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BOUR'S THEOREM AND LIGHTLIKE PROFILE CURVE

By

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Abstract. In this work, we show that a generalized helicoid with lightlike profile curve is isometric to a rotation surface with lightlike profile curve by Bour's theorem in Minkowski 3−space. In addition, we determine these surfaces if they are minimal and have the same Gauss map with an additional condition.

Introduction.

It is well known that the right helicoid (resp. catenoid) is the only ruled (resp. rotation) surface which is minimal in Euclidean space. And, these surfaces have interesting properties. That is, they are both members of a one−parameter family of isometric surfaces. Moreover, by this isometric transformation, minimality is preserved.

On the other hand, if we focuse on the ruled and rotational characters, we have the following generalization.

Bour's Theorem. *A generalized helicoid is isometric to a rotation surface so that helices on the helicoid correspond to parallel circles on the rotation surface* [1].

In this generalization, original property that they are minimal is not generally kept.

So, in [3], T. Ikawa determined pairs of surfaces of Bour's theorem with an additional condition that they have the same Gauss map.

Ikawa classified the spacelike and timelike surfaces as *(axis, profile curve)−type* in [4]. He proved an isometric relation between a spacelike (timelike) generalized helicoid and a spacelike (timelike) rotation surface of spacelike (timelike) axis $(S,S),(S,T),(T,S)$ and (T,T) − *type* by Bour's theorem.

Güler and Vanli [2] showed that a generalized helicoid and a rotation sur-

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face with lightlike axis have isometric relation by Bour's theorem in Minkowski 3−space. They classified the spacelike (timelike) helicoidal (rotation) surfaces with lightlike axis of (L, S) and $(L, T) - type$.

The main purpose of this study is finding an isometric relation between a timelike generalized helicoid and a timelike rotation surface with lightlike profile curve of $(S, L), (T, L)$ and $(L, L) - type$ by Bour's theorem.

time like surface		1. AXIS's type		
		space	time	light
2. PROFILE CURVE's type	space	Ikawa result	Ikawa result	Güler-Vanli result
	time	Ikawa result	Ikawa resut	Güler-Vanli result
	light	Güler new result	Güler new result	Güler new result
space like surface		1. AXIS's type		
		space	time	light
2. PROFILE CURVE's type	space	Ikawa result	Ikawa result	Güler-Vanli result
	time	Ikawa result	Ikawa resut	Güler-Vanli result
	light	no exist	no exist	no exist

Table 1 Types of the surfaces.

In this paper, the author study the parts of "Güler new result". The parts of "Ikawa result" are [4], the parts of "Güler-Vanli result" are [2]. The component of the first fundamental form $E_H = 0$ then $Q_H < 0$. So, the type of the surfaces are timelike. Hence we can write "no exist" to the type of spacelike surfaces.

We summarized this paper in Table 2 as follow

type	*relations*	*conditions*
(S, L)	$e_H = e_R$	*in Th.* 3.2
(S, L)	$H_H = 0,\ H_R = 0$	$\Im(u) = 0,\ \wp(u) = 0$
(T, L)	$e_H = e_R$	*in Th.* 3.4
(T, L)	$H_H = 0,\ H_R = 0$	$\chi(u) = 0,\ \pi(u) = 0$
(L, L)	$e_H = e_R$	*in Th.* 3.6
(L, L)	$H_H = 0,\ H_R = 0$	$\varsigma(u) = 0,\ \kappa(u) = 0$

Table 2 Relations of the timelike surfaces.

In section 1, we recall some basic notions of the Lorentzian geometry and give the definition of the helicoidal surfaces and the rotation surfaces. Rotation surfaces with lightlike profile curve are given to find a pair of surfaces that have the same Gauss map in section 2. In section 3, we study an isometric relation between a timelike generalized helicoid and a timelike rotation surface with lightlike profile curve by Bour's theorem. Then, we give these surfaces have zero mean curvature, have the same Gauss map with an additional condition.

1. Preliminaries.

In this section, we will obtain some rotation and helicoidal surfaces with lightlike profile curve in Minkowski 3–space.

In the rest of this paper we shall identify a vector (a, b, c) with its transpose $(a, b, c)^{\ t}$. First we say that a surface in $\mathbb{R}^3_1$ is spacelike (timelike) if the discriminant $Q = EG - F^2$ is positive (negative), where E, F, G are coefficients of its first fundamental form.

The Minkowski 3–space $\mathbb{R}^3_1$ is the Euclidean space $\mathbb{E}^3$ provided with the inner product

$$\langle \overrightarrow{x}, \overrightarrow{y} \rangle_L = x_1y_1 + x_2y_2 - x_3y_3$$

where $\overrightarrow{x} = (x_1, x_2, x_3)$, $\overrightarrow{y} = (y_1, y_2, y_3) \in \mathbb{E}^3$. We say that a Lorentzian vector $\overrightarrow{x}$ in $\mathbb{R}^3_1$ is spacelike (resp. lightlike and timelike) if $\overrightarrow{x} = 0$ or $\langle \overrightarrow{x}, \overrightarrow{x} \rangle_L > 0$ (resp. $\overrightarrow{x} \neq 0$; $\langle \overrightarrow{x}, \overrightarrow{x} \rangle_L = 0$ and $\langle \overrightarrow{x}, \overrightarrow{x} \rangle_L < 0$). The norm of the vector $\overrightarrow{x} \in \mathbb{R}^3_1$ is defined by $\| \overrightarrow{x} \| = \sqrt{|\langle \overrightarrow{x}, \overrightarrow{x} \rangle_L|}$. Lorentzian vector product $\overrightarrow{x} \times \overrightarrow{y}$ of $\overrightarrow{x}$ and $\overrightarrow{y}$ is defined as follows:

$$\overrightarrow{x} \times \overrightarrow{y} = \begin{vmatrix} e_1 & e_2 & -e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}.$$

If the axis l is spacelike (resp. timelike, lightlike) in Minkowski 3–space $\mathbb{R}^3_1$, then we may suppose that l is the line spanned by the vectors $(1, 0, 0)$ (resp. $(0, 0, 1)$, $(0, 1, 1)$). The semi–orthogonal matrices given as follow is the subgroup of the Lorentzian group that fixes the above vectors as invariant

$$A_1(v) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh(v) & \sinh(v) \\ 0 & \sinh(v) & \cosh(v) \end{pmatrix}, \ A_2(v) = \begin{pmatrix} \cos(v) & -\sin(v) & 0 \\ \sin(v) & \cos(v) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$A_3(v) = \begin{pmatrix} 1 & -v & v \\ v & 1 - \frac{v^2}{2} & \frac{v^2}{2} \\ v & -\frac{v^2}{2} & 1 + \frac{v^2}{2} \end{pmatrix}, \ v \in \mathbb{R}.$$

Now we define a non-degenerate rotation surface and generalized helicoid in $\mathbb{R}^3_1$. For an open interval $I \subset \mathbb{R}$, let $\gamma : I \longrightarrow \Pi$ be a curve in a plane Π in $\mathbb{R}^3_1$, and let ℓ be a straight line in Π which does not intersect the curve γ. A *rotation surface* in $\mathbb{R}^3_1$ is defined as a non degenerate surface rotating a curve γ around a line ℓ (these are called the *profile curve* and the *axis*, respectively). Suppose that when a profile curve γ rotates around the axis ℓ, it simultaneously displaces parallel to ℓ so that the speed of displacement is proportional to the speed of rotation. Then the resulting surface is called the *generalized helicoid* with axis ℓ and *pitch* a.

The matrices A can be found by solving the following equations simultaneously; $A\ell = \ell$, $A^t\varepsilon A = \varepsilon$ where $\epsilon = diag(1, 1, -1)$ and $\det A = 1$.

Since the surface is non-degenerate, we may assume that the profile curve γ lies in the $x_1x_2x_3-$space without loss of generality and its parametrization is given by $\gamma(u) = (f(u), g(u), h(u))$ where $f(u), g(u)$ and $h(u)$ are differentiable functions on I such that for all $u \in \mathbb{R} \setminus \{0\}$.

A helicoidal surface in Minkowski 3$-$space with the spacelike axis which is spanned by $(1, 0, 0)$, and which has pitch $a \in \mathbb{R} \setminus \{0\}$ is as follows

$$H(u, v) = A_1(v) \cdot \gamma(u) + a\,(v, 0, 0)\,.$$

When $a = 0$, $H(u, v)$ reduces to a rotation surface.

We classified a surface by types of axis and profile curve, and write as (axis's type, profile curve's type)$-$type; for example, $(S, L)-type$ mean that the surface has a spacelike axis and a lightlike profile curve.

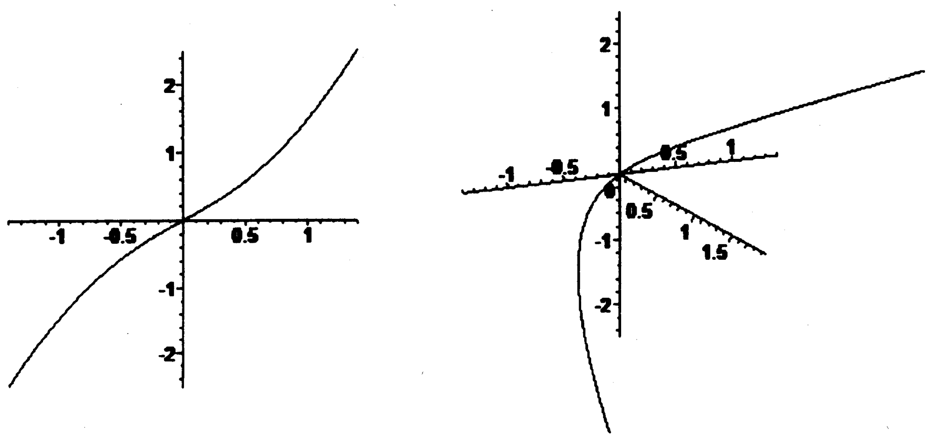


Figure 1. a-b Lightlike profile curve.

One example of the lightlike profile curves (see Fig.1) is $\gamma(u) = (u^2, u, \int \sqrt{4u^2+1}du + c)$. Since $\gamma(u)$ lightlike curve then $< \frac{d}{du}\gamma(u), \frac{d}{du}\gamma(u) >_L = f'^2 + g'^2 - h'^2 = 0$ and $h = \int \sqrt{f'^2 + g'^2}du + c$ $(c = const.)$.

2. Rotation surface.

We give rotation surfaces of lightlike profile curves that are used to have main theorems in this paper. γ profile curve is *lightlike*, then the rotation (helicoidal) surfaces are timelike with *spacelike* (resp. *timelike* or *lightlike*) axis and they are (S, L), (T, L) or (L, L) $-$*type* respectively.

When the axis ℓ is spacelike, there is a Lorentz transformation by which the axis ℓ is transformed to the x_1-axis of $\mathbb{R}^3_1$. If the profile curve is $\gamma(u) = (u^2, u, h(u))$, then the rotation surface $R(u, v)$ can be written as

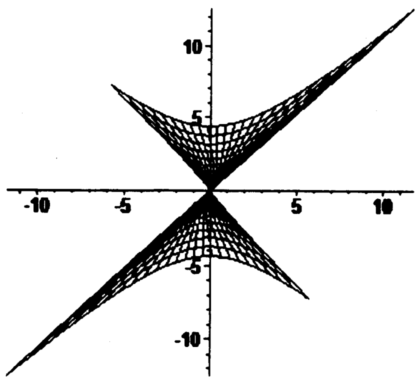
$$(1) \qquad R(u,v) = \begin{pmatrix} u^2 \\ u\cosh(v) + h(u)\sinh(v) \\ u\sinh(v) + h(u)\cosh(v) \end{pmatrix}.$$

PROPOSITION 2.1. *If a (S,L)−type rotation surface is of null profile curve, then the surface is (see Fig. 2)*

$$(2) \qquad R(u,v) = \begin{pmatrix} u^2 \\ u\cosh(v) + \frac{u\sqrt{4u^2+1}}{2}\sinh(v) + \frac{\sinh^{-1}(2u)}{4}\sinh(v) \\ u\sinh(v) + \frac{u\sqrt{4u^2+1}}{2}\cosh(v) + \frac{\sinh^{-1}(2u)}{4}\cosh(v) \end{pmatrix}$$

where $u, v \in \mathbb{R} \setminus \{0\}$.

From (1), we can calculate the mean curvature H and solving a differential equation $H = 0$, we can find the function $h(u)$ on the profile curve of the rotation surface.



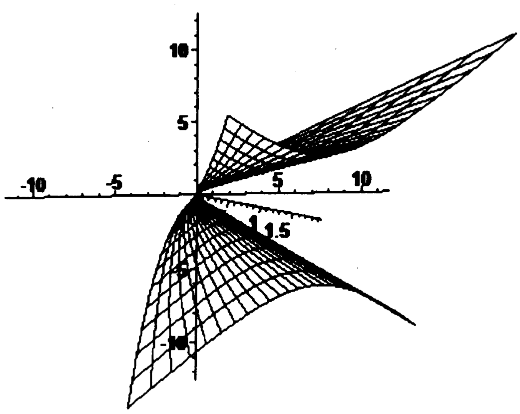


Figure 2. a-b Rotation surface with (S,L)-type.

When the axis ℓ is timelike, there is a Lorentz transformation by which the axis ℓ is transformed to the x_3-axis of $\mathbb{R}^3_1$. If the profile curve is $\gamma(u) =$

$(u^2, u, h(u))$, then the rotation surface $R(u,v)$ can be written as

$$R(u,v) = \begin{pmatrix} u^2\cos(v) - u\sin(v) \\ u^2\sin(v) + u\cos(v) \\ h(u) \end{pmatrix}. \tag{3}$$

PROPOSITION 2.2. *If a (T,L)−type rotation surface is of null profile curve, then the surface is (see Fig. 3)*

$$R(u,v) = \begin{pmatrix} u^2\cos(v) - u\sin(v) \\ u^2\sin(v) + u\cos(v) \\ \frac{1}{2}u\sqrt{4u^2+1} + \frac{1}{4}\sinh^{-1}(2u) \end{pmatrix} \tag{4}$$

where $u, v \in \mathbb{R} \setminus \{0\}$.

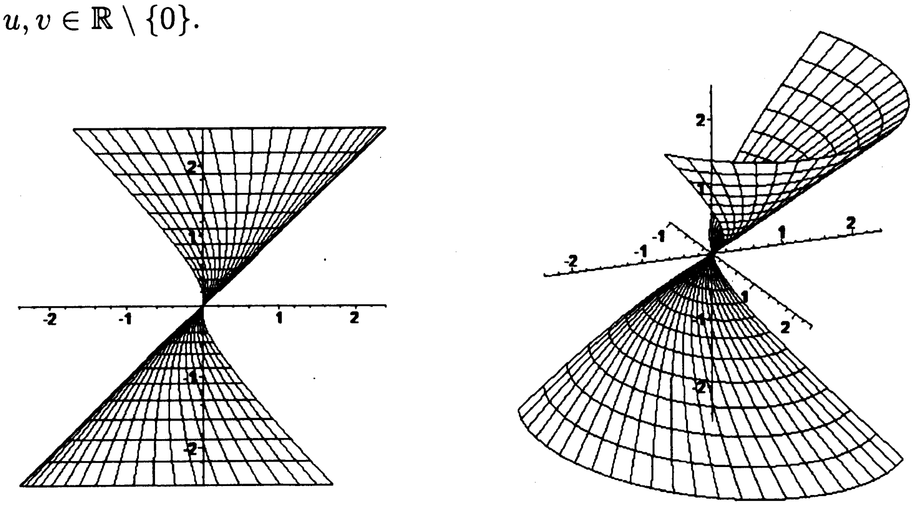


Figure 3. a-b Rotation surface with (T,L)-type.

When the axis ℓ is lightlike, there is a Lorentz transformation by which the axis ℓ is transformed to the x_2x_3-axis of $\mathbb{R}^3_1$. If the profile curve is $\gamma(u) = (u^2, u, h(u))$, then the rotation surface $R(u,v)$ can be written as

$$R(u,v) = \begin{pmatrix} u^2 - uv + hv \\ u^2v + \left(1 - \frac{v^2}{2}\right)u + \frac{v^2}{2}h \\ u^2v - \frac{v^2}{2}u + \left(1 + \frac{v^2}{2}\right)h \end{pmatrix}. \tag{5}$$

PROPOSITION 2.3. *If a (L,L)−type rotation surface is of null profile curve, then the surface is (see Fig. 4)*

$$R(u,v) = \begin{pmatrix} u^2 - uv + \frac{uv\sqrt{4u^2+1}}{2} + \frac{v\sinh^{-1}(2u)}{4} \\ u^2v + \left(1 - \frac{v^2}{2}\right)u + \frac{uv^2\sqrt{4u^2+1}}{4} + \frac{v^2\sinh^{-1}(2u)}{8} \\ u^2v - \frac{v^2}{2}u + \left(1 + \frac{v^2}{2}\right)\left(\frac{u\sqrt{4u^2+1}}{2} + \frac{\sinh^{-1}(2u)}{4}\right) \end{pmatrix} \tag{6}$$

where $u, v \in \mathbb{R} \setminus \{0\}$.

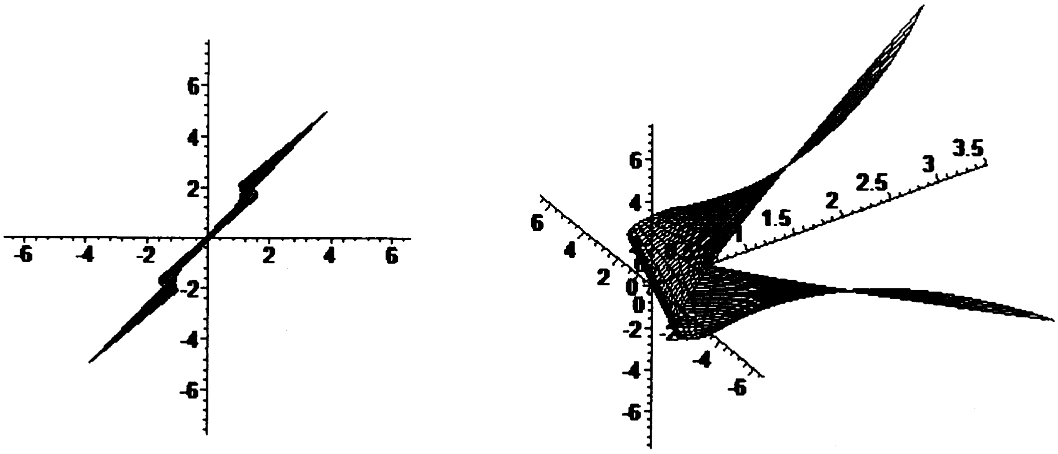


Figure 4. a-b Rotation surface with (L,L)-type.

3. Bour's theorems for the lightlike profile curves.

In this section, we study an isometric relation between a generalized helicoid and a rotation surface of lightlike profile curves.

Case 1. $(S, L) - type$.

First of all, we consider the $(S, L)-type$ surfaces, namely, the axis is spacelike $(1,0,0)$ vector and the profile curve is lightlike.

THEOREM 3.1. *A generalized helicoid*

$$(7) \qquad \begin{pmatrix} u^2 + av \\ u\cosh(v) + h_H \sinh(v) \\ u\sinh(v) + h_H \cosh(v) \end{pmatrix}$$

is isometric to a rotation surface

$$(8) \quad \begin{pmatrix} -u^2 + h_R \\ \sqrt{-u^2+h^2}\cosh(v - \int \frac{2au+h-uh'}{a^2-u^2+h^2}du) + h_R \sinh(v - \int \frac{2au+h-uh'}{a^2-u^2+h^2(u)}du) \\ \sqrt{-u^2+h^2}\sinh(v - \int \frac{2au+h-uh'}{a^2-u^2+h^2}du) + h_R \cosh(v - \int \frac{2au+h-uh'}{a^2-u^2+h^2}du) \end{pmatrix}$$

so that helices on the generalized helicoid correspond to parallel circles on the *rotation surface where* $h = h_H = \frac{1}{2}u\sqrt{4u^2+1} + \frac{1}{4}\sinh^{-1}(2u)$,

$$\frac{(2u_R^2 - h_R^2)h_R'^2 - 2u_R h_R h_R' + 4u_R^2 h_R^2 - (4u_R^2+1)u_R^2}{-u_R^2 + h_R^2}$$
$$= \left(\frac{\left(2au - \frac{1}{2}u\sqrt{4u^2+1} + \log\left(2u + \sqrt{4u^2+1}\right)\right)}{\left(-2u^2 + \frac{1}{2}\right)u - \frac{1}{4}\sqrt{4u^2+1} + \log\left(2u + \sqrt{4u^2+1}\right)}\right)^2$$

and $u_R = \sqrt{-u^2+h^2}$, $a, u, v \in \mathbb{R} \setminus \{0\}$.

Proof. We assume that the profile curve is $\gamma(u) = (u^2, u, h(u))$. Since a generalized helicoid is given by rotating the profile curve around the axis and simultaneously displacing parallel to the axis, so that the speed of displacement is proportional to the speed of rotation, from (1), we have the following representation of a generalized helicoid (Fig. 5)

$$H(u_H, v_H) = \begin{pmatrix} u_H^2 + av_H \\ u_H\cosh(v_H) + h(u_H)\sinh(v_H) \\ u_H\sinh(v_H) + h(u_H)\cosh(v_H) \end{pmatrix}$$

where a is a constant. For a moment, we assume that $h_H' \neq 0$.

The coefficients of the first fundamental form and the line element of the generalized helicoid as above are given by

$$E_H = 4u_H^2 + 1 - h_H'^2, \quad F_H = 2au_H + h_H - u_H h_H', \quad G_H = a^2 - u_H^2 + h_H,$$

$$(9) \qquad ds_H^2 = \left(4u_H^2 + 1 - h_H'^2\right)du_H^2 + 2(2au_H + h_H - u_H h_H')du_H dv_H + \left(a^2 - u_H^2 + h_H^2\right)dv_H^2.$$

Since $\gamma(u)$ is a lightlike curve and $Q_H < 0$ then $H(u_H, v_H)$ is a timelike surface.Helices in $H(u_H, v_H)$ are curves defined by $u_H = const.$, so curves in $H(u_H, v_H)$ that are orthogonal to helices supply the orthogonal condition as follow

$$(2au_H + h_H - u_H h_H')\,du_H + \left(a^2 - u_H^2 + h_H^2\right)dv_H = 0.$$

Thus we obtain

$$v_H = -\int \frac{2au_H + h_H - u_H h_H'}{a^2 - u_H^2 + h_H^2}du_H + c$$

where c is constant. Hence if we put

$$\bar{v}_H = v_H + \int \frac{2au_H + h_H - u_H h_H'}{a^2 - u_H^2 + h_H^2}du_H$$

then curves that are orthogonal to helices are given by $\bar{v}_H = const.$. Substituting the equation

$$dv_H = d\bar{v}_H - \frac{2au_H + h_H - u_H h'_H}{a^2 - u_H^2 + h_H^2} du_H$$

into the line element, we have

$$ds_H^2 = \left(E_H - \frac{\left(2au_H + h_H - u_H h'_H\right)^2}{a^2 - u_H^2 + h_H^2} \right) du_H^2 + \left(a^2 - u_H^2 + h_H^2\right) d\bar{v}_H^2. \tag{10}$$

By putting

$$\bar{u}_H = \int \sqrt{E_H - \frac{\left(2au_H + h_H - u_H h'_H\right)^2}{a^2 - u_H^2 + h_H^2}} du_H, \; f_H(\bar{u}_H) = \sqrt{a^2 - u_H^2 + h_H^2},$$

(10) reduces to

$$ds_H^2 = d\bar{u}_H^2 + f_H^2(\bar{u}_H) d\bar{v}_H^2. \tag{11}$$

On the other hand, an (S, L)−type rotation surface

$$\begin{pmatrix} u_R^2 \\ u_R \cosh(v_R) + h(u_R) \sinh(v_R) \\ u_R \sinh(v_R) + h(u_R) \cosh(v_R) \end{pmatrix} \tag{12}$$

has the line element

$$ds_R^2 = \left(4u_R^2 + 1 - h_R'^2\right) du_R^2 + \left(-u_R^2 + h_R^2\right) d\bar{v}_R^2. \tag{13}$$

Hence, if we put

$$\bar{u}_R = \sqrt{4u_R^2 + 1 - h_R'^2 - \frac{\left(h_R - u_R h'_R\right)^2}{-u_R^2 + h_R^2}}, \quad f_R(\bar{u}_R) = \sqrt{-u_R^2 + h_R^2}, \quad \bar{v}_R = v_R,$$

then (13) reduces to

$$ds_R^2 = d\bar{u}_R^2 + f_R^2(\bar{u}_R) d\bar{v}_R^2. \tag{14}$$

Comparing (10) with (13), if

$$\bar{u}_H = \bar{u}_R, \; \bar{v}_H = \bar{v}_R, \; f_H(\bar{u}_H) = f_R(\bar{u}_R),$$

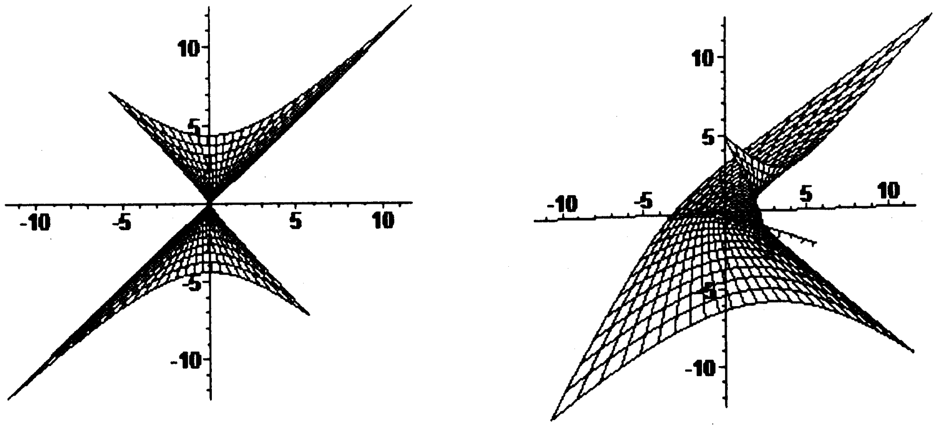


Figure 5. a-b Timelike helicoidal surface with (S,L)-type.

then we have an isometry between $H(u_H, v_H)$ and $R(u_R, v_R)$. Therefore it follows that

$$\int \sqrt{E_H - \frac{F_H^2}{G_H}} du_H = \int \sqrt{(4u_R^2 + 1 - h_R'^2) - \frac{(h_R - u_R h_R')^2}{-u_R^2 + h_R^2}} du_R$$

$$= \int \sqrt{4u_R^2 + 1 - h_R'^2 - \frac{(h_R - u_R h_R')^2}{-u_R^2 + h_R^2} \cdot \frac{-u_H + h_H h_H'}{\sqrt{a^2 - u_H^2 + h_H^2}}} du_H$$

and we have

$$\left(-\frac{F_H}{-u + h_H h_H'} \right)^2 = \frac{(2u_R^2 - h_R^2) h_R'^2 - 2u_R h_R h_R' + 4u_R^2 h_R^2 - (4u_R^2 + 1) u_R^2}{-u_R^2 + h_R^2}.$$

□

THEOREM 3.2. *In Theorem 3.1, if two surfaces have the same Gauss map then, we have*

$$2(u - hh') = h_R h_R',$$

$$(2uh - a)\sinh(v) + (2u^2 - ah')\cosh(v)$$
$$= -2(-u + hh')\left[\sqrt{-u^2 + h^2}\cosh(v_R) + h_R \sinh(v_R)\right]$$

and

$$-(2uh - a)\cosh(v) - (2u^2 - ah')\sinh(v)$$
$$= -2(-u + hh')\left[\sqrt{-u^2 + h^2}\sinh(v_R) + h_R \cosh(v_R)\right]$$

where $h = \int \sqrt{4u^2 + 1} du$, h_R *in Th. 3.1.*, $v_R = v - \int \frac{2au + h - uh'}{a^2 - u^2 + h^2} du$ *and* $u, a \in R \setminus \{0\}$. *If these surfaces are minimal then, we have* $\Im(u) = 0$ *and* $\wp(u) = 0$ *where* $\Im(u)$ *and* $\wp(u)$ *are functions in* H_H, H_R *respectively.*

Proof. First we consider a helicoid (7). Differentiating H_u and H_v, we obtain

$$H_{uu} = (2, h'' \sinh(v), h'' \cosh(v)),$$

$$H_{uv} = (0, \sinh(v) + h' \cosh(v), \cosh(v) + h' \sinh(v)),$$

$$H_{vv} = (0, u\cosh(v) + h\sinh(v), u\sinh(v) + h\cosh(v)).$$

By virtue of the first and second fundamental forms

$$E_H = 4u^2 + 1 - h'^2, \quad F_H = 2au + h - uh', \quad G_H = a^2 - u^2 + h^2,$$

$$L_H = \frac{2(u - hh') + \Psi h''}{\sqrt{|Q_H|}}, \quad M_H = \frac{\Theta + \Gamma}{\sqrt{|Q_H|}}, \quad N_H = \frac{\xi + \Omega}{\sqrt{|Q_H|}},$$

the Gauss map and the mean curvature of the generalized helicoid is given by

$$e_H = \frac{1}{\sqrt{|Q_H|}} \begin{pmatrix} u - hh' \\ (2uh - a)\sinh(v) + (2u^2 - ah')\cosh(v) \\ -(2uh - a)\cosh(v) - (2u^2 - ah')\sinh(v) \end{pmatrix}, \tag{15}$$

$$H_H = \frac{\Im(u)}{2\,(Q_H)^{3/2}} \tag{16}$$

where

$$\begin{aligned}\Im(u) = &-\left(4u^2 + 1 - h'\right) \cdot (\xi + \Omega) - (a^2 - u^2 + h^2) \cdot (2(u - hh') + \Psi h'') \\ &+ 2\left(2au + h - uh'\right) \cdot (\Theta + \Gamma),\end{aligned}$$

$$\Psi(u) = (2uh - a) \cdot \left(\sinh^2(v) + \cosh^2(v)\right) + 2\left(2u^2 - ah'\right) \cdot \sinh(v)\cosh(v),$$

$$\Theta(u) = 2\left[(2uh - a)h' + \left(2u^2 - ah'\right)\right] \cdot \sinh(v)\cosh(v),$$

$$\Gamma(u) = \left[(2uh - a) + \left(2u^2 - ah'\right)h'\right] \cdot \left(\sinh^2(v) + \cosh^2(v)\right),$$

$$\xi(u) = \left[2(2uh - a)u + \left(2u^2 - ah'\right)h\right] \cdot \sinh(v)\cosh(v),$$

and

$$\Omega(u) = \left[(2uh - a)h + \left(2u^2 - ah'\right)u\right] \cdot \left(\sinh^2(v) + \cosh^2(v)\right).$$

Next we calculate the Gauss map and the mean curvature of the rotation surface. Since

$$R_u = \begin{pmatrix} 2(-u + hh') \\ A\cosh(v_R) + B\sinh(v_R) \\ A\sinh(v_R) + B\cosh(v_R) \end{pmatrix}, \quad R_v = \begin{pmatrix} 0 \\ \sqrt{-u^2 + h^2}\sinh(v_R) + h_R\cosh(v_R) \\ \sqrt{-u^2 + h^2}\cosh(v_R) + h_R\sinh(v_R) \end{pmatrix},$$

the Gauss map of the rotation surface is given by

$$e_R = \frac{1}{\sqrt{|Q_R|}} \begin{pmatrix} -u + hh' + h_R h'_R \\ -2(-u + hh')\left[\sqrt{-u^2 + h^2}\cosh(v_R) + h_R\sinh(v_R)\right] \\ -2(-u + hh')\left[\sqrt{-u^2 + h^2}\sinh(v_R) + h_R\cosh(v_R)\right] \end{pmatrix}. \tag{17}$$

where $v_R = v - \int \frac{2au + h - uh'}{a^2 - u^2 + h^2} du$, $E_R = 4(-u + hh')^2 + \left(A^2 - B^2\right)$, $F_R = Ah_R - B\sqrt{-u^2 + h^2}$, $G_R = u - hh' + h_R^2$, $Q_R = u - hh' + h_R h'_R - 4(-u + hh')^2\left(-u^2 + h^2 + h_R^2\right)$, $A = \frac{-u + hh'}{\sqrt{-u^2 + h^2}} - \frac{2au + h - uh'}{a^2 - u^2 + h^2} h_R$ and $B = h'_R - \frac{(2au + h - uh')\sqrt{-u^2 + h^2}}{a^2 - u^2 + h^2}$.
Moreover, we have

$$R_{uu} = \begin{pmatrix} 2(-1 + h'^2 + hh'') \\ \rho(u)\cosh(v_R) + \eta(u)\sinh(v_R) \\ \rho(u)\sinh(v_R) + \eta(u)\cosh(v_R) \end{pmatrix}, \quad R_{uv} = \begin{pmatrix} 0 \\ A\sinh(v_R) + B\cosh(v_R) \\ A\cosh(v_R) + B\sinh(v_R) \end{pmatrix},$$

$$R_{vv} = \begin{pmatrix} 0 \\ \sqrt{-u^2 + h^2}\cosh(v_R) + h_R\sinh(v_R) \\ \sqrt{-u^2 + h^2}\sinh(v_R) + h_R\cosh(v_R) \end{pmatrix},$$

where $\rho(u) = A_u + \frac{2au + h - uh'}{a^2 - u^2 + h^2} B$, $\eta(u)\frac{2au + h - uh'}{a^2 - u^2 + h^2} A + B_u$.

By the straight calculation, we have the coefficients of the second fundamental form as follows

$$L_R = \frac{C}{\sqrt{|Q_R|}}, \quad M_R = \frac{-2(-u + hh')\left(B\sqrt{-u^2 + h^2} - Ah_R\right)}{\sqrt{|Q_R|}}, \quad N_R = 0.$$

Hence the mean curvature is

$$H_R = \frac{\wp(u)}{2(Q_R)^{3/2}} \tag{18}$$

where $\wp(u) = -(u - hh' + h_R^2)C + 4(hh' - u)\left(B\sqrt{h^2 - u^2} - Ah_R\right)^2$ and $C = 2(-1 + h'^2 + hh'')(-u + hh' - h_R h'_R) - 2(-u + hh') \cdot [\sqrt{-u^2 + h^2}\rho(u) - \eta(u)h_R]$.

If the generalized helicoid and the rotation surface have the same Gauss map, comparing (15) and (17), then we obtain an interesting differential equations as follow

$$2(u - hh') = h_R h'_R,$$

$$\begin{aligned}&(2uh - a)\sinh(v) + (2u^2 - ah')\cosh(v)\\&\quad = -2(-u + hh')\left[\sqrt{-u^2 + h^2}\cosh(v_R) + h_R\sinh(v_R)\right]\end{aligned}$$

and

$$\begin{aligned}&-(2uh - a)\cosh(v) - (2u^2 - ah')\sinh(v)\\&\quad = -2(-u + hh')\left[\sqrt{-u^2 + h^2}\sinh(v_R) + h_R\cosh(v_R)\right].\end{aligned}$$

From (16) and (18), if

$$\begin{aligned}&-\left(4u^2 + 1 - h'\right)\cdot(\xi + \Omega) - (a^2 - u^2 + h^2)\cdot(2(u - hh') + \Psi h'')\\&\quad +2(2au + h - uh')\cdot(\Theta + \Gamma) = 0,\end{aligned}$$

and

$$-(u - hh' + h_R^2)C + 4(hh' - u)\left(B\sqrt{h^2 - u^2} - Ah_R\right)^2 = 0$$

then, these equations means that the generalized helicoid and the rotation surface are minimal. □

We continue to show the other case $(T, L) - type$. The techniques of proofs of the next theorems are similar.

Case 2. *$(T, L) - type$.*

We assume that the axis l is timelike $(0, 0, 1)$ vector and the profile curve is lightlike.

THEOREM 3.3. *A generalized helicoid*

$$\begin{pmatrix} u^2\cos(v) - u\sin(v)\\ u^2\sin(v) + u\cos(v)\\ h_H + av\end{pmatrix} \tag{19}$$

is isometric to a rotation surface

(20)

$$\begin{pmatrix} (u^4+u^2-a^2)\cos(v+\int\frac{-u^2+ah'}{u^4+u^2-a^2}du) - \sqrt{u^4+u^2-a^2}\sin(v+\int\frac{-u^2+ah'}{u^4+u^2-a^2}du)\\ (u^4+u^2-a^2)\sin(v+\int\frac{-u^2+ah'}{u^4+u^2-a^2}du) + \sqrt{u^4+u^2-a^2}\cos(v+\int\frac{-u^2+ah'}{u^4+u^2-a^2}du)\\ h_R\end{pmatrix}$$

so that helices on the generalized helicoid correspond to parallel circles on the rotation surface where $h_H = \frac{1}{2}u\sqrt{4u^2+1} + \frac{1}{4}\sinh^{-1}(2u)$,

$$\begin{aligned} h_R = & -2a\sinh^{-1}(2u) + \sqrt{2}a\tanh^{-1}\left(\frac{\sqrt{2}u}{\sqrt{4u^2+1}}\right) - \frac{\sqrt{2}}{4}\tan^{-1}(\sqrt{2}u) \\ & + \frac{2}{\sqrt{-3+4a^2}\sqrt{2-2\sqrt{-3+4a^2}}}\tan^{-1}\left(\frac{2u}{\sqrt{2-2\sqrt{-3+4a^2}}}\right) \\ & - \frac{2}{\sqrt{-3+4a^2}\sqrt{2+2\sqrt{-3+4a^2}}}\tan^{-1}\left(\frac{2u}{\sqrt{2+2\sqrt{-3+4a^2}}}\right) \\ & + u^5 + \frac{5}{3}u^3 + \left(-5a^2 + \frac{7}{2}\right)u + c_2, \end{aligned}$$

and $c_2 \in \mathbb{R}$, $a, u, v \in \mathbb{R} \setminus \{0\}$.

Proof. We assume that the profile curve is $\gamma(u) = (u^2, u, h(u))$. From (3), we have the following representation of a generalized helicoid (see Fig. 6)

$$H(u_H, v_H) = \begin{pmatrix} u_H^2\cos(v_H) - u_H\sin(v_H) \\ u_H^2\sin(v_H) + u_H\cos(v_H) \\ h(u_H) + av_H \end{pmatrix}$$

where $a \neq 0$ is a constant.

The coefficients of the first fundamental form of the generalized helicoid are given by

$$E_H = 4u_H^2 + 1 - h_H'^2, F_H = -u_H^2 + ah'_H,\ G_H = u_H^4 + u_H^2 - a^2.$$

Since $\gamma(u)$ lightlike curve and $Q_H < 0$ then $H(u_H, v_H)$ is a timelike surface. Thus we obtain

$$ds_H^2 = \left(4u_H^2 + 1 - h_H'^2 - \frac{\left(-u_H^2 + ah'_H\right)^2}{u_H^4 + u_H^2 - a^2}\right)du_H^2 + \left(u_H^4 + u_H^2 - a^2\right)d\bar{v}_H^2. \tag{21}$$

On the other hand, an $(T, L)-$type rotation surface

$$\begin{pmatrix} u_R^2\cos(v_R) - u_R\sin(v_R) \\ u_R^2\sin(v_R) + u_R\cos(v_R) \\ h(u_R) \end{pmatrix} \tag{22}$$

has the line element

$$ds_R^2 = \left(4u_R^2 + 1 - h_R'^2 - \frac{u_R^2}{1+u_H^2}\right)du_R^2 + \left(u_R^4 + u_R^2\right)d\bar{v}_R^2. \tag{23}$$

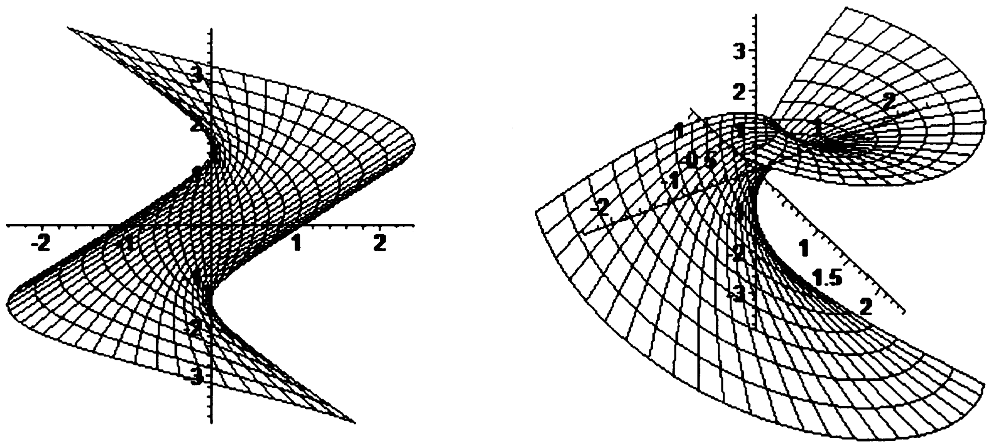


Figure 6. a-b Timelike helicoidal surface with (T,L)-type.

Comparing (21) with (23), if $\bar{u}_H = \bar{u}_R$, $\bar{v}_H = \bar{v}_R$, $f_H(\bar{u}_H) = f_R(\bar{u}_R)$, then we have an isometry between $H(u_H, v_H)$ and $R(u_R, v_R)$. Therefore it follows that

$$\int\sqrt{4u_H^2+1-h_H'^2-\frac{(-u_H^2+ah'_H)^2}{u_H^4+u_H^2-a^2}}du_H = \int\sqrt{4u_R^2+1-h_R'^2-\frac{u_R^4}{1+u_R^2}}du_R$$
$$= \int\sqrt{4u_R^2+1-h_R'^2}\frac{2u_H^3+u_H}{\sqrt{u_H^4+u_H^2-a^2}}du_H$$

and we obtain

$$h'_R$$
$$=\int\left(-\frac{(4u_H^2+1-h_H'^2)(u_H^4+u_H^2-a^2)-(-u_H^2+ah'_H)^2}{u_H^2(2u_H^2+1)}-\frac{(u_H^4+u_H^2-a^2)^2}{(1+u_H^4+u_H^2-a^2)}\right)du_H$$
$$+4u_H\left(\frac{1}{5}u_H^4+\frac{1}{3}u_H^2-a^2+1\right).$$

where $a, u, v \in \mathbb{R}\setminus\{0\}$. □

THEOREM 3.4. *In Theorem 3.3, if two surfaces have the same Gauss map then, we have*

$$a = \mu u_R\left(\sin\left(\int\frac{-u^2+ah'}{u^4+u^2-a^2}du\right) - u_R\cos\left(\int\frac{-u^2+ah'}{u^4+u^2-a^2}du\right)\right) + u^2h',$$

$$-h'u = \mu u_R\left(\cos\left(\int\frac{-u^2+ah'}{u^4+u^2-a^2}du\right) + u_R\sin\left(\int\frac{-u^2+ah'}{u^4+u^2-a^2}du\right)\right) - 2au$$

and

$$\left(1-2u^2\right)u=\left(\lambda-u_R\delta\right)u_R.$$

where $u,a\in\mathbb{R}\setminus\{0\}$. *If these surfaces are minimal then, we have* $\chi(u)=0$ *and* $\pi(u)=0$ *where* $\chi(u)$ *and* $\pi(u)$ *are functions in* H_H, H_R *respectively.*

Proof. We consider a helicoid (19). By virtue of the first and second fundamental forms

$$E_H=4u^2+1-h'^2,\quad F_H=-u^2+ah',\quad G_H=u^4+u^2-a^2,$$

$$L_H=\frac{4u\left(2a-h'\right)\sin(v)\cos(v)+2(a-u^2h')}{\sqrt{\left[-\left(u^2+1\right)\left(4u^2+h'^2\right)+2ah'-4a^2\right]u^2-a^2}},$$

$$M_H=\frac{\left(2a-h'\right)\left[2u^2\left(\cos^2(v)-\sin^2(v)\right)-2u\sin(v)\cos(v)\right]+\left(a-u^2h'\right)}{\sqrt{\left[-\left(u^2+1\right)\left(4u^2+h'^2\right)+2ah'-4a^2\right]u^2-a^2}},$$

$$N_H=\frac{-\left(2a-h'\right)u^2\left[\left(\cos^2(v)-\sin^2(v)\right)+2u\sin(v)\cos(v)\right]+\left(a-u^2h'\right)u^2}{\sqrt{\left[-\left(u^2+1\right)\left(4u^2+h'^2\right)+2ah'-4a^2\right]u^2-a^2}},$$

the Gauss map and the mean curvature of the generalized helicoid is given by

$$e_H=\frac{1}{\sqrt{|Q_H|}}\begin{pmatrix}\left(2a-h'\right)u\sin(v)+\left(a-u^2h'\right)\cos(v)\\ \left(2a-h'\right)u\cos(v)+\left(a-u^2h'\right)\sin(v)\\ -2u^3+u\end{pmatrix}, \tag{24}$$

$$H_H=\frac{\chi(u)}{2\left\{Q_H\right\}^{3/2}} \tag{25}$$

where

$$\begin{aligned}\chi(u)=\{&\left(4u^2+1-h'^2\right)\cdot\left[-2u^3\left(2a-h'\right)\sin(v)\cos(v)+\left(a-u^2h'\right)u^2\right.\\ &\left.-u^2\left(2a-h'\right)\left(\cos^2(v)-\sin^2(v)\right)\right]+2\left(u^4+u^2-a^2\right)\cdot\left[2u\left(2a-h'\right)\sin(v)\cos(v)\right.\\ &\left.+\left(a-u^2h'\right)\right]-2\left(-u^2+ah'\right)\cdot\left[2u^2\left(2a-h'\right)\left(\cos^2(v)-\sin^2(v)\right)\right.\\ &\left.-2u\left(2a-h'\right)\sin(v)\cos(v)+\left(a-u^2h'\right)\right]\}.\end{aligned}$$

Next we have the coefficients of the first fundamental form of the rotation surface as follow

$$E_R=\lambda^2+\delta^2-\mu^2,\quad F_R=u_R\left(u_R\lambda-\delta\right),\quad G_R=u_R^4+u_R^2.$$

The Gauss map of the rotation surface is

(26)
$$e_R = \frac{1}{\sqrt{|Q_R|}} \begin{pmatrix} \left(\sin(v + \int \frac{-u^2+ah'}{u^4+u^2-a^2}du) - u_R \cos(v + \int \frac{-u^2+ah'}{u^4+u^2-a^2}du)\right) u_R\mu \\ \left(\cos(v + \int \frac{-u^2+ah'}{u^4+u^2-a^2}du) + u_R \sin(v + \int \frac{-u^2+ah'}{u^4+u^2-a^2}du)\right) u_R\mu \\ (\lambda - u_R\delta)\, u_R \end{pmatrix}.$$

where $u_R = \sqrt{u^4+u^2-a^2}$, $Q_R = \left[\left(u_R^2+1\right)\mu^2 - \left(-\lambda + u_R\delta\right)^2\right] u_R^2$, $\lambda(u) = -u^2 + ah' + \frac{2u^3+u}{u_R}$, $\delta(u) = 2\left(2u^3+u\right) - \frac{-u^2+ah'}{u_R}$ and $\mu(u) = \frac{u^2-2a\sqrt{4u^2+1}}{2u^2+1} + \frac{5u_R^4+8u_R^2+4}{u_R^2+1}$.

The coefficients of the second fundamental form as follow

$$L_R = \frac{1}{\sqrt{|Q_R|}}\{[2\left(u_R\beta+\omega\right)\sin(v_R)\cos(v_R) + \beta\left(\cos(v_R)-\sin(v_R)\right)$$
$$-u_R\omega\left(\cos^2(v_R)-\sin^2(v_R)\right)]\mu - [\lambda - u_R\delta]\,\mu_u\}u_R,$$

$$M_R = \frac{\left[\left(u_R\lambda+\delta\right)\left(\cos^2(v_R)-\sin^2(v_R)\right) + 2\left(-\lambda+u_R\delta\right)\sin(v_R)\cos(v_R)\right]\mu u_R}{\sqrt{|Q_R|}},$$

$$N_R = \frac{\left[\left(u_R^2-1\right)\left(\cos^2(v_R)-\sin^2(v_R)\right) - 4u_R\sin(v_R)\cos(v_R)\right]u_R}{\sqrt{|Q_R|}}.$$

where $\beta(u) = \lambda_u + \frac{-u^2+ah'}{u^4+u^2-a^2}\delta$, $\omega(u) = -\frac{-u^2+ah'}{u^4+u^2-a^2}\lambda + \delta_u$, $v_R = v + \int \frac{-u^2+ah'}{u^4+u^2-a^2}du$.

Hence the mean curvature is

(27)
$$H_R = \frac{\pi(u)}{2\left\{Q_R\right\}^{3/2}}.$$

where

$$\pi(u) = \{\left(\lambda^2+\delta^2-\mu^2\right)\cdot\left[\left(u_R^2-1\right)\left(\cos^2(v_R)-\sin^2(v_R)\right)-4u_R\sin(v_R)\cos(v_R)\right]u_R$$
$$+\left(u_R^4+u_R^2\right)\cdot(\{[2\left(u_R\beta+\omega\right)\sin(v_R)\cos(v_R)+\beta\left(\cos(v_R)-\sin(v_R)\right)$$
$$-u_R\omega\left(\cos^2(v_R)-\sin^2(v_R)\right)]\mu-[\lambda-u_R\delta]\,\mu_u\}u_R)-2\left[u_R\left(u_R\lambda-\delta\right)\right]$$
$$\cdot\left[\left(u_R\lambda+\delta\right)\left(\cos^2(v_R)-\sin^2(v_R)\right)+2\left(-\lambda+u_R\delta\right)\sin(v_R)\cos(v_R)\right]\mu u_R\}.$$

If the generalized helicoid and the rotation surface have the same Gauss map, comparing (24) and (26), we have

$$a = \mu u_R \left(\sin(\int \frac{-u^2 + ah'}{u^4 + u^2 - a^2} du) - u_R \cos(\int \frac{-u^2 + ah'}{u^4 + u^2 - a^2} du) \right) + u^2 h',$$

$$-h'u = \mu u_R \left(\cos(\int \frac{-u^2 + ah'}{u^4 + u^2 - a^2} du) + u_R \sin(\int \frac{-u^2 + ah'}{u^4 + u^2 - a^2} du) \right) - 2au$$

and

$$\left(1 - 2u^2\right) u = \left(\lambda - u_R \delta\right) u_R.$$

If $\chi(u) = 0$ and $\pi(u) = 0$ then, the generalized helicoid and the rotation surface are minimal from (25) and (27). □

Case 3. $(L, L) - type$.

We assume that the axis l is lightlike $(0, 1, 1)$ vector and the profile curve is lightlike.

THEOREM 3.5. *A generalized helicoid*

$$\begin{pmatrix} u^2 - uv + hv \\ u^2 v + \left(1 - \frac{v^2}{2}\right) u + \frac{v^2}{2} h + av \\ u^2 v - \frac{v^2}{2} u + \left(1 + \frac{v^2}{2}\right) h + av \end{pmatrix} \tag{28}$$

is isometric to a rotation surface

$$\begin{pmatrix} u_R^2 - u_R v_R + h_R v_R \\ u_R^2 v_R + \left(1 - \frac{v_R^2}{2}\right) u_R + \frac{v_R^2}{2} h_R \\ u_R^2 v_R - \frac{v_R^2}{2} u_R + \left(1 + \frac{v_R^2}{2}\right) h_R \end{pmatrix} \tag{29}$$

so that helices on the generalized helicoid correspond to parallel circles on the rotation surface where $u_R = -u + h$, $v_R = v + \int \left(2u + \frac{(u^2+a)(1-h')}{-u+h}\right)^2 du$, $h_H = \frac{1}{2} u \sqrt{4u^2 + 1} + \frac{1}{4} \sinh^{-1}(2u)$,

$$h_R'^2 + \frac{\left(-u_H + h_H\right)^3 \left(1 - h_R'\right)}{-u_H + h_H + h_R} \left(\frac{\left(-u_H + h_H\right)\left(1 - h_R'\right)}{-u_H + h_H + h_R} + 4 \right) \tag{30}$$

$$+ \left\{ \left(\left(4u_H^2 + 1 - h_H'^2\right) - \frac{\left[(-u_H + h_H) 2u_H + (1 - h_H') u_H^2\right]^2}{\left(-u_H + h_H\right)^2} \right) \frac{1}{\left(-1 + h_H'\right)^2} \right\} - 1 = 0,$$

$u, v, a \in \mathbb{R} \setminus \{0\}$.

Proof. We assume that the profile curve is $\gamma(u) = (u^2, u, h(u))$. From (5), we have the following representation of a generalized helicoid (see Fig. 7)

$$H(u_H, v_H) = \begin{pmatrix} u_H^2 - u_H v_H + h_H v_H \\ u_H^2 v_H + \left(1 - \frac{v_H^2}{2}\right) u_H + \frac{v_H^2}{2} h_H + a v_H \\ u_H^2 v_H - \frac{v_H^2}{2} u_H + \left(1 + \frac{v_H^2}{2}\right) h_H + a v_H \end{pmatrix}$$

where $a \neq 0$ is a constant.

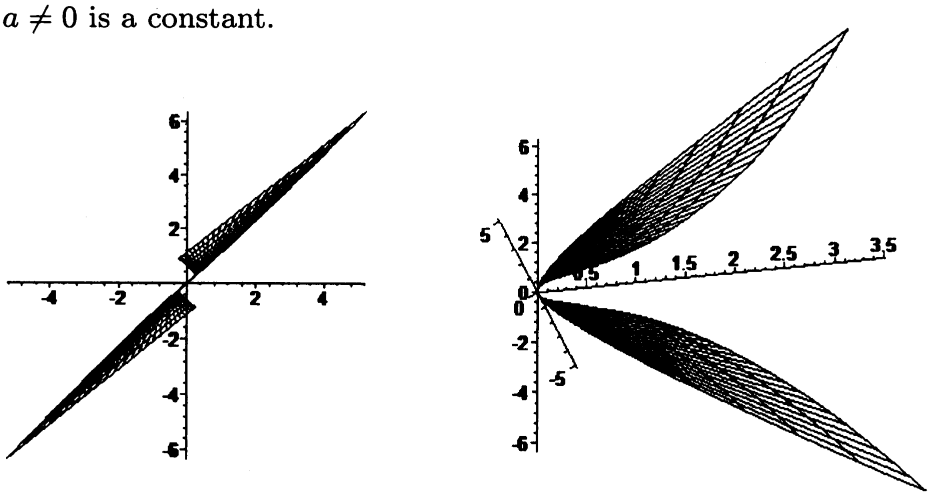


Figure 7. a-b Timelike helicoidal surface with (L,L)-type.

The coefficients of the first fundamental form are given by

$$E_H = 4u_H^2 + 1 - h_H'^2,\ F_H = (-u_H + h_H)2u_H + (u_H^2 + a)(1 - h_H'),\ G_H = (-u_H + h_H)^2.$$

Since $\gamma(u)$ is a lightlike curve and $Q_H < 0$ then $H(u_H, v_H)$ is a timelike surface and

$$(31)\quad ds_H^2 = \left(4u_H^2 + 1 - h_H'^2 - \frac{\left((-u_H + h_H)2u_H + (u_H^2 + a)(1 - h_H')\right)^2}{(-u_H + h_H)^2}\right) du_H^2 + (-u_H + h_H)^2 d\bar{v}_H^2.$$

On the other hand, an $(L, L)-$type rotation surface has the line element

$$(32)\qquad ds_R^2 = \left(4u_R^2 + 1 - h_R'^2 - \frac{\left[(-u_R + h_R)2u_R + (1 - h_R')u_R^2\right]^2}{(-u_R + h_R)^2}\right) du_R^2 + (-u_R + h_R)^2 d\bar{v}_R^2.$$

Comparing (31) with (32), if $\bar{u}_H = \bar{u}_R$, $\bar{v}_H = \bar{v}_R$, $f_H(\bar{u}_H) = f_R(\bar{u}_R)$,then we have an isometry between $H(u_H, v_H)$ and $R(u_R, v_R)$. Therefore we obtain

$$h_R'^2 + \frac{\left(-u + \int \sqrt{4u^2+1}du\right)^3 (1-h_R')}{-u + \int \sqrt{4u^2+1}du + h_R} \left(\frac{\left(-u + \int \sqrt{4u^2+1}du\right)(1-h_R')}{-u + \int \sqrt{4u^2+1}du + h_R} + 4 \right)$$
$$+ \left[\frac{\left(-u + \int \sqrt{4u^2+1}du\right) 2u + \left(1 - \sqrt{4u^2+1}du\right) u^2}{\left(1 - \sqrt{4u^2+1}du\right)\left(-u + \int \sqrt{4u^2+1}du\right)} \right]^2 - 1 = 0.$$

□

THEOREM 3.6. *In Theorem 3.5, if two surfaces have the same Gauss map then, we have*

$$\left(-u^2 + uv - hv - a\right) h' + vh + \left(u^2 - uv + a\right) = \alpha_1(u)h_R' + \alpha_2(u)h_R + \alpha_3(u)$$

$$\left(\sigma(u) - u + h\right) h' + \frac{v^2}{2} h + \vartheta(u) = \beta_1(u)h_R' + \beta_2(u)h_R + \beta_3(u)$$

and

$$\left(\sigma(u) - v^2\right) h' + \left(\frac{v^2}{2} - 1\right) h + \vartheta(u) = \theta_1(u)h_R' + \theta_2(u)h_R + \theta_3(u).$$

where $\alpha_1(u) = u_R v_R - v_R h_R - u_R^2$, $\alpha_2(u) = -u_R v_R^2 v_R' + v_R u_R' + v_R^2 v_R' h_R + u_R^2 v_R v_R'$, $\alpha_3(u) = -u_R v_R u_R' + u_R^2 u_R'$, $\sigma(u) = -u^2 v + \frac{v^2}{2} h - av$, $\vartheta(u) = -2u^3 - 2au + u^2 v - \frac{uv^2}{2} + av$, $\beta_1(u) = 2^{-1} u_R v_R^2 - u_R - 2^{-1} v_R^2 h_R + h_R - u_R^2 v_R$, $\beta_2(u) = 2^{-1} v_R^2 u_R'$, $\beta_3(u) = 2^{-1} u_R v_R^2 u_R' - 2u_R^3 u_R' + u_R^2 v_R u_R'$, $\theta_1(u) = 2^{-1} u_R v_R^2 - 2^{-1} v_R^2 h_R - u_R^2 v_R$, $\theta_2(u) = -u_R v_R v_R' + 2^{-1} v_R^2 u_R' + v_R v_R' h_R + u_R'$, $\theta_3(u) = 2^{-1} u_R v_R^2 u_R' - u_R u_R' - 2u_R^3 u_R' + u_R^2 v_R u_R'$, $u_R = -u + h$, $v_R = v + \int \left(2u + \frac{(u^2+a)(1-h')}{-u+h}\right)^2 du$, $h = \frac{1}{2} u \sqrt{4u^2+1} + \frac{1}{4} \sinh^{-1}(2u)$, $u, v, a \in \mathbb{R} \setminus \{0\}$. *If these surfaces are minimal then, we have* $\varsigma(u) = 0$ *and* $\kappa(u) = 0$ *where* $\varsigma(u)$ *and* $\kappa(u)$ *are functions in* H_H, H_R *respectively.*

Proof. We consider a helicoid (28). The Gauss map and the mean curvature of the generalized helicoid is given by

$$(33) \qquad e_H = \frac{1}{\sqrt{|Q_H|}} \begin{pmatrix} \left(-u^2 + uv - hv - a\right) h' + vh + \left(u^2 - uv + a\right) \\ \left(\sigma(u) - u + h\right) h' + \frac{v^2}{2} h + \vartheta(u) \\ \left(\sigma(u) - v^2\right) h' + \left(\frac{v^2}{2} - 1\right) h + \vartheta(u) \end{pmatrix},$$

$$H_H = \frac{\varsigma(u)}{2\left\{Q_H\right\}^{3/2}} \tag{34}$$

where $\sigma(u) = -u^2v + \frac{v^2}{2}h - av$, $\vartheta(u) = -2u^3 - 2au + u^2v - \frac{uv^2}{2} + av, \varsigma(u) = (4u^2+1-h'^2)(-u+h)[(-u+h+v^2)h'+h]+(-u+h)^2\cdot(u^2-uv+hv+a)(1-h')+(2v+\frac{v^2}{2}h'')[(-u+h+v^2)h'+h]-[(\sigma(u)-v^2)h'+(\frac{v^2}{2}-1)h+\vartheta(u)]h''-2[2u\,(-u+h)+(u^2+a)(1-h')]\cdot\{(-1-h')^2(u^2-uv+hv+a)+(2u-v+vh')[(-u+h+v^2)h'+h]\}$.

Next we have the coefficients of the first fundamental form of the rotation surface as follow

$$\begin{aligned} E_R = {} & -h_R'^2 + \left(v_R^3 v_R' h_R - 2u_R^2 v_R'\right) h_R' + \left(v_R'^2 + 3v_R^2 v_R'^2\right) h_R^2 + (4u_R u_R' v_R \\ & -2v_R'^2 u_R + 2v_R v_R'^2 u_R^2 - v_R^3 v_R' u_R' + 2v_R v_R' u_R' - 2v_R^2 v_R'^2 u_R \\ & +4v_R^2 v_R' u_R u_R')h_R + v_R'^2 u_R^2 - 2u_R^2 v_R' u_R' + u_R'^2 + 4u_R^2 u_R'^2, \end{aligned}$$

$$\begin{aligned} F_R = {} & v_R' h_R'^2 - u_R^2 h_R' + (u_R^2 v_R v_R' - v_R^2 v_R' u_R + v_R v_R' h_R \\ & -2v_R' u_R + 2u_R u_R')h_R^2 + (-u_R^2 + v_R')u_R^2, \end{aligned}$$

$$G_R = \left(-u_R + h_R\right)^2$$

and we have $Q_R = -v_R'^2 h_R'^4 + 2u_R^2 v_R' h_R'^3 + (-4u_R u_R' v_R' h_R - u_R^2 - 2u_R^2 v_R'^2 + 4u_R v_R'^2 h_R - 2v_R v_R'^2 u_R^2 h_R + 2u_R^2 v_R' u_R' - 2v_R v_R'^2 h_R^2 + 2v_R^2 u_R v_R'^2 h_R + 2u_R h_R - u_R^4 - h_R^2)h_R'^2 + (4u_R^3 u_R' h_R - 2u_R^4 u_R' - 2v_R^3 v_R' u_R h_R^2 + v_R^3 v_R' u_R^2 h_R + v_R^3 v_R' h_R^3 - 2u_R^2 v_R' h_R^2 - 2u_R^3 v_R^2 v_R' h_R + 2u_R^4 v_R v_R' h_R + 2u_R^2 v_R v_R' h_R^2)h_R' + (v_R'^2 + 2v_R^2 v_R'^2)h_R^4 + (2v_R^3 v_R'^2 u_R + 4v_R v_R'^2 u_R + 4u_R u_R' v_R' + 2v_R v_R' u_R' + 2u_R^2 v_R v_R'^2 - 8u_R v_R^2 v_R'^2 - v_R^3 u_R' v_R' + 4u_R v_R^2 u_R' v_R' - 4u_R v_R'^2 - 2u_R^2 v_R^2 v_R'^2 - 4v_R u_R v_R' u_R')h_R^3 + (2v_R'^2 u_R^2 + 2v_R v_R' u_R^2 u_R' + u_R^2 - 2u_R^2 u_R' v_R' - 4v_R u_R v_R' u_R' + 2v_R^3 u_R^3 v_R'^2 + 3v_R^2 u_R^2 v_R'^2 - 4u_R^2 v_R^2 u_R' v_R' - u_R^4 v_R^2 v_R'^2 - 4u_R^3 v_R v_R' u_R' - u_R^2 v_R^4 v_R'^2 - 2u_R^2 v_R v_R'^2 + 2u_R v_R^3 v_R' u_R')h_R^2 + (-u_R^2 v_R^3 u_R' v_R' - 2u_R u_R'^2 - 4u_R^3 u_R'^2 + 2u_R^2 v_R v_R' u_R' + 2u_R^3 v_R^2 v_R' u_R' + 2u_R^4 v_R v_R' u_R')h_R + 3u_R^4 u_R'^2 + u_R^2 u_R'^2$, where $u_R := -u + h$, $u_R' := \frac{\partial}{\partial u} u_R$, $v_R := v + \int \left(2u + \frac{(u^2+a)(1-h')}{-u+h}\right)^2 du$, $v_R' := \frac{\partial}{\partial u} v_R$.

The Gauss map of the rotation surface is

$$e_R = \frac{1}{\sqrt{|Q_R|}} \begin{pmatrix} \alpha_1 h_R' + \alpha_2 h_R + \alpha_3 \\ \beta_1 h_R' + \beta_2 h_R + \beta_3 \\ \theta_1 h_R' + \theta_2 h_R + \theta_3 \end{pmatrix} \tag{35}$$

where $\alpha_1(u) = u_R v_R - v_R h_R - u_R^2$, $\alpha_2(u) = -u_R v_R^2 v_R' + v_R u_R' + v_R^2 v_R' h_R + u_R^2 v_R v_R'$, $\alpha_3(u) = -u_R v_R u_R' + u_R^2 u_R'$, $\beta_1(u) = 2^{-1} u_R v_R^2 - u_R - 2^{-1} v_R^2 h_R + h_R - u_R^2 v_R$, $\beta_2(u) = 2^{-1} v_R^2 u_R'$, $\beta_3(u) = 2^{-1} u_R v_R^2 u_R' - 2u_R^3 u_R' + u_R^2 v_R u_R'$, and $\theta_1(u) = 2^{-1} u_R v_R^2 - 2^{-1} v_R^2 h_R - u_R^2 v_R$, $\theta_2(u) = -u_R v_R v_R' + 2^{-1} v_R^2 u_R' + v_R v_R' h_R + u_R'$, $\theta_3(u) = 2^{-1} u_R v_R^2 u_R' - u_R u_R' - 2u_R^3 u_R' + u_R^2 v_R u_R'$.

Hence the mean curvature is

$$H_R = \frac{\kappa(u)}{2\left\{Q_R\right\}^{3/2} } \tag{36}$$

where $\kappa(u) = [-h'^2_R + \left(v^3_R v'_R h_R - 2u^2_R v'_R\right) h'_R + \left(v'^2_R + 3v^2_R v'^2_R\right) h^2_R + (4u_R u'_R v_R -$
$2v'^2_R u_R + 2v_R v'^2_R u^2_R - v^3_R v'_R u'_R + 2v_R v'_R u'_R - 2v^2_R v'^2_R u_R + 4v^2_R v'_R u_R u'_R) h_R + v'^2_R u^2_R -$
$2u^2_R v'_R u'_R + u'^2_R + 4u^2_R u'^2_R] \cdot \left[\left(-u^2_R + 2u_R h_R - h^2_R\right) h'_R + h^2_R - 2u_R h_R + u^2_R\right] +$
$\left(-u_R + h_R\right)^2 \cdot [(u^2_R v_R u'_R + 2^{-1} v^2_R u'_R h_R - 2^{-1} u_R v^2_R u'_R - 2^{-1} u_R v^3_R v'_R h_R +$
$u^2_R v^2_R v'_R h_R - u^2_R v_R h'_R + 2^{-1} v^3_R v'_R h^2_R + 2^{-1} u^2_R v^2_R h'_R - 2^{-1} v^2_R h_R h'_R + 2^{-2} u_R v^4_R h_R -$
$2^{-2} v^4_R u'_R h_R) h''_R + (-u_R v_R v'_R + v_R v'_R h_R - 2^{-1} v^3_R v'_R h_R + 2^{-1} v^2_R h_R + u^2_R v_R +$
$2^{-1} u_R v^3_R v'_R - 2u^2_R v'_R - u^2_R v^2_R v'_R - 2^{-1} u_R v^2_R) h'^2_R + (-u^2_R v_R v'_R - v_R v'_R h^2_R +$
$2^{-1} u_R v^2_R u'_R + 2u^3_R u'_R - u^3_R v_R + u^2_R v_R v'_R h_R - u'_R h_R + u_R u'_R - 2^{-1} v^2_R v'^2_R h^2_R +$
$u^2_R v^2_R v'^2_R - 2u^2_R u'^2_R - 2^{-1} v^2_R u'^2_R h_R - 2^{-1} u_R v^3_R u'_R v'_R + v_R v''_R h^2_R - 3u_R v'^2_R h_R + u^2_R v_R v''_R -$
$2^{-1} v^3_R v''_R h^2_R + 3u_R v^3_R v'_R h_R + u^2_R v'^2_R - 2u^3_R u''_R + 2v'^2_R h^2_R + 2^{-1} u_R v_R v'^2_R h_R +$
$u^2_R v_R v'^2_R h_R - 2u_R v_R v''_R h_R + 4u_R u'_R v'_R h_R - 2u^3_R v_R u'_R v'_R - u^2_R v^2_R v''_R h_R +$
$2^{-1} u_R v^3_R v''_R h_R - v^3_R u'_R v'_R h_R + u^2_R v_R u''_R - 2^{-1} u_R v^2_R u''_R + 2^{-1} v^2_R u''_R h_R) h'_R - v_R v'^3_R h^3_R +$
$(-2^{-1} v^2_R u'_R v'^2_R - u'_R v'^2_R - 2^{-1} v^3_R u'_R v''_R - 4u_R v_R u'_R v'^2_R - 2^{-1} v^3_R v'_R u''_R + u_R v^2_R v'^2_R +$
$u_R v^2_R v'_R v''_R + 2u_R v_R v'^3_R + u_R v^3_R v''_R) h^2_R + (2u^2_R v_R u'_R v'^2_R + u_R v_R u'_R v''_R + 2u^2_R v^2_R u'_R v'_R -$
$2^{-1} u^2_R v^2_R v'^2_R - u^2_R v^3_R u'_R v'_R - 2^{-1} v^2_R u'_R u''_R - 4u_R u'^2_R v'_R + 2u_R u'_R v'^2_R + v^3_R u'^2_R v'_R +$
$2^{-2} v^4_R u'_R u''_R + u_R v^3_R u'^2_R + u^2_R v^3_R u''_R + 2^{-1} u^3_R v^2_R v''_R - 2^{-1} u^2_R v^3_R v''_R - 2^{-2} u_R v^4_R u''_R -$
$2u^3_R u'_R v'^2_R - 2u_R v^2_R u'^2_R v'_R + 2u^2_R v_R u'^2_R v'_R - u^2_R v^2_R v'_R u''_R + 2u^3_R v_R v'_R u''_R + u^3_R v_R v'_R v''_R -$
$2^{-1} u^2_R v^2_R u'_R v''_R - 2u^3_R v_R u'_R v''_R - u_R v^3_R u'_R u''_R + 2^{-1} u_R v^3_R v'_R u''_R) h_R - u^2_R u'_R v'^2_R +$
$2u^2_R u'^3_R + 2^{-1} u_R v^2_R u'_R u''_R + 2u^2_R u'^2_R v'_R + u^3_R u'_R v''_R - u^2_R v_R u'_R u''_R - u^2_R v_R u'_R v''_R +$
$2u^3_R u'_R u''_R] - 2 \cdot [v'_R h'^2_R - u^2_R h'_R + (u^2_R v_R v'_R - v^2_R v'_R u_R + v_R v'_R h_R - 2v'_R u_R +$
$2u_R u'_R) h^2_R + (-u^2_R + v'_R) u^2_R] \cdot [-u^2_R h'^2_R + (2u_R u'_R h_R - 2^{-1} u'_R v_R h_R + 2^{-1} u_R v^3_R h_R +$
$2^{-1} u_R v^2_R h_R - u^2_R v_R h_R - 3u_R h_R + u^2_R + 2h^2_R + u^2_R v_R v'_R h_R) h'_R - v_R v'_R h^3_R +$
$(2u_R u'_R v_R v'_R - 2^{-1} v^2_R u'_R - u'_R + 2u_R v_R v'_R + u_R v^2_R) h^2_R + (-u_R v^2_R u'^2_R - 2u_R u'^2_R +$
$u^2_R v^2_R u'_R - 2^{-1} u_R v^3_R u'_R + 2^{-1} v^3_R u'^2_R - 2^{-1} u^2_R v^2_R + u^2_R v_R u'_R v'_R - 2u^3_R u'_R + 2u_R u'_R) h_R -$
$u^2_R u'_R + u^2_R u'^2_R]$.

If the generalized helicoid and the rotation surface have the same Gauss map, comparing (33) and (35),we obtain same Gauss map conditions in this theorem. If $\varsigma(u) = 0$ and $\kappa(u) = 0$ from (34) and (36) then, the generalized helicoid and the rotation surface are minimal. □

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