

ON HOMOLOGICAL CLASSIFICATION OF MONOIDS BY CONDITION (E') OF RIGHT ACTS

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Abstract. Valdis Laan in (On a generalization of strong flatness, Acta Comment. Univ. Tartuensis **2** (1998), 55-60.) introduced Condition (E') , a generalization of Condition (E) . In this paper we continue the investigation of Condition (E') and give a classification of monoids by comparing this condition of their acts with other properties. We give also a classification of monoids for which all (monocyclic, cyclic) right acts satisfy Condition (E') and in particular for idempotent monoids and monoids S with $E(S) = \{1\}$. A classification of monoids over which all monocyclic right acts are weakly pullback flat will be given too.

1. Introduction

Throughout this paper S will denote a monoid. We refer the reader to [3] and [4] for basic results, definitions and terminology relating to semigroups and acts over monoids and to [1], [8] for definitions and results on flatness which are used here.

A monoid S is said to be *left collapsible* if for any $p, q \in S$ there exists $r \in S$ such that $rp = rq$.

A right S -act A satisfies Condition (P) if for all $a, a' \in A$, $s, s' \in S$, $as = a's'$ implies that there exist $a'' \in A$, $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $us = vs'$. It satisfies Condition (P_E) if whenever $a, a' \in A$, $s, s' \in S$, and $as = a's'$, there exist $a'' \in A$ and $u, v, e^2 = e, f^2 = f \in S$ such that $ae = a''ue$, $a'f = a''vf$, $es = s$, $fs' = s'$ and $us = vs'$. It is shown in [2] that Condition (P_E) implies weak flatness, but the converse is not true. A right S -act A satisfies Condition (E) if for all $a \in A$, $s, s' \in S$, $as = as'$ implies that there exist $a' \in A$, $u \in S$ such that $a = a'u$ and $us = us'$.

We use the following abbreviations,

weak pullback flatness = WPF

weak kernel flatness = WKF

principal weak kernel flatness = $PWKF$

translation kernel flatness = TKF

weak homoflatness = (WP)

principal weak homoflatness = (PWP)

weak flatness = WF

principal weak flatness = PWF

2. Characterization of monoids by Condition (E') of right acts

DEFINITION. A right S -act A satisfies Condition (E') if for all $a \in A$, $s, t, z \in S$, $as = at$ and $sz = tz$ imply that there exist $a' \in A$, $u \in S$ such that $a = a'u$ and $us = ut$.

Note that Condition (E) implies Condition (E') , but the converse is not true in general, for if S is a non-trivial group, then Θ_S satisfies Condition (E') , but it does not satisfy Condition (E) , otherwise by ([4, III, 14.3]), S is left collapsible, and so $|S| = 1$, which is not true.

Note also that Condition (E') does not imply torsion freeness in general, for if $S = (N, \cdot)$, where N is the set of natural numbers, and if $A_S = N \coprod^{2N} N$, then

$$A_S = \{(a, x) \mid a \in 2N_0 + 1\} \dot{\cup} 2N \dot{\cup} \{(a, y) \mid a \in 2N_0 + 1\}.$$

It can easily be seen that

$$\{(a, x) \mid a \in 2N_0 + 1\} \dot{\cup} 2N \cong N_N \cong \{(a, y) \mid a \in 2N_0 + 1\} \dot{\cup} 2N.$$

Since N_N satisfies Condition (E) , then $\{(a, x) \mid a \in 2N_0 + 1\} \dot{\cup} 2N$ and $\{(a, y) \mid a \in 2N_0 + 1\} \dot{\cup} 2N$ satisfy Condition (E) , and so A_S satisfies Condition (E) . Now if Condition (E') implies torsion freeness, then $2 = (1, x)2 = (1, y)2$ implies that $(1, x) = (1, y)$, which is a contradiction. Now it is natural to ask for monoids over which Condition (E') of acts implies torsion freeness, and other properties that imply torsion freeness.

It is shown in [1] that the necessary and sufficient condition for a monoid S to be a group is that all right S -acts be WPF , WKF , $PWKF$, TKF , or satisfy Conditions (PWP) , (WP) , (P) , but by the following theorem we show that the necessary and sufficient condition for a monoid S to be a group is that all right S -acts satisfying Condition (E') be WPF , WKF , $PWKF$, TKF , or satisfy Conditions (PWP) , (WP) , (P) .

THEOREM 2.1. *For any monoid S the following statements are equivalent:*

- (1) *All right S -acts satisfying Condition (E') are WPF.*
- (2) *All right S -acts satisfying Condition (E') satisfy Condition (P) .*
- (3) *All right S -acts satisfying Condition (E') are WKF.*
- (4) *All right S -acts satisfying Condition (E') are PWKF.*
- (5) *All right S -acts satisfying Condition (E') are TKF.*
- (6) *All right S -acts satisfying Condition (E') satisfy Condition (WP) .*
- (7) *All right S -acts satisfying Condition (E') satisfy Condition (PWP) .*
- (8) *S is a group.*

Proof. Implications $(1) \Rightarrow (2) \Rightarrow (6) \Rightarrow (7)$ and $(1) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (7)$ are obvious.

$(8) \Rightarrow (1)$. By ([1, proposition 9]), it is obvious.

$(7) \Rightarrow (8)$. Let I be a proper right ideal of S and let

$$A_S = S \coprod_I S = \{(a, x) \mid a \in S \setminus I\} \dot{\cup} I \dot{\cup} \{(a, y) \mid a \in S \setminus I\}.$$

Then it is obvious that $\{(a, x) \mid a \in S \setminus I\} \dot{\cup} I$ and $\{(a, y) \mid a \in S \setminus I\} \dot{\cup} I$ are subacts of A_S isomorphic to S_S . Since S_S is free, then it satisfies Condition (E') . Thus $\{(a, x) \mid a \in S \setminus I\} \dot{\cup} I$ and $\{(a, y) \mid a \in S \setminus I\} \dot{\cup} I$ satisfy Condition (E') , and so A_S satisfies Condition (E') . Thus by assumption A_S satisfies Condition (PWP) . If $t \in I$, then $t = (1, x)t = (1, y)t$, and so there exist $a \in A$ and $u, v \in S$ such that $(1, x) = au, (1, y) = av$ and $ut = vt$. Thus $(1, x) = au$ implies for some $s \in S \setminus I$, that $a = (s, x)$. Similarly, $a = (s', y)$ for some $s' \in S \setminus I$, and so we have a contradiction. Thus S has no proper right ideal, that is for every $s \in S, sS = S$, and so S is a group as required. \square

Now we have ([10, Theorem 2.5]), as a corollary of Theorem 2.1, as follows:

COROLLARY 2.2. *For any monoid S the following statements are equivalent:*

- (1) *All right S -acts satisfying Condition (E') are free.*
- (2) *All right S -acts satisfying Condition (E') are projective generators.*
- (3) *All right S -acts satisfying Condition (E') are projective.*
- (4) *All right S -acts satisfying Condition (E') are strongly flat.*
- (5) $S = \{1\}$.

Proof. Implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$ are obvious.

(4) \Rightarrow (5). By assumption all right S -acts satisfying Condition (E') are WPF . Thus by Theorem 2.1, S is a group, and so all right S -acts satisfy Condition (E') . Thus all right S -acts are strongly flat, and so by ([4, IV, 10.5]), $S = \{1\}$.

(5) \Rightarrow (1). It is obvious. \square

Qiao and Liu in ([10, Theorem 2.9]), showed that right collapsible monoids for which all right S -acts satisfying Condition (E') are (principally weakly, weakly) flat are regular monoids. Now by the following theorem we extend these results to any monoid. Moreover, we show that monoids for which all right S -acts satisfying Condition (E') satisfy Condition (P_E) are regular too.

THEOREM 2.3. *For any monoid S the following statements are equivalent:*

- (1) *All right S -acts satisfying Condition (E') satisfy Condition (P_E) .*
- (2) *All right S -acts satisfying Condition (E') are flat.*
- (3) *All right S -acts satisfying Condition (E') are WF .*
- (4) *All right S -acts satisfying Condition (E') are PWF .*
- (5) *S is regular.*

Proof. Implications (2) \Rightarrow (3) \Rightarrow (4) are obvious.

(4) \Rightarrow (5). Since Condition (E) implies Condition (E') , then by ([9, Theorem 3]), it is obvious.

(5) \Rightarrow (2). Suppose that S is regular, ${}_S M$ is a left S -act and A_S a right S -act that satisfies Condition (E') . Let $a \otimes m = a' \otimes m'$ in $A_S \otimes {}_S M$ for $a, a' \in A_S$ and $m, m' \in {}_S M$. We show that $a \otimes m = a' \otimes m'$ holds also in $A_S \otimes {}_S (Sm \cup Sm')$. Since $a \otimes m = a' \otimes m'$ in $A_S \otimes {}_S M$, then we have a tossing

$$\begin{array}{rclcl}
 & & s_1 m_1 & = & m \\
 as_1 & = & a_1 t_1 & s_2 m_2 & = t_1 m_1 \\
 a_1 s_2 & = & a_2 t_2 & s_3 m_3 & = t_2 m_2 \\
 & \dots & & & \dots \\
 a_{k-1} s_k & = & a' t_k & m' & = t_k m_k
 \end{array}$$

of length k , where $s_1, \dots, s_k, t_1, \dots, t_k \in S$, $a_1, \dots, a_{k-1} \in A_S$, $m_1, \dots, m_k \in {}_S M$.

If $k = 1$, then we have

$$\begin{array}{rclcl}
 as_1 & = & a' t_1 & s_1 m_1 & = m \\
 & & & m' & = t_1 m_1.
 \end{array}$$

Since S is regular, then the equality $as_1 = a't_1$ implies that $a't_1 = a't_1s'_1s_1$, for $s'_1 \in V(s_1)$. Since A_S satisfies Condition (E') and $t_1 \cdot s'_1 = t_1s'_1s_1 \cdot s'_1$, then there exist $a'' \in A_S$ and $u \in S$ such that $a' = a''u$ and $ut_1 = ut_1s'_1s_1$. From the last equality we obtain $um' = ut_1m_1 = ut_1s'_1s_1m_1 = ut_1s'_1m$. Since $m = s_1m_1$, then $s_1s'_1m = m$, and so we get

$$a \otimes m = a \otimes s_1s'_1m = as_1 \otimes s'_1m = a't_1 \otimes s'_1m = a''ut_1 \otimes s'_1m =$$

$$a'' \otimes ut_1s'_1m = a'' \otimes um' = a''u \otimes m' = a' \otimes m'$$

in $A_S \otimes_S (Sm \cup Sm')$.

Now we suppose that $k \geq 2$ and that the required equality holds for tossing of length less than k . From $as_1 = a_1t_1$ we obtain equalities $a_1t_1 = a_1t_1s'_1s_1$ for $s'_1 \in V(s_1)$ and $as_1 = as_1t'_1t_1$ for $t'_1 \in V(t_1)$. Since A_S satisfies Condition (E') and $t_1 \cdot s'_1 = t_1s'_1s_1 \cdot s'_1$, $s_1 \cdot t'_1 = s_1t'_1t_1 \cdot t'_1$, then there exist $a''_1, a''_2 \in A_S$ and $u_1, u_2 \in S$ such that $a_1 = a''_1u_1$, $u_1t_1 = u_1t_1s'_1s_1$ and $a = a''_2u_2$, $u_2s_1 = u_2s_1t'_1t_1$. Thus we have the following tossings

$$\begin{array}{ll} a''_2u_2s_1 & = a''_1u_1t_1 \\ u_2s_1m_1 & = u_2m \\ u_1s_2m_2 & = u_1t_1m_1. \end{array}$$

of length 1 and

$$\begin{array}{ll} a''_1u_1s_2 & = a_2t_2 \\ \dots & \\ a_{k-1}s_k & = a't_k \\ u_1s_2m_2 & = u_1t_1m_1 \\ s_3m_3 & = t_2m_2 \\ \dots & \\ m' & = t_k m_k \end{array}$$

of length $k - 1$.

From the tossing of length 1, we have $a''_2 \otimes u_2m = a''_1 \otimes u_1s_2m_2$ in $A_S \otimes_S M$. By inductive hypothesis we have $a''_2 \otimes u_2m = a''_1 \otimes u_1s_2m_2$ in $A_S \otimes_S (Su_2m \cup Su_1s_2m_2)$. Since

$$u_1s_2m_2 = u_1t_1m_1 = u_1t_1s'_1s_1m_1 = u_1t_1s'_1m \in Sm,$$

then we have $a''_2 \otimes u_2m = a''_1 \otimes u_1s_2m_2$ in $A_S \otimes_S (Sm \cup Sm')$.

Also from tossing of the length $k - 1$, we have $a''_1 \otimes u_1t_1m_1 = a' \otimes m'$ in $A_S \otimes_S M$. By inductive hypothesis we have $a''_1 \otimes u_1t_1m_1 = a' \otimes m'$ in $A_S \otimes_S (Su_1t_1m_1 \cup Sm')$. Since $u_1t_1m_1 = u_1t_1s'_1m \in Sm$, then $a''_1 \otimes u_1t_1m_1 = a' \otimes m'$ in $A_S \otimes_S (Sm \cup Sm')$. Thus we have

$$a \otimes m = a''_2u_2 \otimes m = a''_2 \otimes u_2m = a''_1 \otimes u_1s_2m_2 = a''_1 \otimes u_1t_1m_1 = a' \otimes m'$$

in $A_S \otimes_S (Sm \cup Sm')$.

(1) \Leftrightarrow (3). Since every regular monoid is left PP , then by ([2, Theorems 2.3, 2.5]), it is obvious. \square

Note that cofreeness of acts does not imply Condition (E') , for if $S = T^1$, where T is a non-trivial right zero semigroup and X is a set with one element, then the cofree right S -act X^S has also one element, and so $X^S \cong \Theta_S$. Since Θ_S does not satisfy Condition (E') , then X^S does not satisfy Condition (E') either. It is now obvious that every property of acts over monoids which is implied by cofreeness does not imply Condition (E') either.

THEOREM 2.4. *For any monoid S the following statements are equivalent:*

- (1) *All principally weakly injective right S -acts satisfy Condition (E') .*
- (2) *All fg -weakly injective right S -acts satisfy Condition (E') .*
- (3) *All weakly injective right S -acts satisfy Condition (E') .*
- (4) *All injective right S -acts satisfy Condition (E') .*
- (5) *All cofree right S -acts satisfy Condition (E') .*
- (6) $(\forall s, t, z \in S)(sz = tz \Rightarrow (\exists u \in S, \rho(s, t) = \ker \lambda_u))$.

Proof. Implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5)$ are obvious.

A similar argument as in ([7, Proposition 2.1]), can be used for implications $(5) \Rightarrow (6)$ and $(6) \Rightarrow (1)$. \square

Note that by ([4, IV, 11.14]) and ([4, III, 17.13]), all (mono) cyclic right S -acts satisfy Condition (E) if and only if $S = \{1\}$ or $S = \{0, 1\}$, but if we replace (E) by (E') , then we have the following theorem.

THEOREM 2.5. *For any monoid S the following statements are equivalent:*

- (1) *All right S -acts satisfy Condition (E') .*
- (2) *All divisible right S -acts satisfy Condition (E') .*
- (3) *All finitely generated right S -acts satisfy Condition (E') .*
- (4) *All cyclic right S -acts satisfy Condition (E') .*
- (5) *All monocyclic right S -acts satisfy Condition (E') .*
- (6) $(\forall s, t, z \in S)(sz = tz \Rightarrow (\exists u \in S, us = ut, 1\rho(s, t)u))$.
- (7) $(\forall s, t, z \in S)(sz = tz \Rightarrow (\exists e \in E(S), es = et, 1\rho(s, t)e))$.
- (8) $(\forall s, t, z \in S)(sz = tz \Rightarrow (\exists e \in E(S), \rho(s, t) = \ker \lambda_e))$.

Proof. Implications $(1) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5)$, and $(1) \Rightarrow (2)$ are obvious.

$(5) \Rightarrow (6)$. Suppose that $sz = tz$ for $s, t, z \in S$. Since by assumption $S/\rho(s, t)$ satisfies Condition (E') and $s\rho(s, t)t$, then there exists $u \in S$ such that $us = ut$ and $1\rho(s, t)u$.

(6) \Rightarrow (7). Suppose that $sz = tz$ for $s, t, z \in S$. Then by assumption there exists $u \in S$ such that $us = ut$ and $1\rho(s, t)u$. Then $(s, t) \in \ker\lambda_u$, and so $\rho(s, t) \subseteq \ker\lambda_u$. Thus $1\rho(s, t)u$ implies that $(1, u) \in \ker\lambda_u$, and so $u^2 = u \in E(S)$.

(7) \Rightarrow (8). Suppose that $sz = tz$, for $s, t, z \in S$. By assumption there exists $e \in E(S)$ such that $es = et$ and $1\rho(s, t)e$. But $es = et$ implies that $(s, t) \in \ker\lambda_e$, and so $\rho(s, t) \subseteq \ker\lambda_e$. Now we suppose that $(s', t') \in \ker\lambda_e$, then $es' = et'$. Since $1\rho(s, t)e$, then $s'\rho(s, t)es'$ and $t'\rho(s, t)et'$. Since $es' = et'$, then $s'\rho(s, t)t'$, and so $(s', t') \in \rho(s, t)$. Thus $\ker\lambda_e \subseteq \rho(s, t)$, and so $\ker\lambda_e = \rho(s, t)$.

(8) \Rightarrow (1). First we show that every cyclic right S -act satisfies Condition (E') . For this we suppose that ρ is a right congruence on S , spt and $sz = tz$, for $s, t, z \in S$. Then by assumption there exists $e \in E(S)$ such that $\ker\lambda_e = \rho(s, t)$. Since $(1, e) \in \ker\lambda_e$ and $(s, t) \in \rho(s, t)$, then $es = et$ and $1\rho(s, t)e$. Since spt , then $\rho(s, t) \subseteq \rho$, and so $1\rho e$. Thus by ([5, Lemma 6]), S/ρ satisfies Condition (E') as required. Now we suppose that A is a right S -act, $as = at$ and $sz = tz$ for $a \in A$ and $s, t, z \in S$. Since aS satisfies Condition (E') , then there exist $w_1, w_2 \in S$ such that $a = (aw_1)w_2$ and $w_2s = w_2t$. If $w_1w_2 = u$, then $a = au$ and $us = ut$, that is A satisfies Condition (E') as required.

(2) \Rightarrow (1). Suppose that A is a right S -act. Let $as = at$ and $sz = tz$ for $a \in A$ and $s, t, z \in S$. By ([4, III, 2.9]), if $D(A)$ is the divisible extension of A , then by assumption $D(A)$ satisfies Condition (E') and hence there exist $b \in D(A)$ and $u \in S$ such that $a = bu$ and $us = ut$. By ([4, III, 2.9]), there exists $n \in N$ such that $b \in A_n$. Now if $b \in A_n \setminus A_{n-1}$, then by ([4, III, 2.10]), there exist $b_1 \in A_{n-1}$, $u_1, v_1 \in S$ and a left cancellable element $c \in S$ such that $a = b_1u_1$ and $cu_1 = v_1u$. Then we have $cu_1s = v_1us = v_1ut = cu_1t$, which implies that $u_1s = u_1t$, continuing this process, by ([4, III, 2.10]), there exist $a' \in A$, $v \in S$ such that $a = a'v$, $vs = vt$, and so A satisfies Condition (E') . \square

COROLLARY 2.6. *For any monoid S with $E(S) = \{1\}$, the following statements are equivalent:*

- (1) *All right S -acts satisfy Condition (E') .*
- (2) *All divisible right S -acts satisfy Condition (E') .*
- (3) *All finitely generated right S -acts satisfy Condition (E') .*
- (4) *All cyclic right S -acts satisfy Condition (E') .*
- (5) *All monocyclic right S -acts satisfy Condition (E') .*
- (6) *S is right cancellative.*

Proof. By Theorem 2.5, it is obvious. \square

It is shown in ([1, Propositions 9, 25]), that all (cyclic) right S -acts are WPF if and only if S is a (group or $S = \{0, 1\}$) group. Until now there were no classification of monoids for which all monocyclic right S -acts are WPF . Now see the following corollary for this classification.

COROLLARY 2.7. *Let S be a monoid. Then all monocyclic right S -acts are WPF if and only if S is a group or $S = \{0, 1\}$.*

Proof. Suppose that all monocyclic right S -acts are WPF . Since WPF is equivalent to the conjunction of Conditions (P) and (E'), all monocyclic right S -acts satisfy Condition (P), and so by ([4, IV, 9.9]), S is a group or a group with a zero adjoined. Now if $S = G^0$, where G is a group, then we claim that $|G| = 1$. Otherwise there exist $s, t \in G$ such that $s \neq t$. By Theorem 2.5, and that weak pullback flatness implies Condition (E'), the equality $s0 = t0$, implies that there exists $e \in E(S)$ such that $\rho(s, t) = \ker \lambda_e$. Since $E(S) = \{0, 1\}$, then either $e = 1$ or $e = 0$. If $e = 1$, then $\rho(s, t) = \ker \lambda_1 = \Delta_S$, and so $s = t$, which is a contradiction. Thus $e = 0$, and so $\rho(s, t) = \ker \lambda_0 = S \times S$. Hence $(0, 1) \in \rho(s, t)$, and so there exist $y_1, \dots, y_n, s_1, \dots, s_n, t_1, \dots, t_n \in S$ such that

$$0 = s_1 y_1 \quad t_2 y_2 = s_3 y_3 \quad \dots \quad t_n y_n = 1$$

$$t_1 y_1 = s_2 y_2 \quad t_3 y_3 = s_4 y_4 \dots$$

and for every $i \in \{1, 2, \dots, n\}$, $\{s_i, t_i\} = \{s, t\}$. Now $0 = s_1 y_1$ implies that $y_1 = 0$, because $s_1 \in G$. Thus $s_2 y_2 = 0$, and so $y_2 = 0$. By continuing this procedure we have, $y_1 = y_2 = \dots = y_n = 0$, and so $1 = t_n y_n = 0$, which is a contradiction. Thus $|G| = 1$, and so $S = \{0, 1\}$

Conversely, if S is a group, then by ([1, Proposition 9]), all monocyclic right S -acts are weakly pullback flat. If $S = \{0, 1\}$, then by ([4, IV, 11.14]), all cyclic right S -acts are strongly flat, and so weakly pullback flat. Thus all monocyclic right S -acts are weakly pullback flat as required. \square

Now we give a characterization of idempotent monoids by Condition (E').

LEMMA 2.8. *Let S be a monoid. If all right Rees factor S -acts satisfy Condition (E'), then for all $e, f \in E(S) \setminus \{1\}$, $ef = e$.*

Proof. Suppose that all right Rees factor acts of S satisfy Condition (E') and let $e, f \in E(S) \setminus \{1\}$. Since $ef, e \in eS$, then $ef \rho_{eS} e$, also $ef \cdot f = e \cdot f$. Hence by

Theorem ([5, Lemma 6]), there exists $u \in S$ such that $1\rho_e su$ and $u \cdot ef = u \cdot e$, because S/eS satisfies Condition (E') . But $eS \neq S$ and so $u = 1$, that is, $ef = e$. \square

THEOREM 2.9. *For an idempotent monoid S the following statements are equivalent:*

- (1) *All right S -acts satisfy Condition (E') .*
- (2) *All finitely generated right S -acts satisfy Condition (E') .*
- (3) *All cyclic right S -acts satisfy Condition (E') .*
- (4) *All monocyclic right S -acts satisfy Condition (E') .*
- (5) *All right Rees factor S -acts satisfy Condition (E') .*
- (6) *All divisible right S -acts satisfy Condition (E') .*
- (7) *All principally weakly injective right S -acts satisfy Condition (E') .*
- (8) *All fg -weakly injective right S -acts satisfy Condition (E') .*
- (9) *All weakly injective right S -acts satisfy Condition (E') .*
- (10) *All injective right S -acts satisfy Condition (E') .*
- (11) *All cofree right S -acts satisfy Condition (E') .*
- (12) *For all $e, f \in S \setminus \{1\}$, $ef = e$.*

Proof. Implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$, $(3) \Rightarrow (5)$, $(6) \Rightarrow (7) \Rightarrow (8) \Rightarrow (9) \Rightarrow (10) \Rightarrow (11)$ and $(1) \Rightarrow (6)$ are obvious.

$(4) \Rightarrow (5)$. By assumption and Theorem 2.5, all cyclic right S -acts satisfy Condition (E') , and so all right Rees factor S -acts satisfy Condition (E') .

$(5) \Rightarrow (12)$. Since $E(S) = S$, then by Lemma 2.8, it is obvious.

$(11) \Rightarrow (12)$. By (6) of Theorem 2.4, (8) of Theorem 2.5, and that $E(S) = S$, it follows that all cyclic right S -acts satisfy Condition (E') , and so by Lemma 2.8, for all $e, f \in S \setminus \{1\}$, $ef = e$.

$(12) \Rightarrow (1)$. Suppose that A_S is a right S -act, $ae = af$ and $eg = fg$, for $a \in A_S$ and $e, f, g \in S$. Then there are four cases that can arise:

Case 1. $e \neq 1, f \neq 1$. Since by assumption $eg = e$ and $fg = f$, then $e = f$. Now $a = a \cdot 1$ and $1 \cdot e = 1 \cdot f$.

Case 2. $e = f = 1$. Then $a = a \cdot 1$ and $1 \cdot e = 1 \cdot f$.

Case 3. $e = 1$ and $f \neq 1$. Then $a = a \cdot f$ and $f \cdot 1 = f \cdot f$.

Case 4. $f = 1$ and $e \neq 1$. It is similar to the case 3.

Thus A_S satisfies Condition (E') . \square

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