Yokohama Mathematical
Journal Vol. 55, 2010

$\mathcal{A}$-SIMPLE MULTIGERMS AND $\mathcal{L}$-SIMPLE MULTIGERMS

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(Received June 2, 2008; Revised September 28, 2009)

Abstract. In this paper, we investigate restrictions on multiplicities and numbers of branches for $\mathcal{G}$-simple multigerms ($\mathcal{G} = \mathcal{A}$ or $\mathcal{L}$).

Introduction

Throughout this paper, let $S = \{s_1, \cdots, s_r\}$ be a finite subset of $\mathbf{R}^n$ with r elements, $f : (\mathbf{R}^n, S) \to (\mathbf{R}^p, 0)$ be a germ of C^∞ mapping at S such that $f(S) = 0$ (called a *multigerm*) and for any i $(1 \leq i \leq r)$ let f_i be the restriction of f to $(\mathbf{R}^n, s_i)$ (called a *branch* of f). The integer r is called the *number of branches of* f. Let C_S (resp. C_0) be the set of C^∞ function-germs $(\mathbf{R}^n, S) \to \mathbf{R}$ (resp. $(\mathbf{R}^p, 0) \to \mathbf{R}$). Let m_S (resp. m_0) be the subset of C_S (resp. C_0) consisting of C^∞ function-germs $(\mathbf{R}^n, S) \to (\mathbf{R}, 0)$ (resp. $(\mathbf{R}^p, 0) \to (\mathbf{R}, 0)$). The sets C_S and C_0 have natural $\mathbf{R}$-algebra structures induced by the $\mathbf{R}$-algebra structure of $\mathbf{R}$. For a multigerm $f : (\mathbf{R}^n, S) \to (\mathbf{R}^p, 0)$, let $f^* : C_0 \to C_S$ be the $\mathbf{R}$-algebra homomorphism defined by $f^*(u) = u \circ f$. Put $Q(f) = C_S/f^*(m_0)C_S$. The dimension of $Q(f)$ as a real vector space is called the *multiplicity* of f, and in the case that $n \leq p$ it is finite in general.

Two multigerms $f,\ g : (\mathbf{R}^n, S) \to (\mathbf{R}^p, 0)$ are said to be $\mathcal{A}$-*equivalent* if there exist germs of C^∞ diffeomorphisms $\varphi : (\mathbf{R}^n, S) \to (\mathbf{R}^n, S)$ with the condition that $\varphi(s_i) = s_i$ for $(1 \leq i \leq r)$ and $\psi : (\mathbf{R}^p, 0) \to (\mathbf{R}^p, 0)$ such that $f = \psi \circ g \circ \varphi^{-1}$. $\mathcal{L}$-*equivalence* (resp. $\mathcal{R}$-equivalence) for f and g is defined in the same way as $\mathcal{A}$-equivalence but such that φ (resp. ψ) is the germ of identity mapping. A multigerm $f : (\mathbf{R}^n, S) \to (\mathbf{R}^p, 0)$ is said to be $\mathcal{A}$-*simple* (resp. $\mathcal{L}$-*simple*) if there exists a finite number of $\mathcal{A}$-equivalence classes (resp. $\mathcal{L}$-equivalence classes) such that for any positive integer d and any C^∞ mapping $F : U \to V$ where $U \subset \mathbf{R}^n \times \mathbf{R}^d$ is a neighbourhood of $S \times 0$, $V \subset \mathbf{R}^p \times \mathbf{R}^d$ is a neighbourhood of $(0, 0)$, $F(x, \lambda) = (f_\lambda(x), \lambda)$ and the germ of f_0 at S is f, there

2000 Mathematics Subject Classification: 58K05, 58K20, 57R45
Key words and phrases: $\mathcal{A}$-simple multigerm – $\mathcal{L}$-simple multigerm– multiplicity – number of branches.

exists a sufficiently small neighbourhood $W_i \subset U$ of $(s_i, 0)$ $(1 \le i \le r)$ such that for every $\{(x_1, \lambda), \cdots, (x_r, \lambda)\}$ with $(x_i, \lambda) \in W_i$ and $F(x_1, \lambda) = \cdots = F(x_r, \lambda)$ the multigerm $f_\lambda : (\mathbf{R}^n, \{x_1, \cdots, x_r\}) \to (\mathbf{R}^p, f_\lambda(x_i))$ lies in one of these finite $\mathcal{A}$-equivalence classes (resp. $\mathcal{L}$-equivalence classes).

Theorem 1. *Let $f : (\mathbf{R}^n, S) \to (\mathbf{R}^p, 0)$ $(n \le p)$ be a multigerm with corank at most one.*

1. *Suppose that $np \ne 1$ and f is $\mathcal{A}$-simple. Then, the following inequality holds.*

$$\dim_{\mathbf{R}} Q(f) \le \frac{p^2 + (n-1)r}{n(p-n) + (n-1)}.$$

2. *Suppose that f is $\mathcal{L}$-simple. Then, the following inequality holds.*

$$\dim_{\mathbf{R}} Q(f) \le \frac{p}{n}.$$

Here, *corank at most one* for an $\mathcal{A}$-simple multigerm $f : (\mathbf{R}^n, S) \to (\mathbf{R}^p, 0)$ means that $\max\{n - \mathrm{rank} Jf_i(s_i) \mid 1 \le i \le r\} \le 1$ holds, where $Jf_i(s_i)$ is the Jacobian matrix of the restriction f_i of f at s_i. Note that there are no upper bounds for $\dim_{\mathbf{R}} Q(f)$ of an $\mathcal{A}$-simple f in the case that $n = p = 1$ since for any positive integer δ the map-germ $f(x) = x^\delta$ is $\mathcal{A}$-simple and of corank at most one.

The author does not know whether or not Theorem 1 still holds without the assumption of corank at most one.

Theorem 2. *Let $f : (\mathbf{R}^n, S) \to (\mathbf{R}^p, 0)$ $(n \le p)$ be a multigerm.*

1. *Suppose that $n \ne p$ and f is $\mathcal{A}$-simple. Then, the number of branches r is restricted in the following way.*

$$r < \frac{p^2}{n(p-n)}.$$

2. *Suppose that f is $\mathcal{L}$-simple. Then, the number of branches r is restricted in the following way.*

$$r \le \frac{p}{n}$$

.

Note that there are no upper bounds for the number of branches of an $\mathcal{A}$-simple f in the case that $n = p$ since for any positive integer r a smooth finite covering with r fibers gives an example of $\mathcal{A}$-simple multigerm in this case. Note also that since $r \le \dim_{\mathbf{R}} Q(f)$ the inequality $r \le \frac{p^2}{n(p-n)}$ for an $\mathcal{A}$-simple multigerm with corank at most one can be obtained from 1 of Theorem 1 as an immediate corollary. Thus, the point of 1 of Theorem 2 is the sharpness of the inequality.

Since the left hand side of the inequality in 1 of Theorem 2 is an integer while the right hand side is a rational number, the sharp inequality in 1 of Theorem 2 suggests that there must exist some special restrictions for the number of branches of an $\mathcal{A}$-simple multigerm when the right hand side is an integer. The rational number $\frac{p^2}{n(p-n)}$ can be an integer only when $p = 2n$ and in this case it attains its minimal value 4. Thus, we may guess that the classical cross ratio and the symplectic cross ratio ([14]) are the very invariants of special restrictions for the number of branches of an $\mathcal{A}$-simple multigerm, and it is impossible to find out such invariants in the case that $p \neq 2n, p > n$.

It seems interesting also to compare 1 of Theorem 1 and 1 of Theorem 2 when the right hand side of the inequality in 1 of Theorem 2 is an integer. The rational number $\frac{p^2+(n-1)r}{n(p-n)+(n-1)}$ for $p = 2n,\ r < 4$ can be an integer only when $n = 1$ and the maximal value it attains is 4. Although there are no $\mathcal{A}$-simple multigerms $f : (\mathbf{R}^n, S) \to (\mathbf{R}^{2n}, 0)$ with $r = 4$ by 1 of Theorem 2, for instance map-germs $x \mapsto (x^4, x^5+x^7)$ (taken from [3]), $\{x \mapsto (x, 0), x \mapsto (x^3, x^4)\}$ and $\{x \mapsto (x, 0), x \mapsto (0, x), x \mapsto (x^2, x^3)\}$ (these two are taken from [9]) give examples of $\mathcal{A}$-simple multigerms satisfying $\dim_{\mathbf{R}} Q(f) = 4$ in the case that $(n, p) = (1, 2)$. Thus, we can not expect the sharpness for the inequality of 1 of Theorem 1.

Not only in the case above, the upper bound for $\dim_{\mathbf{R}} Q(f)$ given in 1 of Theorem 1 is the best possible bound in the classification results of $\mathcal{A}$-simple map-germs listed here ([4], [5], [6], [7], [8], [9], [10], [13], [15], [17]), and the upper bound for r is also the best possible bound in the classification results ([5], [6], [9], [17]). However, if $n = r = 1$ and p is greater than 5, then the upper bound in Theorem 1 is not the best estimate since the effect of $\mathcal{A}$-moduli sets in $\mathcal{K}$-simple orbits can not be disregarded as shown in [1].

For $\mathcal{L}$-simple multigerms, by 2 of Theorem 1 we see that if $n \leq p < 2n$ then any $\mathcal{L}$-simple map-germ with corank at most one must be an immersive monogerm (i.e. an immersion germ with only one branches), and we can not expect to improve 2 of Theorem 1 and 2 of Theorem 2 to hold the sharp inequality $r < \frac{p}{n}$.

Next, we remark briefly that there exist $\mathcal{A}$-simple multigerms which are not $\mathcal{L}$-simple even when $p \geq 2n$.

Proposition 1. *Let $p > 1$, $r \geq p$ and let $f : (\mathbf{R}, S) \to (\mathbf{R}^p, 0)$ be an immersion such that $\sum_{i=1}^{p} j^1 f_{j_i}(s_{j_i})(\mathbf{R}) = \mathbf{R}^p$, where $j_i \in \{1, \cdots, r\}$ ($j_i \neq j_k$ if $i \neq k$) and the 1-jet $j^1 f_i(s_i)$ is being regarded as a linear mapping. Then, we have the following:*

1. *Suppose that $r = p$. Then, f is $\mathcal{L}$-simple.*
2. *Suppose that $r = p + 1$. Then, f is $\mathcal{A}$-simple.*
3. *Suppose that $r \geq p + 2$. Then, f is not $\mathcal{A}$-simple.*

Proposition 2. *Let $f : (\mathbf{R}^n, S) \to (\mathbf{R}^{2n}, 0)$ be an immersion such that f_i is transversally intersecting with f_j for any i, j $(1 \leq i, j \leq r,\ i \neq j)$.*

1. *Suppose that $r = 2$. Then, f is $\mathcal{L}$-simple.*
2. *Suppose that $r = 3$. Then, f is $\mathcal{A}$-simple.*
3. *Suppose that $r \geq 4$. Then, f is not $\mathcal{A}$-simple.*

Note that under the situation of Proposition 1 (resp. Proposition 2), f is not $\mathcal{L}$-simple if $r = p + 1$ (resp. $r = 3$) by 2 of Theorem 1, and thus an f given in 2 of Proposition 1 (resp. 2 of Proposition 2) is an $\mathcal{A}$-simple map-germ which is not $\mathcal{L}$-simple.

All results in this paper hold also in complex holomorphic category.

In §1, several preparations are given. Theorems 1 and 2 and propositions 1 and 2 are proved in §2, §3, §4 and §5 respectively.

1. Preliminaries

Most notions and notations defined in this section are due to Mather ([11], [12]) and already common in singularity theory of C^∞ mappings. For details of them, we recommend an excellent survey [16] to the readers. Although in [16] r is always 1, it is very useful to understand the geometric meaning of the notions intruduced in this section.

Two multigerms $f,\ g : (\mathbf{R}^n, S) \to (\mathbf{R}^p, 0)$ are said to be $\mathcal{K}$-*equivalent* if there exist a germ of C^∞ diffeomorphism $\varphi : (\mathbf{R}^n, S) \to (\mathbf{R}^n, S)$ such that $\varphi(s_i) = s_i$ for $(1 \leq i \leq r)$ and a germ of C^∞ mappings $M : (\mathbf{R}^n, S) \to GL(p, \mathbf{R})$ such that $f(x) = M(x) g \circ \varphi^{-1}(x)$. Note that the multiplicities are $\mathcal{K}$-invariant for multigerms. The $\mathcal{C}$-*equivalence* for f and g is defined as the $\mathcal{K}$-equivalence of them such that φ is the germ of identity mapping.

For a multigerm $f : (\mathbf{R}^n, S) \to (\mathbf{R}^p, 0)$, let $\theta_S(f)$ be the C_S-module consisting of germs of C^∞ vector fields along f. We may identify $\theta_S(f)$ with $\underbrace{C_S \times \cdots \times C_S}_{p \text{ tuples}}$.
We put $\theta_S(n) = \theta_S(id._{(\mathbf{R}^n,S)})$ and $\theta_0(p) = \theta(id._{(\mathbf{R}^p,0)})$, where $id._{(\mathbf{R}^n,S)}$ (resp. $id._{(\mathbf{R}^p,0)}$) is the germ of the identity mapping of $(\mathbf{R}^n, S)$ (resp. $(\mathbf{R}^p, 0)$). For a $k \in \{0, 1, \cdots, \infty\}$, an element of $m_S^k \theta_S(n)$ or $m_0^k \theta_0(p)$ is a germ of C^∞ vector field along the germ of the identity mapping such that the terms of the Taylor series of it up to $(k - 1)$ are zero.

For a given multigerm $f : (\mathbf{R}^n, S) \to (\mathbf{R}^p, 0)$, the $\mathcal{G}$-equivalence class of f is denoted by $\mathcal{G}(f)$, where $\mathcal{G}$ is one of $\mathcal{L}, \mathcal{A}, \mathcal{C}$ and $\mathcal{K}$. For the f, we define tf and ωf in the following way:

$$tf : \theta_S(n) \to \theta_S(f), \quad tf(a) = df \circ a,$$
$$\omega f : \theta_0(p) \to \theta_S(f), \quad \omega f(b) = b \circ f,$$

where df is the differential of f. For the f, we put

$$T\mathcal{R}(f) = tf(m_S \theta_S(n)),$$
$$T\mathcal{L}(f) = \omega f(m_0 \theta_0(p)),$$
$$T\mathcal{A}(f) = tf(m_S \theta_S(n)) + \omega f(m_0 \theta_0(p)),$$
$$T\mathcal{C}(f) = f^*(m_0)\theta_S(f),$$
$$T\mathcal{K}(f) = tf(m_S \theta_S(n)) + f^*(m_0)\theta_S(f).$$

For a given multigerm f, we may identify $Q(f)^n$ as $\theta_S(n)/f^*(m_0)\theta_S(n)$ and $Q(f)^p$ as $\theta_S(f)/f^*(m_0)\theta_S(f)$. Under this identification, Wall's homomorphism of $Q(f)$-modules (p. 508 of [16]) is the following:

$$\bar{t}f : Q(f)^n \to Q(f)^p, \quad \bar{t}f([a]) = [tf(a)],$$

where $[a] = a + f^*(m_0)\theta_S(n)$ and $[tf(a)] = tf(a) + f^*(m_0)\theta_S(f)$. Let $\delta(f)$ (resp. $\gamma(f)$) be the dimension of $Q(f)$ (resp. the dimension of the kernel of $\bar{t}f$).

For a given multigerm $f : (\mathbf{R}^n, S) \to (\mathbf{R}^p, 0)$ and a positive integer i, we put ${}_iQ(f) = f^*(m_0^i)C_S/f^*(m_0^{i+1})C_S$ and ${}_i\delta(f) = \dim_{\mathbf{R}} {}_iQ(f)$. We may identify ${}_iQ(f)^n$ as $f^*(m_0^i)\theta_S(n)/f^*(m_0^{i+1})\theta_S(n)$ and ${}_iQ(f)^p$ as $f^*(m_0^i)\theta_S(f)/f^*(m_0^{i+1})\theta_S(f)$. Under this identification, we let ${}_i\gamma(f)$ be the dimension of the kernel of the following homomorphism of $Q(f)$-modules.

$${}_i\bar{t}f : {}_iQ(f)^n \to {}_iQ(f)^p, \quad {}_i\bar{t}f([a]) = [tf(a)].$$

Then, we see easily that $\delta(f) \le {}_i\delta(f) \le p^i\, \delta(f)$, and thus ${}_i\delta(f) < \infty$ if $\delta(f) < \infty$. Similarly $\gamma(f) \le {}_i\gamma(f) \le p^i\, \gamma(f)$ and thus ${}_i\gamma(f) < \infty$ if $\gamma(f) < \infty$. Note that ${}_iQ(f)$ is not isomorphic to ${}_iQ(F)$, where F is an unfolding of f. However, in the case that $n = 1$ we see easily that

$${}_1\delta(F) = (1+q)\, {}_1\delta(f) \text{ and } {}_1\gamma(F) = (1+q)\, {}_1\gamma(f),$$

where q is the number of parameters for the unfolding F.

The Taylor series ignoring terms of degree higher than k at points of S for a multigerm $f : (\mathbf{R}^n, S) \to (\mathbf{R}^p, 0)$ is called *k jet of f at S* and is denoted by $j^k f(S)$. We put

$$J^k(n,p) = \{j^k f(0) \mid f : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)\ C^\infty\}.$$

The jet space suitable for multigerms $(\mathbf{R}^n, S) \to (\mathbf{R}^p, 0)$ is the following multijet space:

$$ {}_rJ^k(n,p) = \{(j^k f_1(s_1), \cdots, j^k f_r(s_r)) \mid f_1(s_1) = \cdots = f_r(s_r)\}. $$

For a multigerm $f : (\mathbf{R}^n, S) \to (\mathbf{R}^p, 0)$, the multijet space ${}_rJ^k(n,p)$ may be identified with the quotient space $m_S\theta_S(f)/m_S^{k+1}\theta_S(f)$. Under this identification we put

$$
\begin{aligned}
T\mathcal{R}^k(j^k f(S)) &= \{[g] \in {}_rJ^k(n,p) \mid g \in T\mathcal{R}(f)\}, \\
T\mathcal{L}^k(j^k f(S)) &= \{[g] \in {}_rJ^k(n,p) \mid g \in T\mathcal{L}(f)\}, \\
T\mathcal{A}^k(j^k f(S)) &= \{[g] \in {}_rJ^k(n,p) \mid g \in T\mathcal{A}(f)\}, \\
T\mathcal{C}^k(j^k f(S)) &= \{[g] \in {}_rJ^k(n,p) \mid g \in T\mathcal{C}(f)\}, \\
T\mathcal{K}^k(j^k f(S)) &= \{[g] \in {}_rJ^k(n,p) \mid g \in T\mathcal{K}(f)\},
\end{aligned}
$$

where $[g] = g + m_S^{k+1}\theta_S(f)$. These are tangent spaces to orbits of actions of well-defined Lie groups corresponding to Mather's groups $\mathcal{R}$, $\mathcal{L}$, $\mathcal{A}$, $\mathcal{C}$ and $\mathcal{K}$. (for details, see [16]).

2. Proof of Theorem 1

Lemma 2.1. *Let $f : (\mathbf{R}^n, S) \to (\mathbf{R}^p, 0)$ be a multigerm.*

1. *Suppose that f is $\mathcal{A}$-simple. Then, there exists a multigerm $g : (\mathbf{R}^n, S) \to (\mathbf{R}^p, 0)$ such that two properties $\mathcal{K}(f) = \mathcal{K}(g)$ and $T\mathcal{K}(g) = T\mathcal{A}(g)$ are satisfied.*
2. *Suppose that f is $\mathcal{L}$-simple. Then, there exists a multigerm $g : (\mathbf{R}^n, S) \to (\mathbf{R}^p, 0)$ such that two properties $\mathcal{C}(f) = \mathcal{C}(g)$ and $T\mathcal{C}(g) = T\mathcal{L}(g)$ are satisfied.*

Proof. First we show 1 of Lemma 2.1. If f satisfies the property that $T\mathcal{K}(f) = T\mathcal{A}(f)$, then just take the f as g. If $T\mathcal{K}(f) \neq T\mathcal{A}(f)$, then since f is $\mathcal{A}$-simple there must exist a multigerm $g \in \mathcal{K}(f)$ such that $\mathcal{A}(f)$ is adjacent to $\mathcal{A}(g)$ and the property $T\mathcal{K}(g) = T\mathcal{A}(g)$ holds.

For the proof of 2 of Lemma 2.1, just replace $\mathcal{A}$ and $\mathcal{K}$ in the proof of 1 of Lemma 2.1 with $\mathcal{L}$ and $\mathcal{C}$. □

Lemma 2.2. *Let $g : (\mathbf{R}^n, S) \to (\mathbf{R}^p, 0)$ be a multigerm such that $\delta(g) < \infty$.*

1. *Suppose that $T\mathcal{K}(g) = T\mathcal{A}(g)$. Then, the following inequality holds:*

$$ (p-n)\,{}_1\delta(g) + {}_1\gamma(g) - \gamma(g) \leq p^2. $$

2. Suppose that $T\mathcal{C}(g) = T\mathcal{L}(g)$. Then, the following inequality holds:

$$_1\delta(g) \leq p.$$

Proof. First we show 1 of Lemma 2.2. Put

$$A = \frac{g^*(m_0)\theta_S(g)}{tg(m_S\theta_S(n)) \cap g^*(m_0)\theta_S(g)}.$$

The assumption $T\mathcal{K}(g) = T\mathcal{A}(g)$ implies that any element φ of $g^*(m_0)\theta_S(g)$ has the form $\varphi = \varphi_1 + \varphi_2$ $(\varphi_1 \in tg(m_S\theta_S(n)), \varphi_2 \in \omega g(m_0\theta_0(p)))$. Then, note that $\varphi_1 = \varphi - \varphi_2$ belongs to $tg(m_S\theta_S(n)) \cap g^*(m_0)\theta_S(g)$ since the vector space $\omega g(m_0\theta_0(p))$ is contained in the vector space $g^*(m_0)\theta_S(g)$. Thus, under the assumption $T\mathcal{K}(g) = T\mathcal{A}(g)$, we see that any element of A has the form $\varphi + tg(m_S\theta_S(n)) \cap g^*(m_0)\theta_S(g)$ $(\varphi \in \omega g(m_0\theta_0(p)))$. Therefore, we see that the minimal number of elements of a generator of A as C_0-module via g is less than or equal to the minimal number of elements of a generator of $\omega g(m_0\theta(p))$ as C_0-module via g and it is clear that the latter number is less than or equal to p^2. Thus, we have the following inequality:

$$\dim_{\mathbf{R}} \frac{A}{g^*(m_0)A} \leq p^2.$$

On the other hand, the left hand side of the above inequality is more than or equal to $p\,{}_1\delta(g) - (n\,{}_1\delta(g) - {}_1\gamma(g)) - \gamma(g)$.

Next we show 2 of Lemma 2.2. The assumption $T\mathcal{C}(g) = T\mathcal{L}(g)$ implies the inequality $p\ {}_1\delta(g) = \dim_{\mathbf{R}}{}_1Q(g)^p \leq p^2$. □

Proof of Theorem 1. First we prove 1 of Theorem 1. Since f is of corank at most one, the multigerm g in 1 of Lemma 2.1 is also of corank at most one. Since g is of corank at most one, g is $\mathcal{A}$-equivalent to an unfolding of a multigerm h of one variable with $(n-1)$ parameters. Note that

$$\delta(f) = \delta(g) = \delta(h).$$

Furthermore, since h is a multigerm of one variable we have that ${}_1\delta(h) = \delta(h)$, ${}_1\gamma(h) = \gamma(h)$ easily and $\gamma(h) = \delta(h) - r$ (for the last equality, refer to p. 508 of [16]). By combining the above equalities and the equalities in §2 we have the following two:

$$\begin{aligned} {}_1\delta(g) &= n\delta(h) = n\delta(f), \\ {}_1\gamma(g) &= n\gamma(h) = n\left(\delta(f) - r\right). \end{aligned}$$

Therefore, for the g in 1 of Lemma 2.1, by 1 of Lemma 2.2 we have the following desired inequality:

$$(p-n)n\delta(f) + n(\delta(f) - r) - (\delta(f) - r) \le p^2.$$

For the proof of 2 of Theorem 1, just replace 1 of Lemma 2.1 and 1 of Lemma 2.2 in the proof of 1 of Theorem 1 with 2 of Lemma 2.1 and 2 of Lemma 2.2. Q.E.D.

3. Proof of Theorem 2

Lemma 3.1. *Let* $f : (\mathbf{R}^n, S) \to (\mathbf{R}^p, 0)$ $(n \le p)$ *be a multigerm.*

1. *Suppose that* f *is* $\mathcal{A}$*-simple. Then, there exists an* $\mathcal{A}$*-simple immersion* $h : (\mathbf{R}^n, S) \to (\mathbf{R}^p, 0)$ *such that* f *is adjacent to* h *and the equality* $T\mathcal{K}(h) = T\mathcal{A}(h)$ *holds.*
2. *Suppose that* f *is* $\mathcal{L}$*-simple. Then, there exists an* $\mathcal{L}$*-simple immersion* $h : (\mathbf{R}^n, S) \to (\mathbf{R}^p, 0)$ *such that* f *is adjacent to* h *and the equality* $T\mathcal{C}(h) = T\mathcal{L}(h)$ *holds.*

Proof. Since $n \le p$ the multigerm f can be deformed to an immersive g by adding sufficiently small linear terms. Since f is $\mathcal{A}$-simple (resp. $\mathcal{L}$-simple), the obtained immersive germ g must be $\mathcal{A}$-simple (resp. $\mathcal{L}$-simple). Applying Lemma 2.1 for the g we obtain the desired multigerm h. □

Lemma 3.2. *Let* $h : (\mathbf{R}^n, S) \to (\mathbf{R}^p, 0)$ *be an immersive multigerm. Then, we have that* $\dim_{\mathbf{R}} T\mathcal{R}^1(j^1h(S)) = n^2r$ *and* $\dim_{\mathbf{R}} T\mathcal{G}^1(j^1h(S)) = npr$ *for* $\mathcal{G} = \mathcal{C}, \mathcal{K}$.

Proof. Since h is immersive, we see that $\dim_{\mathbf{R}} T\mathcal{R}^1(j^1h(S))$ is equal to r multiplied by the dimension of the space of linear isomorphisms of $\mathbf{R}^n$, which is n^2r. For $\dim_{\mathbf{R}} T\mathcal{K}^1(j^1h(S))$, note that $T\mathcal{K}(h) = T\mathcal{C}(h)$ since h is immersive. By definition of $T\mathcal{C}(h)$ we have that $\dim_{\mathbf{R}} T\mathcal{C}^1(j^1h(S)) = p\sum_{i=1}^{r} \operatorname{rank}(Jh_i(s_i))$. Therefore, for an immersive h the desired equality holds. □

Lemma 3.3. *Let* $h : (\mathbf{R}^n, S) \to (\mathbf{R}^p, 0)$ *be an immersive multigerm. Then,* $\dim_{\mathbf{R}} T\mathcal{A}^1(j^1h(S)) < n^2r + p^2$.

Proof. Put $V = T\mathcal{R}^1(j^1h(S)) \cap T\mathcal{L}^1(j^1h(S))$. First we show that $\dim_{\mathbf{R}} V$ is positive. For any non-zero real number α we let $\varphi_{\alpha,i} : (\mathbf{R}^n, s^i) \to (\mathbf{R}^n, s^i)$ be given by $\varphi_{\alpha,i}(x) = \alpha(x - s_i) + s_i$. Let $\varphi_\alpha : (\mathbf{R}^n, S) \to (\mathbf{R}^n, S)$ be the multigerm whose restriction to $(\mathbf{R}^n, s_i)$ is $\varphi_{\alpha,i}$. Furthermore, let $\psi_\alpha : (\mathbf{R}^p, 0) \to (\mathbf{R}^p, 0)$ be given by $\psi_\alpha(X) = \alpha X$ for any non-zero real number α. Then, we have that $j^1(h \circ \varphi_\alpha)(S) = j^1(\psi_\alpha \circ h)(S)$. Thus, we see that $\dim_{\mathbf{R}} V$ must be positive.

By Lemma 3.2, $\dim_{\mathbf{R}} T\mathcal{L}^1(j^1h(S)) \leq p^2$ and $\dim_{\mathbf{R}} V > 0$ we have the following:

$$\begin{aligned}\dim_{\mathbf{R}} T\mathcal{A}^1(j^1h(S)) &= \dim_{\mathbf{R}} T\mathcal{R}^1(j^1h(S)) + \dim_{\mathbf{R}} T\mathcal{L}^1(j^1h(S)) - \dim_{\mathbf{R}} V \\ &\leq n^2r + p^2 - \dim_{\mathbf{R}} V < n^2r + p^2.\end{aligned}$$

□

Proof of Theorem 2. For the $\mathcal{A}$-simple (resp. $\mathcal{L}$-simple) h obtained in Lemma 3.1, by lemmata 3.2 and 3.3 we see that $npr < n^2r+p^2$ (resp. $npr \leq p^2$). Q.E.D.

4. Proof of Proposition 1

Since 3 of Proposition 1 is a direct corollary of 1 of Theorem 2, it suffices to prove 1 and 2 of Proposition 1. One can prove 1 of Proposition 1 in the following way.

Proof. By composing a germ of C^∞ diffeomorphism $(\mathbf{R}^p, 0) \to (\mathbf{R}^p, 0)$ if necessary, from the first we may assume that f has the following form:

$$f_1(x) = (x, 0, \cdots, 0), \cdots, f_p(x) = (0, \cdots, 0, x).$$

By putting $ax\frac{\partial}{\partial X_j} = aX_i \circ f_i \frac{\partial}{\partial X_j}$ for any f_i $(1 \leq i \leq p)$, we see that the equality

$$T\mathcal{C}^1(j^1f(S)) = T\mathcal{L}^1(j^1f(S))$$

holds, where $(X_1, \cdots, X_p) \in \mathbf{R}^p$ and $\frac{\partial}{\partial X_j}$ is the j-th fundamental vector field of $\mathbf{R}^p$. Therefore, we have that

$${}_1Q(f)^p = \{[g] \mid g \in T\mathcal{L}(f)\},$$

where $[g] = g + f^*(m_0^2)\theta_S(f)$, and thus, by Malgrange preparation theorem (for instance, see [2]) we have that

$$T\mathcal{C}(f) = T\mathcal{L}(f).$$

This equality shows that there are no $\mathcal{L}$-equivalence classes to which the immersion f is adjacent. Hence, f must be $\mathcal{L}$-simple. □

One can prove 2 of Proposition 1 in the following way.

Proof. By composing a germ of C^∞ diffeomorphism $(\mathbf{R}^p, 0) \to (\mathbf{R}^p, 0)$ if necessary, from the first we may assume that f has the following form:

$$f_1(x) = (x, 0, \cdots, 0), \cdots, f_p(x) = (0, \cdots, 0, x), f_{p+1}(x) = (a_1x, \cdots, a_px),$$

where $a_1 \cdots a_p \neq 0$. For any f_i $(1 \leq i \leq p)$ we put $ax\frac{\partial}{\partial X_i} = df_i \circ (ax\frac{\partial}{\partial x})$ and $ax\frac{\partial}{\partial X_j} = aX_i \circ f_i \frac{\partial}{\partial X_j}$ if $j \neq i$ and for f_{p+1} we put $ax\frac{\partial}{\partial X_j} = \frac{a}{a_j} X_j \circ f_{p+1} \frac{\partial}{\partial X_j}$. Then, we see that the equality

$$T\mathcal{K}^1(j^1 f(S)) = T\mathcal{A}^1(j^1 f(S))$$

holds, where $(X_1, \cdots, X_p) \in \mathbf{R}^p$ and $\frac{\partial}{\partial X_j}$ is the j-th fundamental vector field of $\mathbf{R}^p$. Therefore, we have that

$${}_1Q(f)^p = \{[g] \mid g \in T\mathcal{A}(f)\},$$

where $[g] = g + f^*(m_0^2)\theta_S(f)$, and thus by Malgrange preparation theorem we have that

$$T\mathcal{K}(f) = T\mathcal{A}(f).$$

This equality shows that there are no $\mathcal{A}$-equivalence classes to which the immersion f is adjacent. Hence, f must be $\mathcal{A}$-simple. □

5. Proof of Proposition 2

Since 3 of Proposition 2 is a direct corollary of 1 of Theorem 2, it suffices to prove 1 and 2 of Proposition 2. One can prove 1 of Proposition 2 in the following way.

Proof. By composing a germ of C^∞ diffeomorphism $(\mathbf{R}^p, 0) \to (\mathbf{R}^p, 0)$ if necessary, from the first we may assume that f has the following form:

$$f_1(x_1, \cdots, x_n) = (x_1, \cdots, x_n, 0, \cdots, 0), f_2(x_1, \cdots, x_n) = (0, \cdots, 0, x_1, \cdots, x_n).$$

By putting $\sum_{k=1}^{n} a_k x_k \frac{\partial}{\partial X_j} = \sum_{j=2(i-1)+1}^{2(i-1)+n} a_j X_j \circ f_i \frac{\partial}{\partial X_j}$ for any f_i $(i = 1, 2)$, we see that the equality

$$T\mathcal{C}^1(j^1 f(S)) = T\mathcal{L}^1(j^1 f(S))$$

holds, where $(X_1, \cdots, X_p) \in \mathbf{R}^p$ and $\frac{\partial}{\partial X_j}$ is the j-th fundamental vector field of $\mathbf{R}^p$. Therefore, we have that

$${}_1Q(f)^p = \{[g] \mid g \in T\mathcal{L}(f)\},$$

where $[g] = g + f^*(m_0^2)\theta_S(f)$, and thus by Malgrange preparation theorem we have that

$$T\mathcal{C}(f) = T\mathcal{L}(f).$$

This equality shows that there are no $\mathcal{L}$-equivalence classes to which the immersion f is adjacent. Hence, f must be $\mathcal{L}$-simple. □

One can prove 2 of Proposition 2 in the following way.

Proof. By composing a germ of C^∞ diffeomorphism $(\mathbf{R}^p, 0) \to (\mathbf{R}^p, 0)$ if necessary, from the first we may assume that f has the following form:

$$f_1(x_1, \cdots, x_n) = (x_1, \cdots, x_n, 0, \cdots, 0),$$
$$f_2(x_1, \cdots, x_n) = (0, \cdots, 0, x_1, \cdots, x_n),$$
$$f_3(x_1, \cdots, x_n) = (x_1, \cdots, x_n, a_1x_1, \cdots, a_nx_n),$$

where $a_1, \ldots, a_p \neq 0$. For f_1 we put $\sum_{k=1}^n b_k x_k \frac{\partial}{\partial X_j} = df_1 \circ (\sum_{k=1}^n b_k x_k \frac{\partial}{\partial x_k})$ $(1 \leq j \leq n)$ and $\sum_{k=1}^n b_k x_k \frac{\partial}{\partial X_j} = \sum_{k=1}^n b_k X_k \circ f_1 \frac{\partial}{\partial X_j}$ $(n+1 \leq j \leq p)$. For f_2 we put $\sum_{k=1}^n b_k x_k \frac{\partial}{\partial X_j} = df_2 \circ (\sum_{k=1}^n b_k x_k \frac{\partial}{\partial x_k})$ $(n+1 \leq j \leq p)$ and $\sum_{k=1}^n b_k x_k \frac{\partial}{\partial X_j} = \sum_{k=1}^n b_k X_k \circ f_2 \frac{\partial}{\partial X_j}$ $(1 \leq j \leq n)$. Finally, for f_3 we put $\sum_{k=1}^n b_k x_k \frac{\partial}{\partial X_j} = \sum_{k=1}^n b_k X_k \circ f_3 \frac{\partial}{\partial X_j}$ $(1 \leq j \leq n)$ and $\sum_{k=1}^n b_k x_k \frac{\partial}{\partial X_j} = \sum_{k=1}^n \frac{b_k}{a_k} X_k \circ f_3 \frac{\partial}{\partial X_j}$ $(n+1 \leq j \leq p)$. Then, we see that the equality

$$T\mathcal{K}^1(j^1 f(S)) = T\mathcal{A}^1(j^1 f(S))$$

holds, where $(X_1, \cdots, X_p) \in \mathbf{R}^p$ and $\frac{\partial}{\partial X_j}$ is the j-th fundamental vector field of $\mathbf{R}^p$. Therefore, we have that

$${}_1Q(f)^p = \{[g] \mid g \in T\mathcal{A}(f)\},$$

where $[g] = g + f^*(m_0^2)\theta_S(f)$, and thus by Malgrange preparation theorem we have that

$$T\mathcal{K}(f) = T\mathcal{A}(f).$$

This equality shows that there are no $\mathcal{A}$-equivalence classes to which the immersion f is adjacent. Hence, f must be $\mathcal{A}$-simple. □

Acknowledgement. The author would like to express his sincere gratitude to the reviewer for his/her careful reading of the manuscript and giving invaluable suggestions.

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