformly take the exponent of the convergence rate strictly smaller than 1. Auffinger and Damron [2, Theorem 2.5] established that the variance of the first passage time has a lower bound with a logarithmic order in the case where the limit shape has flat edges. For Theorem 2.5 of [2], they require not only (1.5) with $\alpha = 2$ but also a bound on the non-random fluctuations at that time. Thanks to Theorem 1.1, we can check their condition whereas (1.5) holds for $\alpha = 2$.

Let $d = 2$ and write $\mathrm{supp}(\nu')$ for the support of the probability measure ν'. Moreover, let $\vec{p}_c$ be the critical parameter for oriented percolation on $\mathbb{Z}^2$. Furthermore, denote by θ_q the unique angle such that the line segment connecting 0 and the point $N_q := (1/2 + \alpha_q/\sqrt{2}, 1/2 - \alpha_q/\sqrt{2}) \in \mathbb{R}^2$ has angle θ_q with the x-axis, where α_q is the asymptotic speed of oriented percolation with parameter q. For details of oriented percolation, we refer the reader to [3]. For $q \ge \vec{p}_c$, $\mathcal{M}_q$ is defined by the set of probability measures ν' satisfying conditions

(C1) $\mathrm{supp}(\nu') \subset [1, \infty)$,
(C2) $\nu'(\{1\}) = q$.

Note that if $\nu \in \mathcal{M}_q$ (in particular, (C1) holds for ν), then we have $\nu(\{0\}) = 0 < p_c$, i.e., (1.3) is satisfied.

We now assume that (1.5) holds for $\alpha = 2$ and the law ν satisfies one of conditions

(a) $\inf \mathrm{supp}(\nu) = 0$ and $\nu(\{0\}) < p_c$,
(b) $\lambda := \inf \mathrm{supp}(\nu) > 0$ and $\nu(\{\lambda\}) < \vec{p}_c$.

In [8, Theorem 2], under the above assumptions Newman and Piza showed that there is a constant $C > 0$ such that for all $n \geq 1$ and $\theta \in [0, 2\pi)$,

$$\mathrm{Var}(T(0, n\xi_\theta)) \geq C \log n, \tag{1.8}$$

where $\xi_\theta := (\cos\theta, \sin\theta) \in \mathbb{R}^2$. This means that the variance of the first passage time diverges as $n \to \infty$ in these cases. On page 980 of [8], they also state that the variance does not diverge for $\theta \in (\theta_q, \pi/2 - \theta_q)$ in the case $\nu \in \mathcal{M}_q$ with $q > \vec{p}_c$. We are now concerned with the divergence of $\mathrm{Var}(T(0, n\xi_\theta))$ for $\theta \in [0, \theta_q)$ in the same situation. If ξ_θ is a coordinate direction, then Zhang [9, Theorem 2] proved (1.8) under assumption (1.4). After that, Auffinger and Damron [2, Theorem 2.5] improved it as follows.

Theorem 1.2. (Auffinger and Damron) *For a given $q \in [\vec{p}_c, 1)$, let $\nu \in \mathcal{M}_q$ and $\theta \in [0, \theta_q)$. Suppose that* (1.5) *holds with $\alpha = 2$ and there exists $\beta < 1$ such that for all large n,*

$$E[T(0, n\xi_\theta)] < n\mu(\xi_\theta) + n^\beta, \tag{1.9}$$

where $\xi_\theta := (\cos\theta, \sin\theta) \in \mathbb{R}^2$. Then, there exists a positive constant $C = C(\theta)$ such that (1.8) *holds for all n.*

If we assume (1.4), then (1.2) yields (1.9) for all angles θ, and (1.8) holds for all $\theta \in [0, \theta_q)$. Under the assumption of Theorem 1.2, (1.6) only guarantees the validity of (1.9) for each coordinate direction ξ_θ. We use Theorem 1.1 to obtain (1.9) for all angles θ. With these observations, the whole picture of divergence for $\mathrm{Var}(T(0, n\xi_\theta))$ is completed under (1.5) with $\alpha = 2$.

Corollary 1.3. *For a given $q \in [\vec{p}_c, 1)$, let $\nu \in \mathcal{M}_q$ and $\theta \in [0, \theta_q)$. Suppose that* (1.5) *holds with $\alpha = 2$. Then,* (1.8) *holds for all n.*

1.3 Organization of the paper

Let us describe how the present article is organized. In Section 2, we introduce truncated weights following the method of Zhang [10]. Since the argument of Sections 2 and 3 in [10] contains an oversight, we will present one of ways to fix this (see Lemma 2.1 below). In addition, we give a method to compare the expectation of the first passage time for the truncated weights with that for the original weights.

In Section 3, we give the proof of Theorem 1.1. To do this, we improve the approach taken in [7, Section 3, page 317] under our assumption (1.5). This section is divided into two subsections. In Subsection 3.1, for the reader's convenience, we explain the outline of Kesten's approach under (1.4), and clarify differences between his and ours. In Subsection 3.2, we present a new method to derive the convergence rate for all directions under low moment conditions.

In the following sections, C_i, $i = 1, 2, \dots,$ are always positive constants depending on d, ν and α.

2. Preliminaries

In this section, we shall introduce truncated weights, following basically the strategy taken in [10]. By assumptions (1.3) and (1.5), we can take $\kappa \in (0,1)$ such that

$$P(\omega(e) < \kappa) \vee P(\omega(e) > \kappa^{-1}) < p_c.$$

From now on, we fix κ as above. Then, an edge $e \in \mathcal{E}$ is said to be *bad* if $\omega(e) < \kappa$, and a site $x \in \mathbb{Z}^d$ is said to be *unhealthy* if some weights of $2d$ adjacent edges of x are larger than κ^{-1}. Let us now introduce two connectivities of paths on $\mathbb{Z}^d$. We say that a path $\gamma = (\gamma_0, \dots, \gamma_l)$ is $\mathbb{Z}^d$- or $*$-connected if for all $i \in [0, l-1]$, $\|\gamma_{i+1} - \gamma_i\|_2$ or $\|\gamma_{i+1} - \gamma_i\|_\infty$ equals 1, respectively. Here$\|\cdot\|_2$ is the ℓ_2-norm. A $\mathbb{Z}^d$-connected path $\gamma = (\gamma_0, \dots, \gamma_l)$ is called bad if each edge $\{\gamma_i, \gamma_{i+1}\}$ is bad. Furthermore, a $*$-connected path $\gamma = (\gamma_0, \dots, \gamma_l)$ is called unhealthy if each site γ_i is unhealthy. Let $\mathcal{C}_-(x)$ be a bad $\mathbb{Z}^d$-connected cluster containing a site x, i.e., the set of all sites connected to x by a bad $\mathbb{Z}^d$-connected path. We also denote by $\mathcal{C}_+(x)$ an unhealthy $*$-connected cluster containing a site x, i.e., the set of all sites connected to x by an unhealthy $*$-connected path.

Fix $\delta < 1/d$. We now define a truncated weight $\sigma(e)$ as follows. If one of the following conditions 1–3 holds, then we set $\sigma(e) := \omega(e)$, otherwise $\sigma(e) := 1$:

1. $\kappa \le \omega(e) \le \kappa^{-1}$,
2. $\omega(e) < \kappa$, and e is connected to a bad $\mathbb{Z}^d$-connected cluster with less than n^δ vertices,
3. $\omega(e) > \kappa^{-1}$, and e is connected to an unhealthy $*$-connected cluster with less than n^δ vertices.

Then, let T_σ be the first passage time on the truncated weights σ. Moreover, for $x \in \mathbb{R}^d$ and $n \ge 1$, let

$$D_n(x) := x + \left[-3^d\kappa^{-1}n^\delta, 3^d\kappa^{-1}n^\delta\right]^d.$$

We now consider the first passage time $T(D_n(0), D_n(n\xi))$ for each ℓ_2-unit vector $\xi \in \mathbb{R}^d$. Note that for all $x \in D_n(0)$ and $y \in D_n(n\xi)$,

$$T(D_n(0), D_n(n\xi)) \le T(x,y) \le T(D_n(0), D_n(n\xi)) + J_n(\xi, \omega), \tag{2.1}$$

where $J_n(\xi,\omega)$ is the sum of $\omega(e)$ over all edges included in $D_n(0)\cup D_n(n\xi)$.

The following lemma is a minor modification of Lemma 8 and (3.23) in [10].

Lemma 2.1. *We can choose κ satisfying that, for each ℓ_2-unit vector $\xi\in\mathbb{R}^d$, there exist constants $\widetilde{C}_1,\widetilde{C}_2>0$ (which depend only on the law ν, d, α, δ and κ) such that*

$$P\big(T(D_n(0),D_n(n\xi))\neq T_\sigma(D_n(0),D_n(n\xi))\big)\le \widetilde{C}_1\exp\{-\widetilde{C}_2 n^\delta\}, \tag{2.2}$$

and, for all $u>0$,

$$\begin{aligned}&P\big(|T_\sigma(D_n(0),D_n(n\xi))-E[T_\sigma(D_n(0),D_n(n\xi))]|\ge un^{1/2+3\delta}\big)\\&\le \widetilde{C}_1\exp\{-\widetilde{C}_2u^2n^\delta\}.\end{aligned} \tag{2.3}$$

Proof. We replace the component $(\log n)^{1+\delta}$ appearing in (1.10) of [10] with n^δ. Then, the proofs of (2.2) and (2.3) follow from the same strategy taken in [10, Sections 2 and 3], and we do not repeat it here. As mentioned in Subsection 1.3, an oversight is contained in the proof of Lemma 8 in [10] and let us present a way to fix it. In the beginning of its proof, the following claim is stated:

> By Proposition 5.8 in [6], with a probability larger than $1-C_1\exp(-C_2n)$, there exists an optimal path γ for $T(D_n(0),D_n(nu))$ with $\#\gamma\le Ln$.

Because we now only assume $m_{\nu,\alpha}<\infty$, this does not directly follow from Proposition 5.8 in [6]. To fix this problem, we replace the phrase "$\#\gamma\le Ln$" with "$\#\gamma\le\exp\{Ln^\delta\}$". Let

$$A_n:=\big\{\text{any optimal path }\gamma\text{ for }T(D_n(0),D_n(n\xi))\text{ satisfies }\#\gamma>\exp\{Ln^\delta\}\big\}.$$

Proposition 5.8 in [6] then shows that there are constants C_1, C_2 and C_3 such that

$$\begin{aligned}&P\Big(\exists\text{ a path }\gamma\text{ from }0\text{ with }\#\gamma\ge\exp\{Ln^\delta\}\text{ but }T(\gamma)\le C_1\exp\{Ln^\delta\}\Big)\\&\le C_2\exp\big\{-C_3\exp\{Ln^\delta\}\big\}.\end{aligned}$$

Chebyshev's inequality hence implies

$$\begin{aligned}P(A_n)&\le C_2(\#D_n(0))\exp\big\{-C_3\exp\{Ln^\delta\}\big\}\\&\qquad+P\big(T(D_n(0),D_n(n\xi))>C_1\exp\{Ln^\delta\}\big)\\&\le C_2(\#D_n(0))\exp\big\{-C_3\exp\{Ln^\delta\}\big\}+C_1^{-1}m_{\nu,1}n\exp\{-Ln^\delta\}\\&\le C_4\exp\{-C_5n^\delta\}\end{aligned} \tag{2.4}$$

for some constants C_4 and C_5. If

$$T(D_n(0), D_n(n\xi)) \neq T_\sigma(D_n(0), D_n(n\xi)),$$

then we have an edge $e \in \gamma$ satisfying that $\#\mathcal{C}_-(v_e) > n^\delta$ or $\#\mathcal{C}_+(v_e) > n^\delta$, where v_e is an endpoint of the edge e. Note that if $e \in \gamma$ with $\#\gamma \leq \exp\{Ln^\delta\}$, then $v_e \in [-\exp\{Ln^\delta\}, \exp\{Ln^\delta\}]^d$ holds. Therefore, we have

$$\begin{aligned} &P\big(T(D_n(0), D_n(n\xi)) \neq T_\sigma(D_n(0), D_n(n\xi))\big) \\ &\leq P(A_n) + \sum_{e \in [-\exp\{Ln^\delta\}, \exp\{Ln^\delta\}]^d} P\big(\#\mathcal{C}_-(v_e) > n^\delta \text{ or } \#\mathcal{C}_+(v_e) > n^\delta\big). \end{aligned}$$

By the choice of κ, Theorem 6.1 of [4] implies that there are constants C_6 and C_7 such that the second term on the right-hand side is bounded above by

$$\sum_{e \in [-\exp\{Ln^\delta\}, \exp\{Ln^\delta\}]^d} 2\exp\{-C_6 n^\delta\} \leq C_7 \exp\{dLn^\delta - C_6 n^\delta\}.$$

This, together with (2.4), gives (2.2) for sufficiently small L. □

We need the following lemma to estimate the difference between the expectations of T and T_σ.

Lemma 2.2. *For each ℓ_2-unit vector $\xi \in \mathbb{R}^d$ there exist constants $\widetilde{C}_3, \widetilde{C}_4 > 0$ (which depend only on ν, d, α, δ and κ) such that*

$$\big|E[T(D_n(0), D_n(n\xi))] - E[T_\sigma(D_n(0), D_n(n\xi))]\big| \leq \widetilde{C}_3 n \exp\{-\widetilde{C}_4 n^\delta\}.$$

Proof. Let $\Gamma := \{T(D_n(0), D_n(n\xi)) \neq T_\sigma(D_n(0), D_n(n\xi))\}$, and set

$$C_8 := \sqrt{d}\widetilde{C}_1^{(\alpha-1)/\alpha} m_{\nu,\alpha}^{1/\alpha}, \qquad C_9 := \widetilde{C}_2(\alpha-1)/\alpha.$$

Using Hölder's inequality and (2.2), we have

$$\begin{aligned} E\left[T(D_n(0), D_n(n\xi))\mathbf{1}_\Gamma\right] &\leq \sqrt{d}\widetilde{C}_1^{(\alpha-1)/\alpha} m_{\nu,\alpha}^{1/\alpha} n \exp\{-n^\delta \widetilde{C}_2(\alpha-1)/\alpha\} \\ &= C_8 n \exp\{-C_9 n^\delta\}. \end{aligned}$$

Therefore,

$$E[T_\sigma(D_n(0), D_n(n\xi))] + C_8 n \exp\{-C_9 n^\delta\} \geq E[T(D_n(0), D_n(n\xi))].$$

Similarly, since $\sigma(e) \leq \omega(e) + 1$ holds for all $e \in \mathcal{E}$,

$$E[T(D_n(0), D_n(n\xi))] + C_{10} n \exp\{-C_{11} n^\delta\} \geq E[T_\sigma(D_n(0), D_n(n\xi))]$$

for some constants C_{10} and C_{11}. Thus, Lemma 2.2 follows by choosing $\widetilde{C}_3 := C_8 \vee C_{10}$ and $\widetilde{C}_4 := C_9 \wedge C_{11}$. □

In the next section, $\widetilde{C}_i$'s are always constants appearing in this section.

3. Proof of Theorem 1.1

3.1 Kesten's approach

Let us first prepare some notations. Fix an ℓ_2-unit vector $\xi \in \mathbb{R}^d$, and for $M \in \mathbb{N}$ let $U_1, \ldots, U_K$ be all the vectors with integer components and $\|U_k\|_\infty = M$, $1 \le k \le K$. Define

$$\Lambda(M,n) := \min\left\{ \sum_{k=1}^{K} p(k) E[T(0,U_k)] \right\} - n\mu(\xi),$$

where the minimum is over all choices of $p(k) \in \mathbb{N}_0$ such that

$$\left\| \sum_{k=1}^{K} p(k) U_k - n\xi \right\|_\infty \le M. \tag{3.1}$$

In [7, pages 317–327], the proof of (1.2) is composed of three steps. The main parts are Steps 1 and 2 of [7, pages 317–326], so that we will explain only these steps here. Step 3 in [7, pages 326–327] will be explained in the proof of Theorem 1.1.

In Step 1 of [7, page 317], Kesten shows that there exists a constant $C_1 > m_{\nu,1}$ such that for $M \in [n^{1/(d+1)}, n]$ and $l \ge 1$,

$$l\Lambda(M,n) - C_1 l M^{1/d} n^{(d-1)/d} \le \Lambda(M, ln) \le C_1 ln. \tag{3.2}$$

His proof works under assumption (1.5).

In Step 2 of [7, page 321], it is proved that there are constants $c, c', C, C' > 0$ such that for large n and M as above and for $l \ge 2$,

$$\begin{aligned} &P\left(a_{0,ln}(\xi) \le ln\mu(\xi) + \frac{l}{2}\Lambda(M,n) \right) \\ &\le ce^{-ln} + \exp\left\{ c' \frac{ln}{M} \log M + ClM^{(2-d)/(2d)} n^{(d-1)/d} - C' \frac{l\Lambda(M,n)^2}{nM^{1/2}} \right\}. \end{aligned} \tag{3.3}$$

We have to modify this estimate under assumption (1.5). In particular, (1.4) is required for bounds (3.12) and (3.11) below. Thus, if (1.5) is assumed instead of (1.4), then we must get a bound similar to (3.3) without (3.11) and (3.12). In fact, this is possible by replacing (3.13) with Lemma 3.1, which is proved in Subsection 3.2.

Let us give a sketch of Kesten's proof of (3.3). Let $\gamma := (v_0, v_1, \dots, v_p)$ be any self-avoiding nearest neighbor path from $v_0 = 0$ to $v_p = [ln\xi]$ with passage time $T(\gamma) \le ln\mu(\xi) + (l/2)\Lambda(M,n)$. In addition, define the indices $\tau_0 := 0$ and

$$\tau_{i+1} := \min\{k \in (\tau_i, p]; \|v_k - v_{\tau_i}\|_\infty = M\}, \qquad i \ge 0,$$

with the convention $\min\emptyset = \infty$. Set $Q := \max\{i \ge 0; \tau_i < \infty\}$ and $a_i := v_{\tau_i}$ for $i \in [0, Q]$. By definition of Q, we have

$$\|v_k - v_{\tau_Q}\|_\infty < M, \qquad \tau_Q < k \le p,$$

and in particular,

$$\|v_{\tau_Q} - ln\xi\|_\infty \le \|v_{\tau_Q} - v_p\|_\infty + \|[ln\xi] - ln\xi\|_\infty \le M. \tag{3.4}$$

Moreover,

$$\|a_i - a_{i-1}\|_\infty = \|v_{\tau_i} - v_{\tau_{i-1}}\|_\infty = M, \qquad 1 \le i \le Q, \tag{3.5}$$

so that $a_i - a_{i-1}$ is one of the U_k's (which appear in the beginning of this section). It holds from [7, pages 322–323] that there exists constants C_2, C_3 such that

$$P(Q \ge C_2 ln/M) \le C_3 e^{-ln}. \tag{3.6}$$

We now fix $Q < C_2 ln/M$ and $a_1, \dots, a_Q$ satisfying (3.4) and (3.5). We denote by $p(k)$ the number of $i \in [1, Q]$ with $a_i - a_{i-1} = U_k$. The $p(k)$'s are fixed at the moment. Then, (3.28)–(3.32) of [7, page 323] enable us to show that for any $\beta \ge 0$,

$$\begin{aligned} &P\left(\begin{matrix}\exists \text{ a self-avoiding path } \gamma \text{ with } v_{\tau_i} = a_i,\ 1 \le i \le Q, \\ \text{and satisfying (3.4) and } T(\gamma) \le ln\mu(\xi) + (l/2)\Lambda(M,n)\end{matrix}\right) \\ &\le \exp\left\{-\frac{\beta l}{2}\Lambda(M,n) + \beta C_1 l M^{1/d} n^{(d-1)/d}\right\} \\ &\quad \times \prod_{k=1}^{K} E\big[\exp\{-\beta(T(0,U_k) - E[T(0,U_k)])\}\big]^{p(k)}. \end{aligned} \tag{3.7}$$

It remains to estimate the product in (3.7). Note that $\sum_{k=1}^{K} p(k) = Q$, which is the number of $(a_i - a_{i-1})$'s, and

$$\begin{aligned} &E\big[\exp\{-\beta(T(0,U_k) - E[T(0,U_k)])\}\big] \\ &\le \exp\left\{C_4 \frac{\beta l}{Q}\Lambda(M,n)\right\} \\ &\quad + \exp\{\beta E[T(0,U_k)]\} P\left(T(0,U_k) - E[T(0,U_k)] \le -\frac{C_4 l}{Q}\Lambda(M,n)\right), \end{aligned} \tag{3.8}$$

where C_4 will be chosen such that for large M and for $n \geq M$ and $l \geq 2d$,

$$\frac{C_4 l}{Q}\Lambda(M,n) \leq \frac{d}{2} M m_{\nu,1} \quad \text{and} \quad C_4 \leq \frac{1}{4}. \tag{3.9}$$

The argument below (3.34) of [7] guarantees the existence of such a C_4. In particular, for $n \geq M$ and $l \geq 2d$,

$$Q \geq \frac{ln}{dM} - 1 \geq \frac{ln}{2dM}. \tag{3.10}$$

We shall estimate the last probability in (3.8). Set $\eta := U_k/\|U_k\|_2$ and $m := \lfloor \|U_k\|_2 \rfloor \in [M, dM]$. Note that $\|[m\eta] - U_k\|_\infty \leq 2$. Assumption (1.4) guarantees that there exist constants $c, C, C' > 0$ such that for $t \geq 0$,

$$P\big(\big|T(0,[m\eta]) - E[T(0,[m\eta])]\big| \geq t\sqrt{m}\big) \leq Ce^{-C't}, \tag{3.11}$$

and for $t \leq cm$,

$$P\big(\big|T(0,[m\eta]) - E[T(0,[m\eta])] - T(0,U_k) + E[T(0,U_k)]\big| \geq t\big) \leq Ce^{-C't}, \tag{3.12}$$

which are (2.49) and (3.36) of [7], respectively. By choosing t suitably (see (3.37) of [7, page 325] for details), these estimates show that for some constants $C_5, C_6 > 0$,

$$\begin{aligned} &P\bigg(T(0,U_k) - E[T(0,U_k)] \leq -\frac{C_4 l}{Q}\Lambda(M,n)\bigg) \\ &\leq C_5 \exp\bigg\{-\frac{C_6}{QM^{1/2}} l\Lambda(M,n)\bigg\}. \end{aligned} \tag{3.13}$$

Therefore, the right-hand side of (3.8) is at most

$$\exp\bigg\{C_4 \frac{\beta l}{Q}\Lambda(M,n)\bigg\} + C_7 \exp\bigg\{\beta d M m_{\nu,1} - \frac{C_6}{QM^{1/2}} l\Lambda(M,n)\bigg\}.$$

for some constant C_7. Choose β such that the two exponents become equal, so that the left-hand side of (3.8) is smaller than

$$C_8^Q \exp\bigg\{C_9 l M^{\delta-(d-2)/(2d)} n^{(d-1)/d} - C_{10}\frac{l^2\Lambda(M,n)^2}{QM^{3/2}}\bigg\}.$$

for some constants C_8, C_9, C_{10}. Hence (3.3) follows by summing the left-hand side of (3.7) over all possible values of Q and $a_1, \dots, a_Q$. (See the first paragraph of [7, page 326] for details.)

With these observations, under (1.5) we must estimate the last probability in (3.8) without (3.11) and (3.12). In fact, this is possible as follows. (See Subsection 3.2 for the proof.)

Lemma 3.1. *Assume* (1.3) *and* (1.5). *For* $\delta \le 1/6$ *there exist constants* $C_{11}, C_{12} > 0$ *such that, for all large* n, *if* $\Lambda(M,n) \ge C_{11} n M^{-(1-d\delta)}$ *and* $Q < C_2 ln/M$, *then*

$$P\left(T(0,U_k) - E[T(0,U_k)] \le -\frac{C_4 l}{Q}\Lambda(M,n)\right) \le 2\widetilde{C}_1 \exp\left\{ -C_{12} M^{-(2-\delta)} \left(\frac{l}{Q}\right)^2 \Lambda(M,n)^2 \right\}.$$

3.2 Proofs of Lemma 3.1 and Theorem 1.1

Let us first give the proof of Lemma 3.1.

Proof of Lemma 3.1. Recall that $\eta := U_k/\|U_k\|_2$ and $m := \lfloor \|U_k\|_2 \rfloor \in [M, dM]$. Note that $\|m\eta - U_k\|_\infty \le 1$ and $0 \in D_m(0)$ and $U_k \in D_m(m\eta)$ hold for large m. By (2.1),

$$T(D_m(0), D_m(m\eta)) \le T(0,U_k) \le T(D_m(0), D_m(m\eta)) + J_m(\xi,\omega).$$

This, together with Lemma 2.2, gives

$$E[T(0,U_k)] \le E[T_\sigma(D_m(0), D_m(m\eta))] + \widetilde{C}_3 m \exp\{-\widetilde{C}_4 m^\delta\} + C_{13} m^{d\delta}$$

for some constant C_{13}. Therefore,

$$\begin{aligned} &P\left(T(0,U_k) - E[T(0,U_k)] \le -\frac{C_4 l}{Q}\Lambda(M,n)\right) \\ &\le P\Big(T(D_m(0), D_m(m\eta)) - E[T_\sigma(D_m(0), D_m(m\eta))] \\ &\qquad \le -\frac{C_4 l}{Q}\Lambda(M,n) + \widetilde{C}_3 m \exp\{-\widetilde{C}_4 m^\delta\} + C_{13} m^{d\delta}\Big). \end{aligned} \tag{3.14}$$

Take $C_{11} := 4d^{d\delta} C_2 (\widetilde{C}_3 \vee C_{13})/C_4$. Since we have assumed $\Lambda(M,n) \ge C_{11} n M^{-(1-d\delta)}$ and $Q < C_2 ln/M$, the choice of n, M and m implies for all large n,

$$\frac{C_4 l}{2Q}\Lambda(M,n) \ge \widetilde{C}_3 m \exp\{-\widetilde{C}_4 m^\delta\} + C_{13} m^{d\delta}.$$

It follows that the right-hand side of (3.14) is smaller than

$$P\left(T(D_m(0), D_m(m\eta)) - E[T_\sigma(D_m(0), D_m(m\eta))] \le -\frac{C_4 l}{2Q}\Lambda(M,n)\right).$$

Thanks to (2.2) and (2.3), this is bounded from above by

$$\begin{aligned}
&\widetilde{C}_1 \exp\{-\widetilde{C}_2 m^{\delta}\} \\
&+ P\biggl(|T_\sigma(D_m(0), D_m(m\eta)) - E[T_\sigma(D_m(0), D_m(m\eta))]| \geq \frac{C_4 l}{2Q}\Lambda(M,n) \biggr) \\
&\leq \widetilde{C}_1 \exp\{-\widetilde{C}_2 m^{\delta}\} + \widetilde{C}_1 \exp\biggl\{ -\biggl(\frac{\widetilde{C}_2 C_4^2}{4} \biggr) \frac{(l/Q)^2 \Lambda(M,n)^2}{m^{1+5\delta}} \biggr\}.
\end{aligned}$$

By (3.9) and $\delta \leq 1/6$, there exists a constant $C_{12} > 0$ such that the right-hand side is smaller than

$$2\widetilde{C}_1 \exp\biggl\{ -C_{12} M^{-(2-\delta)} \biggl(\frac{l}{Q} \biggr)^2 \Lambda(M,n)^2 \biggr\}.$$

Hence the proof is complete. □

Finally, we prove Theorem 1.1.

Proof of Theorem 1.1. Let us first show that there exist constants $C_{14}, C_{15}, C_{16} > 0$ such that, for all large n, if $\Lambda(M,n) \geq C_{11} n M^{-(1-d\delta)}$, then

(3.15)
$$\begin{aligned}
&P\biggl(a_{0,ln}(\xi) \leq ln\mu(\xi) + \frac{l}{2}\Lambda(M,n) \biggr) \\
&\leq C_{14} e^{-ln} + \exp\biggl\{ C_{15} \frac{ln}{M} \log M + C_{15} l M^{\delta-(d-1)/d} n^{(d-1)/d} - C_{16} \frac{l\Lambda(M,n)^3}{n^2 M^{1-\delta}} \biggr\},
\end{aligned}$$

which is the counterpart of (3.3) under (1.5). From Lemma 3.1, the right-hand side of (3.8) is at most

$$\exp\biggl\{ C_4 \frac{\beta l}{Q} \Lambda(M,n) \biggr\} + 2\widetilde{C}_1 \exp\biggl\{ \beta d M m_{\nu,1} - C_{12} M^{-(2-\delta)} \biggl(\frac{l}{Q} \biggr)^2 \Lambda(M,n)^2 \biggr\}.$$

Finally, we choose β such that the two exponents above here become equal, i.e.,

$$\beta = C_{12} M^{-(2-\delta)} \biggl(\frac{l}{Q} \biggr)^2 \Lambda(M,n)^2 \biggl(d M m_{\nu,1} - \frac{C_4 l}{Q} \Lambda(M,n) \biggr)^{-1}.$$

In particular, by (3.9),

$$\beta \leq C_{12} M^{-(2-\delta)} \biggl(\frac{l}{Q} \biggr)^2 \Lambda(M,n)^2 \biggl(\frac{d}{2} M m_{\nu,1} \biggr)^{-1} \leq C_{17} M^{-(1-\delta)}$$

for some constant C_{17}. By (3.9) and (3.10), the left-hand side of (3.7) is smaller than

$$\exp\{\beta C_1 l M^{1/d} n^{(d-1)/d}\} \times \prod_{k=1}^{K} \left((2\widetilde{C}_1 + 1) \exp\left\{ \left(C_4 - \frac{1}{2} \right) \frac{\beta l}{Q} \Lambda(M,n) \right\} \right)^{p(k)}$$
$$\leq (2\widetilde{C}_1 + 1)^{C_2 ln/M} \exp\left\{ C_1 C_{17} l M^{\delta-(d-1)/d} n^{(d-1)/d} - C_{18} \frac{l \Lambda(M,n)^3}{n^2 M^{1-\delta}} \right\}$$

for some constant C_{18}. Therefore, bound (3.15) follows by summing the left-hand side of (3.7) over all possible values of Q and $a_1, \dots, a_Q$. See the first paragraph in [7, page 326] for details.

We complete the proof of Theorem 1.1 following basically Step 3 of [7, pages 326–327]. Pick

$$\delta := 1/(d+4). \tag{3.16}$$

Here, note that $\delta < 1/d$. We first treat the case $\Lambda(M,n) \geq C_{11} n M^{-(1-d\delta)}$. Choose

$$M := \lfloor n^{1/(d\delta+1)} \rfloor. \tag{3.17}$$

If we have

$$C_{16} \frac{l \Lambda(M,n)^3}{n^2 M^{1-\delta}} > C_{15} \frac{ln}{M} \log M + C_{15} l M^{\delta-(d-1)/d} n^{(d-1)/d}, \tag{3.18}$$

then by (3.15),

$$\lim_{l \to \infty} P\left(a_{0,ln}(\xi) \leq ln\mu(\xi) + \frac{l}{2} \Lambda(M,n) \right) = 0.$$

However, this contradicts to (1.1), and (3.18) fails to hold. This means that

$$\Lambda(M,n) \leq C_{19} \left\{ n M^{-\delta/3} (\log M)^{1/3} + n^{1-1/(3d)} M^{1/(3d)} \right\}$$

for some constant C_{19}. By (3.16), $\Lambda(M,n)$ is smaller than

$$2C_{19} n M^{-\delta/3} (\log M)^{1/3} \leq C_{20} n^{1-1/(6d+12)} (\log n)^{1/3}$$

for some constant C_{20}. This, together with the definition of $\Lambda(M,n)$, enables us to take $p(k) \geq 0$ satisfying (3.1) and

$$\sum_{k=1}^{\nu} p(k) E[T(0,U_k)] \leq n\mu(\xi) + C_{20} n^{1-1/(6d+12)} (\log n)^{1/3}.$$

Now set $\rho = \sum_{k=1}^{\nu} p(k)$ and let $u_1, \dots, u_\rho$ be the sites defined by $u_i - u_{i-1} = U_k$ for $\sum_{j=1}^{k-1} p(j) < i \le \sum_{j=1}^{k} p(j)$. Note that $u_\rho = \sum_{k=1}^{\nu} p(k)U_k$. Subadditivity of the first passage time gives

$$E[a_{0,n}(\xi)] \le \sum_{i=1}^{\rho} E[T(u_{i-1}, u_i)] + E[T(u_\rho, n\xi)].$$

By the choice of $u_1, \dots, u_\rho$,

$$\begin{aligned}\sum_{i=1}^{\rho} E[T(u_{i-1}, u_i)] &= \sum_{k=1}^{\nu} p(k)E[T(0, U_k)] \\ &\le n\mu(\xi) + C_{20}n^{1-1/(6d+12)}(\log n)^{1/3}.\end{aligned}$$

In addition, by (3.1),

$$E[T(u_\rho, n\xi)] \le d\|[n\xi] - u_\rho\|_\infty E[\omega(0)] \le d(M+1)E[\omega(0)],$$

and (1.7) immediately follows in the case $\Lambda(M, n) \ge C_{11}nM^{-(1-d\delta)}$.

In the case $\Lambda(M, n) < C_{11}nM^{-(1-d\delta)}$, the definition of $\Lambda(M, n)$ implies

$$n\mu(\xi) + C_{11}nM^{-(1-d\delta)} > \min\left\{\sum_{k=1}^{K} p(k)E[T(0, U_k)]\right\},$$

where the minimum is taken over all choices of $p(k)$ satisfying (3.1). Subadditivity of the first passage time shows that

$$\begin{aligned}\sum_{k=1}^{K} p(k)E[T(0, U_k)] &\ge \sum_{k=1}^{K} E\left[T\left(\sum_{j=1}^{k-1} p(j)U_j, \sum_{j=1}^{k} p(j)U_j\right)\right] \\ &\ge -d(M+1)m_{\nu,1} + E[a_{0,n}(\xi)].\end{aligned}$$

With these observations,

$$E[a_{0,n}(\xi)] \le n\mu(\xi) + C_{11}nM^{-(1-d\delta)} + d(M+1)m_{\nu,1}.$$

This, together with (3.16) and (3.17), is bounded from above by

$$n\mu(\xi) + (C_{11} + 2dm_{\nu,1})n^{1-1/(d+2)}(\log n)^{1/3}.$$

Since $n^{1-1/(6d+12)} \ge n^{1-1/(d+2)}$, (1.7) is valid in all cases. □

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