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ON CONNECTED QUANDLES WITH THE PROFILE $\{1, \ell, \ell\}$ or $\{1, \ell, \ell, \ell\}$

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Abstract. In this paper, we study a finite connected quandle with the profile $\{1, \ell, \ell\}$ or $\{1, \ell, \ell, \ell\}$. In particular, we study an affine quandle with the above profile. We prove that the automorphism group of a connected affine quandle with the above profile acts doubly transitively on itself.

1. Introduction

Definition 1.1. A quandle is a set Q with a binary operation $* : Q \times Q \to Q$ satisfying the following three axioms.

(Q1) For any $a \in Q$, $a * a = a$.

(Q2) For any pair $a, b \in Q$, there exists a unique $c \in Q$ such that $c * a = b$.

(Q3) For any triple $a, b, c \in Q$, $(a * b) * c = (a * c) * (b * c)$.

Example 1.2. Let A be a finite abelian group, and $T \in \mathrm{Aut}(A)$. We endow A with a quandle structure $x * y = T(x) + (1 - T)(y)$ for $x, y \in A$. We denote this quandle $\mathrm{Aff}(A, T)$, which is called an affine quandle.

Let $(Q, *)$ and $(Q', *')$ be two quandles. A map $f : Q \to Q'$ is said to be a homomorphism if $f(a * b) = f(a) *' f(b)$ for any $a, b \in Q$. If a homomorphism is bijective as a map, then it is said to be an isomorphism. An isomorphism from a quandle Q to Q itself is said to be an automorphism of Q.

The map $r_c : Q \to Q; x \mapsto x * c$ is a bijection for any $c \in Q$ by axiom (Q2) and we have $r_c(a * b) = (a * b) * c = (a * c) * (b * c) = r_c(a) * r_c(b)$ for any pair $a, b \in Q$ by axiom (Q3), so r_c is an automorphism.

Let $\mathrm{Aut}(Q)$ be the group of all automorphisms of Q. The inner group of a quandle Q is the subgroup of $\mathrm{Aut}(Q)$ generated by the maps r_c for all $c \in Q$.

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We write Inn(Q) for the inner group of Q. A quandle Q is said to be connected if Inn(Q) acts transitively on Q.

Note that an affine quandle Aff(A, T) is connected if and only if $1 - T$ is in Aut(A).

Example 1.3. Let p be a prime, $t \in \mathbb{N}$, let $\mathbb{F}$ be a field of p^t elements and let X be a non-zero element of $\mathbb{F}$. Since the map $\mathbb{F} \to \mathbb{F}; x \mapsto Xx$ is an automorphism of the additive group $(\mathbb{F}, +)$, by Example 1.2, $\mathbb{F}$ is regarded as an affine quandle with $x * y = Xx + (1 - X)y$ for $x, y \in \mathbb{F}$. If further X is not the identity element of $\mathbb{F}$, then the map $\mathbb{F} \to \mathbb{F}; x \mapsto x - Xx$ is also an automorphism and this affine quandle is connected.

Let Q be a finite quandle of order n. We write its elements as $1, 2, \ldots, n$. Since the map r_c is a bijection, it can be regarded as a permutation on the set $\{1, 2, \ldots, n\}$.

In general, when a permutation σ on $\{1, 2, \ldots, n\}$ can be written as the product of disjoint cycles $(i_{1,1} \cdots i_{1,\ell_1})(i_{2,1} \cdots i_{2,\ell_2}) \cdots (i_{k,1} \cdots i_{k,\ell_k})$, we call the multiple set of the length of the cycles $\{\ell_1, \ell_2, \ldots, \ell_k\}$ the pattern of σ. In [4], P. Lopes and D. Roseman defined the profile of a quandle with n elements to be the sequence of the patterns of $r_1, r_2, \ldots, r_n$. In the case of a connected quandle Q of order n, it is easily seen that r_i and r_j are mutually conjugate for any pair i, j with $1 \leq i < j \leq n$. Therefore, r_i and r_j have the same pattern. In this paper, we call this common pattern the profile of Q for short.

L. Vendramin proved that the inner group of a quandle acts doubly transitively on itself if and only if a quandle is of cyclic type. In this paper, we study the doubly transitive property of the automorphism group of a connected quandle with the above profile.

In [3], C. Hayashi asked a question (ibid. Question 1.6), which is equivalent to the case of $\{1, \ell, \ell\}$ of the following.

Question 1.4. If a finite connected quandle Q has a profile of the form $\{1, \ell, \ell\}$ or $\{1, \ell, \ell, \ell\}$, where $\ell \geq 2$, then Aut(Q) acts doubly transitively on Q.

This question motivates our research. The following is the main theorem in this paper.

Theorem 1.5. *Let A be a finite abelian group, and $T \in$Aut(A). If an affine* *quandle* Aff(A, T) *has a profile of the form* $\{1, \ell, \ell\}$ *or* $\{1, \ell, \ell, \ell\}$*, where* $\ell \geq 2$*, then there exist a finite field* $\mathbb{F}$ *of the characteristic* p*, a generator* X *of* $\mathbb{F}$ *over* $\mathbb{F}_p$ *and an isomorphism* $(\mathbb{F}, +) \simeq A$ *via which the map* $\mathbb{F} \to \mathbb{F}; a \mapsto Xa$ *corresponds to* T *on* A*.*

Corollary 1.6. *For an affine quandle, Question* 1.4 *is solved affirmatively.*

Theorem 1.7. *Question* 1.4 *is solved affirmatively for a quandle whose order is less than or equal to* 35*.*

We prove Theorem 1.5, Corollary 1.6 and Theorem 1.7 in Section 3.

2. Simple quandles

In this section, we review the definition of a simple quandle, refer to its properties and explain some propositions.

Definition 2.1. A quandle Q is called simple if any surjective quandle homomorphism on Q has trivial image or is bijective.

The following results are consequences of Proposition 2 and Proposition 3 in [2].

Proposition 2.2. ([2]) *If* Q *is a connected quandle and* $f : Q \to P$ *is a surjective quandle homomorphism, then* P *is connected.*

Proposition 2.3. ([2]) *If* $f : Q \to P$ *is a surjective quandle homomorphism and* P *is connected, then there exists a quandle isomorphism* $g : f^{-1}(a) \to f^{-1}(b)$ *for* $a, b \in P$*.*

By using these propositions, we prove the following proposition.

Proposition 2.4. *A connected quandle with the profile* $\{1, \ell, \ell\}$ *or* $\{1, \ell, \ell, \ell\}$*, where* $\ell \geq 2$*, is simple.*

Proof. Let Q be a connected quandle with the profile $\{1, \ell, \ell\}$. To prove it by contradiction, we suppose that Q is not simple. Then, there exists a surjective homomorphism $f : Q \to P$ of quandles such that $|P| \neq 1, |Q|$. By Proposition 2.2, P is a connected quandle. We fix an element $x \in Q$ and let $y = f(x)$. For any $a \in Q$ satisfying $f(a) = y$, we have that $f(a * x) = f(a) * f(x) = y * y = y$. Therefore, x and $a * x$ belong to the same fiber $f^{-1}(y)$. Writing the map r_x as the form $r_x = (x_1 \cdots x_\ell)(x_{\ell+1} \cdots x_{2\ell})(x)$, we have that $x_1, \ldots, x_\ell$ belong to the same fiber and $x_{\ell+1}, \ldots, x_{2\ell}$ belong to the same fiber. Therefore, the order of the fiber $f^{-1}(y)$ must be $1, \ell, \ell+1, 2\ell$ or $2\ell+1$. On the other hand, by the assumption $|P| \neq 1, |Q|$ and Proposition 2.3, the order of any fiber is a nontrivial divisor of $|Q| = 2\ell+1$. However, $1, \ell, \ell+1, 2\ell$ and $2\ell+1$ are not a nontrivial divisor of $|Q|$, which is a contradiction. Therefore, we have that Q is simple.

In the case of the profile $\{1, \ell, \ell, \ell\}$, similarly as in the case of the profile $\{1, \ell, \ell\}$, we suppose a connected quandle with the profile $\{1, \ell, \ell, \ell\}$ is not simple. Then, the order of its any fiber must be $1, \ell, \ell+1, 2\ell, 2\ell+1, 3\ell$ or $3\ell+1$. Since these are not a nontrivial divisor of $|Q| = 3\ell+1$, we get a contradiction. Therefore, we have that a connected quandle with the profile $\{1, \ell, \ell, \ell\}$ is simple. □

The following theorem is a part of Theorem 3.9 in [1].

THEOREM 2.5. ([1]) *Let Q be a simple and connected quandle and let p be a prime. Then, the following are equivalent.*

(1) *Q has p^t elements, for some $t \in \mathbb{N}$.*

(2) *Q is an affine quandle $\mathrm{Aff}(\mathbb{F}_p^t, T)$, where $T \in \mathrm{GL}(t, \mathbb{F}_p)$ acts irreducibly.*

By using this theorem, we prove the following theorem.

THEOREM 2.6. *Let Q be a simple and connected quandle and let p be a prime. Then, the following are equivalent.*

(1) *Q has p^t elements, for some $t \in \mathbb{N}$.*

(2) *Q is affine quandle defined by $(\mathbb{F}, +)$ in Example 1.3.*

Proof. For a non-identity element $a \in \mathbb{F}_p^t$, we consider the map $\mathbb{F}_p[X] \to \mathbb{F}_p^t; f(X) \mapsto f(T) \cdot a$. This map is surjective, the kernel of this map is generated by $g(X)$ and $\mathbb{F}_p[X]/(g(X)) \simeq \mathbb{F}_p^t$ is irreducible now. Therefore, $g(X)$ is irreducible and $\mathbb{F}_p[X]/(g(X))$ is a field. We regard this field as $\mathbb{F}$ in Example 1.3 and there exists an isomorphism $(\mathbb{F}, +) \simeq (\mathbb{F}_p^t, +)$ via which the map $\mathbb{F} \to \mathbb{F}; x \mapsto Xx$ corresponds to T on $\mathbb{F}_p^t$. By Theorem 2.5, the proof is complete. □

3. Proof

In this section, we prove Theorem 1.5, Corollary 1.6 and Theorem 1.7. We give two proofs of Theorem 1.5. The first proof appeals to a result of Andruskiewitsch and Graña [1], which is based on the classification of simple finite groups. On the other hand, since we work only with an abelian group, we can give a more direct proof of Theorem 1.5, which uses only elementary divisor theory. This is the second proof.

3.1. First we prove that for an affine quandle $\mathrm{Aff}(A,T)$ with the profile $\{1,\ell,\ell\}$, an abelian group A is of the form $\mathbb{Z}/p\mathbb{Z}\times\cdots\times\mathbb{Z}/p\mathbb{Z}$, where p is an odd prime.

Let Q be an affine quandle $\mathrm{Aff}(A,T)$ with the profile $\{1,\ell,\ell\}$. Since Q has the profile $\{1,\ell,\ell\}$, the pattern of the map r_0 is $\{1,\ell,\ell\}$, where 0 is the identity element of A.

We have $r_0(x)=T(x)+(1-T)(0)=T(x)$ for each $x\in A$, that is, $r_0=T$, so r_0 is of the form $r_0=(x\ T(x)\cdots T^{\ell-1}(x))(y\ T(y)\cdots T^{\ell-1}(y))(0)$. Therefore, the cardinality of the set of the orders of the elements in $A\setminus\{0\}$ is at most two. Moreover, there are ℓ elements of each order when the cardinality of the set of the orders is exactly two. By the fundamental theorem of finite abelian groups, A satisfying these conditions is of the form $\mathbb{Z}/p\mathbb{Z}\times\cdots\times\mathbb{Z}/p\mathbb{Z}$, where p is an odd prime.

3.2. Now we prove Theorem 1.5 in the case of the profile $\{1,\ell,\ell\}$. By Proposition 2.4, a quandle with the profile $\{1,\ell,\ell\}$ is a simple quandle. By using the facts that the order of Q is a prime power by 3.1 and that Q is a simple quandle and Theorem 2.6, the proof of Theorem 1.5 in the case of the profile $\{1,\ell,\ell\}$ is complete.

In the next subsection, we give the direct proof in the case of $\{1,\ell,\ell\}$ in Theorem 1.5.

3.3. We prove Theorem 1.5 in the case of $\{1,\ell,\ell\}$ directly. We use the elementary divisor theory in this direct proof.

By 3.1, A is of the form $\mathbb{Z}/p\mathbb{Z}\times\cdots\times\mathbb{Z}/p\mathbb{Z}$, so regard A as an $\mathbb{F}_p[X]$-module with the action $\mathbb{F}_p[X]\curvearrowright A; f\cdot a:=f(T)\cdot a$. By the elementary divisor theory, A is isomorphic to the underlying additive group of the ring

$$R:=\mathbb{F}_p[X]/(f_1)\oplus\cdots\oplus\mathbb{F}_p[X]/(f_d),$$

where $f_1,\cdots,f_d\in\mathbb{F}_p[X]\setminus\mathbb{F}_p$. We write $R_i=\mathbb{F}_p[X]/(f_i)$ for each i with $1\leq i\leq d$. Since T is an automorphism, X must be an invertible element in R_i for each i and the order of $(X,\ldots,X)$ must be ℓ in $R^\times$.

We claim $d=1$. Assume that $d\geq 2$, and the automorphism on R corresponding to T on A can be described as $R\to R; a\mapsto(X,\ldots,X)a$. We consider the orbit decomposition of this automorphism. We have

$$\begin{aligned}R=\{(0,\ldots,0)\}&\coprod\{(1,\ldots,1),(X,\ldots,X),\ldots,(X^{\ell-1},\ldots,X^{\ell-1})\}\\&\coprod\{(1,0,\ldots,0),(X,0,\ldots,0),\ldots,(X^{\ell-1},0,\ldots,0)\}\\&\coprod\{(0,\ldots,0,1),(0,\ldots,0,X),\ldots,(0,\ldots,0,X^{\ell-1})\}\coprod\cdots.\end{aligned}$$

In particular, there are at least three distinct orbits of order ℓ, which contradicts the assumption that $\mathrm{Aff}(A,T)$ has the profile $\{1,\ell,\ell\}$.

Therefore, we can write $R=\mathbb{F}_p[X]/(f)$. Let the degree of f be m. We claim that f is irreducible. We may assume that $f=g^k$, where g is irreducible of degree c. Now, we have $|R^\times| = p^m - p^{m-c}$. Since the element X is of the order $\ell = \frac{1}{2}(p^m-1)$ in $R^\times$, we have $\frac{1}{2}(p^m-1) | p^m - p^{m-c} = p^{m-c}(p^c-1)$, hence $\frac{1}{2}(p^m-1)|p^c-1$. Since $\frac{1}{2}(p^m-1)=\frac{1}{2}(p^{kc}-1)=\frac{1}{2}(p^c-1)(p^{(k-1)c}+p^{(k-2)c}+\cdots+p^c+1)$, we have $\frac{1}{2}(p^{(k-1)c}+p^{(k-2)c}+\cdots+p^c+1)\leq 1$, that is, $p^{(k-1)c}+p^{(k-2)c}+\cdots+p^c+1\leq 2$. When $k\geq 2$, this contradicts the fact that p is a prime. Therefore, $k=1$, that is, f is irreducible and we get $A\simeq R=\mathbb{F}_p[X]/(f)$ is a field. The direct proof in the case of the profile $\{1,\ell,\ell\}$ is complete.

3.4. In the case of the profile $\{1,\ell,\ell,\ell\}$, we can get a proof similarly as in 3.2, by using the fact that a quandle with the profile $\{1,\ell,\ell,\ell\}$ is also a simple quandle by Proposition 2.4 and Theorem 2.6. Now, we give the direct proof similarly as in 3.3.

Let Q be an affine quandle $\mathrm{Aff}(A,T)$ with the profile $\{1,\ell,\ell,\ell\}$. The map r_0 is of the form $r_0 = (x\ T(x)\cdots T^{\ell-1}(x))(y\ T(y)\cdots T^{\ell-1}(y))(z\ T(z)\cdots T^{\ell-1}(z))(0)$. Therefore, the cardinality of the set of the orders of the elements in $A\setminus\{0\}$ is at most three. Moreover, there are ℓ elements of each order when the cardinality of the set of the orders is exactly three, and there are ℓ and 2ℓ elements of each order when it is exactly two. First, we claim that A is a p-group.

To prove it by contradiction, suppose that there exist different primes p and q which are divisors of $|A|$. Then, A has elements of the order p and the order q. The number of the elements of the order p is p^m-1 and that of the order q is q^n-1, where $m,n\in\mathbb{N}$, and these numbers are different. By the assumption, we may assume that the number of the elements of the order p is ℓ and the number of the elements of the order q is 2ℓ, so any element of $A\setminus\{0\}$ is either of the order p or of the order q. However, in fact, there exists an element of the order pq, which is a contradiction. Therefore, A is a p-group and the claim follows.

Next, we claim that A satisfying the above conditions is of the form $\mathbb{Z}/p\mathbb{Z}\times\cdots\times\mathbb{Z}/p\mathbb{Z}$, where p is a prime. By the assumption that the cardinality of the set of the orders of the elements in $A\setminus\{0\}$ is at most three, the fact that A is a p-group and using the fundamental theorem of finite abelian group, we write

$$A=\underbrace{\mathbb{Z}/p\mathbb{Z}\times\cdots\times\mathbb{Z}/p\mathbb{Z}}_{a_1\ \text{times}}\times\underbrace{\mathbb{Z}/p^2\mathbb{Z}\times\cdots\times\mathbb{Z}/p^2\mathbb{Z}}_{a_2\ \text{times}}\times\underbrace{\mathbb{Z}/p^3\mathbb{Z}\times\cdots\times\mathbb{Z}/p^3\mathbb{Z}}_{a_3\ \text{times}}.$$

It is sufficient that we prove $a_2=a_3=0$. Now, $|A|=3\ell+1=p^{a_1+2a_2+3a_3}$, so $\ell=\frac{1}{3}(p^{a_1+2a_2+3a_3}-1)$. It is easily seen that the number of the elements of the order p is $p^{a_1+a_2+a_3}-1$. Assume that $a_3 \geq 1$ or $a_2 \geq 2$. Then, we have that $p^{a_1+a_2+a_3}-1 < \ell = \frac{1}{3}(p^{a_1+2a_2+3a_3}-1)$, which contradicts the fact that there exist ℓ elements whose orders are p. Therefore, $a_3 = 0$ and $a_2 \leq 1$. Assume that $a_2 = 1$. Then, it is easily seen that $p^{a_1+1}-1 < 2\ell = \frac{2}{3}(p^{a_1+2}-1)$. If $p^{a_1+1}-1 = \ell = \frac{1}{3}(p^{a_1+2}-1)$, then $p = 2$ and $a_1 = 0$, so we have $|A| = 4$ and $\ell = 1$ which contradicts the assumption $\ell \geq 2$. Therefore, when $a_2 = 1$, the number of the elements of the order p is not ℓ, 2ℓ and 3ℓ. Thus, $a_2 = 0$. The claim follows.

Therefore, A is of the form $\mathbb{Z}/p\mathbb{Z} \times \cdots \times \mathbb{Z}/p\mathbb{Z}$. Furthermore, A is isomorphic to the underlying additive group R in 3.3. Assume that $d \geq 2$. We consider the orbit decomposition of the map $R \to R; a \mapsto (X, \ldots, X)a$. We have

$$\begin{aligned} R = \{(0,\ldots,0)\} &\coprod \{(1,\ldots,1),(X,\ldots,X),\ldots,(X^{\ell-1},\ldots,X^{\ell-1})\} \\ &\coprod \{(1,0,\ldots,0),(X,0,\ldots,0),\ldots,(X^{\ell-1},0,\ldots,0)\} \\ &\coprod \{(0,\ldots,0,1),(0,\ldots,0,X),\ldots,(0,\ldots,0,X^{\ell-1})\} \coprod \cdots . \end{aligned}$$

The order of X must be ℓ in $R_1^\times$ and in $R_d^\times$, since $\mathrm{Aff}(A,T)$ has the profile $\{1,\ell,\ell,\ell\}$. Then, there exists another orbit $\{(1,0,\ldots,0,X),(X,0,\ldots,0,X^2),\ldots,(X^{\ell-1},0,\ldots,0,1)\}$, which contradicts the fact that the number of the orbits of the length ℓ is three. Therefore, we can write $R = \mathbb{F}_p[X]/(f)$. Let the degree of f be m. We may assume that $f = g^k$, where g is irreducible of degree c. Similarly as in the case of $\{1,\ell,\ell\}$ in 3.3, since the element X is of the order $\ell = \frac{1}{3}(p^m-1)$ in $R^\times$, we have $p^{(k-1)c}+p^{(k-2)c}+\cdots+p^c+1 \leq 3$. When $k \geq 3$, this contradicts the fact that p is a prime. When $k = 2$, only $p = 2$ and $c = 1$ satisfy this condition. Then, we have $m = 2$, $|R| = 4$ and $\ell = 1$ which contradicts the assumption $\ell \geq 2$. Therefore, $k = 1$, that is, f is irreducible and we get $A \simeq R = \mathbb{F}_p[X]/(f)$ is a field.

3.5. To prove Corollary 1.6, we prove the following proposition, the statement of which is obtained by replacing $\mathrm{Inn}(Q)$ by $\mathrm{Aut}(Q)$ of Proposition 3.3 in [6].

Proposition 3.1. *Let Q be a quandle and assume that $|Q| \geq 3$. Then the following conditions are mutually equivalent. Here, $\mathrm{Aut}(Q)_q$ is the stabilizer of q in $\mathrm{Aut}(Q)$.*

(1) $\mathrm{Aut}(Q)$ *acts doubly transitively on Q.*
(2) *For every $q \in Q$, the action of $\mathrm{Aut}(Q)_q$ on $Q \setminus \{q\}$ is transitive.*
(3) *Q is connected, and there exists $q \in Q$ such that the action of $\mathrm{Aut}(Q)_q$ on $Q \setminus \{q\}$ is transitive.*

The proof of Proposition 3.1 is the same as Proposition 3.3 in [6]. Now, we prove Corollary 1.6.

Let Q be an affine quandle $\mathrm{Aff}(A,T)$ with the profile $\{1,\ell,\ell\}$ or $\{1,\ell,\ell,\ell\}$. We prove that $\mathrm{Aut}(Q)$ acts doubly transitively on Q. By 2.3 and 2.4, we have $Q \simeq \mathrm{Aff}(R,\varphi)$ and r_0 is of the form $r_0 = (1\ X \cdots X^{\ell-1})(a\ Xa \cdots X^{\ell-1}a)(0)$ or $r_0 = (1\ X \cdots X^{\ell-1})(a\ Xa \cdots X^{\ell-1}a)(b\ Xb \cdots X^{\ell-1}b)(0)$, respectively, where $R = \mathbb{F}_p[X]/(f)$, $f \in \mathbb{F}_p[X] \setminus \mathbb{F}_p$ is irreducible and $\varphi : R \to R; \alpha \mapsto X\alpha$. Let ψ_a be the map $R \to R; \alpha \mapsto a\alpha$. We have that ψ_a is a quandle automorphism of $\mathrm{Aff}(R,\varphi)$. In fact, for any $x, y \in R$, $\psi_a(x * y) = \psi_a(Xx + (1-X)y) = a(Xx + (1-X)y) = X(ax) + (1-X)(ay) = (ax) * (ay) = \psi_a(x) * \psi_a(y)$.

In the case of the profile $\{1,\ell,\ell,\ell\}$, let ψ_b be the map $R \to R; \alpha \mapsto b\alpha$. We have that ψ_b is a quandle automorphism of $\mathrm{Aff}(R,\varphi)$ similarly as above.

Since r_0 and ψ_a (and ψ_b in the case of $\{1,\ell,\ell,\ell\}$) are in $\mathrm{Aut}(Q)_0$, we have $\mathrm{Aut}(Q)_0$ acts transitively on $Q \setminus \{0\}$ in both cases. Here, $\mathrm{Aut}(Q)_0$ is the stabilizer of the identity element 0 of A in $\mathrm{Aut}(Q)$. By Proposition 3.1 (3) $\Rightarrow$ (1), $\mathrm{Aut}(Q)$ acts doubly transitively on Q. The proof is complete.

3.6. We prove Theorem 1.7.

We have the complete list of non-isomorphic connected quandles whose order is less than or equal to 35 by Rig which is available at `http://code.google.com/p/rig`. By the list [5], we can list up all non-affine quandles of the order $2\ell+1$ or $3\ell+1$. In Table 1 and Table 2, we enumerate non-affine quandles $Q_{n,m}$ of the order $2\ell+1$ or $3\ell+1$, where $Q_{n,m}$ is the m-th quandle of the order n in their list. Also, by Rig, we have its profiles. According to these tables, the profile of a non-affine quandle of the order $2\ell+1$ or $3\ell+1$ is not $\{1,\ell,\ell\}$ or $\{1,\ell,\ell,\ell\}$. Therefore, we see that a connected quandle with the profile $\{1,\ell,\ell\}$ or $\{1,\ell,\ell,\ell\}$ whose order is less than or equal to 35 is affine. The proof is complete by Corollary 1.6.

Table 1 Connected and non-affine quandles of the order $2\ell+1$

Order	Connected quandle	Non-affine quandle	$Q_{n,m}$	Profile
15	7	4	$Q_{15,2}$	$\{1,1,1,2,2,2,2,2,2\}$
			$Q_{15,5}$	$\{1,2,2,10\}$
			$Q_{15,6}$	$\{1,2,2,2,2,2,2,2\}$
			$Q_{15,7}$	$\{1,1,1,1,1,1,1,2,2,2,2\}$
21	9	4	$Q_{21,6}$	$\{1,1,1,1,1,2,2,2,2,2,2,2,2\}$
			$Q_{21,7}$	$\{1,2,2,2,14\}$
			$Q_{21,8}$	$\{1,2,2,2,2,2,2,2,2,2,2\}$
			$Q_{21,9}$	$\{1,1,1,1,1,1,1,1,1,1,1,2,2,2,2,2\}$
27	65	35	$Q_{27,1}$	$\{1,1,1,2,2,2,2,2,2,2,2,2,2,2,2\}$
			$Q_{27,2}$	$\{1,1,1,4,4,4,4,4,4\}$
			$Q_{27,7}$ $Q_{27,9}$ $Q_{27,11}$ $Q_{27,12}$ $Q_{27,16}$ $Q_{27,35}$ $Q_{27,36}$ $Q_{27,41}$ $\vdots$ $Q_{27,46}$ $Q_{27,56}$ $\vdots$ $Q_{27,59}$	$\{1,2,2,2,2,6,6,6\}$
			$Q_{27,8}$ $Q_{27,10}$ $Q_{27,13}$ $Q_{27,15}$	$\{1,2,2,2,2,2,2,2,2,2,2,2,2,2\}$
			$Q_{27,14}$	$\{1,1,1,2,2,2,6,6,6\}$
			$Q_{27,27}$ $Q_{27,28}$	$\{1,2,8,8,8\}$
			$Q_{27,37}$ $\vdots$ $Q_{27,40}$ $Q_{27,60}$ $Q_{27,61}$	$\{1,2,6,18\}$
			$Q_{27,53}$ $Q_{27,54}$ $Q_{27,55}$	$\{1,2,2,2,2,18\}$
33	11	2	$Q_{33,10}$	$\{1,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2\}$
			$Q_{33,11}$	$\{1,2,2,2,2,2,22\}$

Table 2 Connected and non-affine quandles of the order $3\ell + 1$

Order	Connected quandle	Non-affine quandle	$Q_{n,m}$	Profile
10	1	1	$Q_{10,1}$	$\{1,1,1,1,2,2,2\}$
28	13	8	$Q_{28,3}$	$\{1,3,3,3,3,3,3,3,3,3\}$
			$Q_{28,4}$	
			$Q_{28,11}$	
			$Q_{28,12}$	
			$Q_{28,5}$	$\{1,3,6,6,6,6\}$
			$Q_{28,6}$	
			$Q_{28,10}$	$\{1,1,1,1,2,2,2,2,2,2,2,2,2,2,2,2\}$
			$Q_{28,13}$	$\{1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,2,2,2,2,2,2\}$

4. Conjecture

Conjecture 4.1. A connected quandle with the profile $\{1,\ell,\ell\}$ or $\{1,\ell,\ell,\ell\}$ is affine.

Remark 4.2.

(1) If we prove that Conjecture 4.1 is true, then Question 1.4 is solved completely by Corollary 1.6.

(2) L. Vendramin proved a connected quandle with profile $\{1,\ell\}$ is affine in [7, Corollary 2]. It would be interesting to consider if the method there could be generalized for our problem.

Proposition 4.3. *For a quandle whose order is less than or equal to* 35 *or a prime power, Conjecture* 4.1 *is true.*

Proof. A connected quandle with the profile $\{1,\ell,\ell\}$ or $\{1,\ell,\ell,\ell\}$ whose order is less than or equal to 35 is affine by Rig and [5] as in 3.6. A quandle with the profile $\{1,\ell,\ell\}$ or $\{1,\ell,\ell,\ell\}$ is a simple quandle by Proposition 2.4 and a simple quandle whose order is a prime power is affine by Theorem 2.5 or Theorem 2.6. $\square$

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