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RECURRENT DIMENSIONS AND DIOPHANTINE CONDITIONS OF DISCRETE DYNAMICAL SYSTEMS GIVEN BY CIRCLE MAPPINGS II

By

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Abstract. In our previous paper we estimated the upper and the lower recurrent dimensions of discrete orbits given by circle mappings and we have given the lower bounds of the gap values between the upper and the lower dimensions by using the parametrizing Diophantine conditions on the irrational rotation numbers. In this paper, estimating the upper bounds of the gap values, we give the exact gap values, which are described by the Diophantine parameters of the rotation numbers.

1. Introduction

In this paper we study recurrent dimensions of discrete dynamical systems given by a circle diffeomorphism $f : S^1 \to S^1$. The rotation number of f is defined by

$$\rho(f) = \lim_{n\to\infty} \frac{\hat{f}^n(x) - x}{n}$$

where $\hat{f} : \mathbf{R} \to \mathbf{R}$ is a lift of f such that $\pi \circ \hat{f} = f \circ \pi, \quad \pi : \mathbf{R} \to \mathbf{R}/\mathbf{Z}(= S^1)$ is a covering map. Our purpose of this paper is to estimate the recurrent dimensions of the discrete orbits $\Sigma_x = \{f^n(x) : n \in \mathbf{N}_0\}$ according to the algebraic properties of $\rho(f)$.

In 1885 Poincaré proved that, if $f : S^1 \to S^1$ is a homeomorphism without periodic points, then there exist a rotation $R_\alpha(x) := x + \alpha(\text{mod } 1)$ and a continuous surjective monotone map $h : S^1 \to S^1$, which satisfies

$$h \circ f = R_\alpha \circ h$$

and α is an irrational number and equal to the rotation number of f. Consequently, $\rho(f)$ is independent of x.

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In this case we say that f is semi-conjugate to the rotation R_α or h is a semi-conjugacy between f and R_α. Furthermore, if h is strictly monotone (one-to-one), we say that f is conjugate to the rotation R_α or h is a conjugacy between f and R_α.

If f is sufficiently smooth, f is conjugate to a rotation. The following theorem was given by Denjoy.

THEOREM 1.1 (Denjoy,1932). *If $f : S^1 \to S^1$ is C^2-diffeomorphism without periodic points, then f is topologically conjugate to a rotation. That is, the conjugacy h between f and the rotation is a homeomorphism.*

The regularity of the conjugacy was studied by many authors. Here we introduce the estimate by Katznelson and Ornstein [2].

We say that g is $C^{m+\delta}$-class where $m \geq 1$ is an integer and $0 \leq \delta < 1$, if g is C^m and its m-th derivative is Hölder continuous with its exponent δ.

THEOREM 1.2 (Katznelson and Ornstein, 1989). *Let $f : S^1 \to S^1$ be a C^k-diffeomorphism, $k > 0$, without periodic points and its rotation number α satisfies the Diophantine condition for $\beta \geq 0$:*

$$|\alpha - \frac{p}{q}| > \frac{C}{q^{2+\beta}} \qquad (*)$$

for all $p/q \in \mathbf{Q}$. Then, if $\beta + 2 < k$, the conjugacy h between f and the rotation R_α is of class $C^{k-1-\beta-\varepsilon}$ for all $\varepsilon > 0$.

In our previous paper [8] we introduced the gaps between the upper and the lower recurrent dimensions as the index parameters, which measure unpredictability levels of the orbits. In [9] and [10] we estimated the gaps of recurrent dimensions for some quasi-periodic orbits by using the order of the parametrizing Diophantine conditions, we say d_0-(D) condition, on their irrational frequencies. In [11], using the order d_0 of (D) condition, we estimated the lower bounds of gaps of recurrent dimensions for the discrete orbit Σ_x, given by a C^k-class function f, in the following cases:
(I) The rotation number satisfies the assumption $\beta + 2 < k$ and the conjugacy h is smooth : C^γ-class, $\gamma \geq 1$,
(II) The rotation number satisfies $2 \leq k \leq \beta + 2$ and h is a homeomorphism.

In this paper, estimating the upper bounds of these gap values, we give the exact gap values of these recurrent dimensions.

Our plan of this paper is as follows. In section 2 we introduce the classifications of irrational numbers to parametrize the Diophantine condition $(*)$ and give definitions of recurrent dimensions. In section 3 we give the exact gaps values in the case (I) and in section 4 we treat the case (II). In section 5 we give some numerical results on the gap values of recurrent dimensions for quasi-periodic orbits given by the rotations according to the classifications of irrational rotation numbers.

2. classification of irrational numbers

Let τ be an irrational number. In our previous papers ([7], [8], [9]) we introduce the following classifications according to (good or bad) levels of approximation by rational numbers.

We say that τ is an α-order Roth number if there exists $\alpha \geq 0$ such that, for every $\beta : \beta > \alpha$, there exists a constant $c_\beta > 0$, which satisfies

$$|\tau - \frac{q}{p}| \geq \frac{c_\beta}{p^{2+\beta}}$$

for all rational numbers $q/p \in \mathbf{Q}$.

Let $\{n_k/m_k\}$ be the Diophantine approximation of τ. Then we call τ an α-order weak Liouville number if there exists an infinite subsequence $\{m_{k_j}\} \subset \{m_k\}$, which satisfies

$$|\tau - \frac{n_{k_j}}{m_{k_j}}| < \frac{c}{m_{k_j}^{2+\alpha}}, \quad \forall j$$

for some constants $c, \alpha > 0$.

Furthermore, we can parametrize the Diophantine condition $(*)$ as follows:

Let $R(\alpha)$ be the set of α-order Roth numbers and $wL(\beta)$ the set of β-order weak Liouville numbers. In [9] we have shown that

$$\begin{aligned}
&R(\alpha) \subset R(\alpha'),\ \alpha \leq \alpha',\ wL(\beta) \subset wL(\beta'),\ \beta \geq \beta',\\
&R(\alpha)^c \subset \bigcap_{\beta<\alpha} wL(\beta), \quad wL(\beta) \subset \bigcap_{\beta>\alpha} R(\alpha)^c,\\
&R(0)^c = \bigcup_{\beta>0} wL(\beta)
\end{aligned}$$

where the complements are considered in the set of all irrational numbers. Thus, for each irrational number τ, there exists a constant d_0, which specifies the levels of (bad or good) approximable properties by rational numbers:

$$\begin{aligned}
(2.1) \qquad &\inf\{\alpha : \tau \text{ is an } \alpha\text{-order Roth number}\}\\
&= \sup\{\beta : \tau \text{ is a } \beta\text{-order weak Liouville number}\} := d_0.
\end{aligned}$$

In our previous paper ([8]) we introduced a d_0-(D) condition for a pair of irrational numbers. For a single irrational case, let us say that τ satisfies a d_0-(D) condition if (2.1) holds.

Definitions of recurrent dimensions:

Define the first ε-recurrent time by

$$M_\varepsilon(x) = \min\{m \in \mathbf{N} : |f^m(x) - x| < \varepsilon\}.$$

and the upper and the lower recurrent dimensions by

$$\overline{D}_x = \limsup_{\varepsilon \to 0} \frac{\log M_\varepsilon(x)}{-\log \varepsilon}, \qquad \underline{D}_x = \liminf_{\varepsilon \to 0} \frac{\log M_\varepsilon(x)}{-\log \varepsilon}.$$

Then we can define the gaps of recurrent dimensions by $G_x = \overline{D}_x - \underline{D}_x$. (See [7] or [8] for further details.)

If the gap values G_x take positive values, we cannot exactly determine or predict the ε-recurrent time of the orbits. Thus we propose the value G_x as the parameter, which measures the unpredictability level of the orbit.

3. smooth conjugacy case

In this section we consider the case where the conjugacy h between the circle map f and the rotaion is C^γ-class, $\gamma \geq 1$. First we note that the metric in S^1 is induced by the covering (quotient) map $\pi : \mathbf{R} \to S^1$ such that

$$|x - y| := \inf_{m \in \mathbf{Z}} |x - y - m|, \quad x, y \in S^1$$

where we use the same notation as that of usual absolute values as far as not being confused.

THEOREM 3.1. *Let $f : S^1 \to S^1$ be a C^3-diffeomorphism without periodic points and its rotation number α satisfies the d_0-(D) condition for $0 \leq d_0 < 1$. Then, for each $x \in S^1$, we have*

$$\underline{D}_x = \frac{1}{1 + d_0}, \quad \overline{D}_x = 1.$$

Consequently, we have

$$G_x = \frac{d_0}{1 + d_0}.$$

Proof. Since we have shown the following estimates in [11]

$$\underline{D}_x \leq \frac{1}{1 + d_0}, \quad \overline{D}_x \geq 1, \quad G_x \geq \frac{d_0}{1 + d_0}.$$

it is sufficient to show that

$$\underline{D}_x \geq \frac{1}{1+d_0}, \quad \overline{D}_x \leq 1.$$

First we estmate the lower bound of the lower recurrent dimension. Since the Diophantine condition $(*)$ in Theorem 1.2 is satisfied with $\beta = 1 - \varepsilon_0 > d_0$ for some sufficiently small $\varepsilon_0 > 0$, the conjugacy h is $C^{1+\varepsilon_0-\varepsilon}$-class for every $\varepsilon > 0$. Thus we can admit C^1-conjugacy $h : h \circ f = R_\alpha \circ h$. Since $f^n(x) = h^{-1} \circ R_\alpha^n \circ h$ and Lipschitz continuity conditions of h and h^{-1}, which are given by the Mean Value Theorem, such that

$$C_1|x-y| \leq |h(x) - h(y)| \leq C_2|x-y|, \quad x, y \in S^1 : |x-y| \leq \frac{1}{2}$$

for some $C_2 > C_1 > 0$, we can take an integer m :

$$\begin{aligned} |f^n(x) - x| &= |h^{-1} \circ R_\alpha^n \circ h(x) - (h^{-1} \circ h)(x)| \\ &\leq C_1^{-1}|\alpha n - m|, \\ |\alpha n - m| &\leq \frac{1}{2}, \end{aligned} \tag{3.1}$$

and also an integer m' :

$$\begin{aligned} |f^n(x) - x| &= |h^{-1} \circ R_\alpha^n \circ h(x) - (h^{-1} \circ h)(x)| \\ &\geq C_2^{-1}|\alpha n - m'|, \\ |\alpha n - m'| &\leq \frac{1}{2}. \end{aligned} \tag{3.2}$$

Let $\{q_k/p_k\}$ be the Diophantine sequence of the rotation number α of f. It follows from d_0-(D) condition and the equivalent relations of Roth numbers (see [9]) that for every $\varepsilon > 0$ the irrational number α becomes a Roth number with its order $d_0 + \varepsilon$, which satisfies

$$p_{k+1} \leq c p_k^{1+d_0+\varepsilon}$$

for some $c := c_\varepsilon > 0$.

Here we use the following elementary property of the Dophantine sequence that

$$\frac{1}{p_k(p_{k+1}+p_k)} < |\alpha - \frac{q_k}{p_k}| < \frac{1}{p_k p_{k+1}} < \frac{1}{p_k^2} \tag{3.3}$$

and

$$\inf_{r \in \mathbf{N}} |\alpha n - r| \geq |\alpha p_k - q_k| \tag{3.4}$$

holds for every $n : 1 \leq n < p_{k+1}$.

It follows from (3. 2), (3. 3) and (3. 4) that we have

$$|f^n(x) - x| \geq C_2^{-1}|\alpha p_k - q_k| \geq \frac{1}{2C_2 p_{k+1}} \geq \frac{1}{2cC_2 p_k^{1+d_0+\varepsilon}} := \varepsilon_k$$

for every $n : 1 \leq n < p_{k+1}$. Thus we can estimate the lower recurrent dimension as follows.

$$\begin{aligned}
\underline{D}_x &= \liminf_{\varepsilon \to \infty} \frac{\log M(\varepsilon)}{-\log \varepsilon} \\
&= \liminf_{k \to \infty} \inf_{\varepsilon_{k+1} \leq \varepsilon \leq \varepsilon_k} \frac{\log M(\varepsilon)}{-\log \varepsilon} \\
&\geq \liminf_{k \to \infty} \frac{\log M(\varepsilon_k)}{-\log \varepsilon_{k+1}} \\
&\geq \lim_{k \to \infty} \frac{\log p_{k+1}}{\log 2c + \log C_2 + (1 + d_0 + \varepsilon) \log p_{k+1}} \\
&= \frac{1}{1 + d_0 + \varepsilon}
\end{aligned}$$

for every $\varepsilon > 0$.

Next we show the upper estimate. It follows from (3. 1) and (3. 3) that we have

$$|f^{p_k}(x) - x| \leq C_1^{-1}|\alpha p_k - q_k| \leq \frac{1}{C_1 p_{k+1}} := \varepsilon_{k+1}.$$

Thus we can estimate the upper recurrent dimension

$$\begin{aligned}
\overline{D}_x &= \limsup_{\varepsilon \to \infty} \frac{\log M(\varepsilon)}{-\log \varepsilon} \\
&= \limsup_{k \to \infty} \sup_{\varepsilon_{k+1} \leq \varepsilon \leq \varepsilon_k} \frac{\log M(\varepsilon)}{-\log \varepsilon} \\
&\leq \limsup_{k \to \infty} \frac{\log M(\varepsilon_{k+1})}{-\log \varepsilon_k} \\
&\leq \lim_{k \to \infty} \frac{\log p_k}{\log C_1 + \log p_k} = 1
\end{aligned}$$

and from the definition of the gap values we obtain the conclusion. □

4. topological conjugate case

Next we consider the case (II) by applying the same argument used in [11]. f has a unique invariant probability measure μ, defined by $\mu(A) = \lambda(h(A))$ where h is the conjugacy between f and the rotation and λ is a Lebesgue measure.

Let $\{q_k/p_k\}$ be the Diophantine sequence of the rotation number α of f and denote

$$m_k(x) = |f^{p_k}(x) - x|,$$
$$\alpha_k = |p_k\alpha - q_k|,$$

then we consider the subsets A', B' of S^1, defined by

$$A' = \{x \in S^1 : \liminf_{k\to\infty} \frac{m_k(x)}{\alpha_k} > 0\},$$
$$B' = \{x \in S^1 : \liminf_{k\to\infty} \frac{\alpha_k}{m_k(x)} > 0\}.$$

In [11] we proved that

$$\lambda(A) = \lambda(B) = 1$$

for

$$A = \{x \in S^1 : \limsup_{k\to\infty} \frac{m_k(x)}{\alpha_k} > 0\},$$
$$B = \{x \in S^1 : \limsup_{k\to\infty} \frac{\alpha_k}{m_k(x)} > 0\}.$$

Here we can also show that

$$\lambda(A') = \lambda(B') = 1.$$

We note that

$$\alpha_k = \int_{S^1} m_k(x) d\mu(x) \tag{4.1}$$

(see [4]).

We can estimate the measure of these subsets:

LEMMA 4.1. *Let $f : S^1 \to S^1$ be a C^2-diffeomorphism. Then we have*

$$\lambda(A') = \lambda(B') = 1. \tag{4.2}$$

Proof. Since f and f^{-1} is differentiable, it follows from the Mean Value Theorem that

$$K_1|x-y| \le |f(x) - f(y)| \le K_2|x-y|, \quad x, y \in S^1 : |x-y| \le \frac{1}{2},$$

for some $K_2 > K_1 > 0$. Thus we can easily show that

$$x \in A' \Longleftrightarrow f(x) \in A',$$
$$x \in B' \Longleftrightarrow f(x) \in B'.$$

Since f is ergodic (cf.[4]), the invariant sets A' and B' have full measures or null measures.

First we show that $\lambda(A') > 0$. Define

$$A'_0 = \{x \in S^1 : \liminf_{k\to\infty} \frac{m_k(x)}{\alpha_k} > c_0\}$$

for sufficiently small $c_0 > 0$ and assume that $\mu(A'_0) = 0$, that is, $\mu(A'^c_0) = 1$. Since

$$A'^c_0 = \{x \in S^1 : \liminf_{k\to\infty} \frac{m_k(x)}{\alpha_k} \le c_0\},$$

for $x \in A'^c_0$ and a sufficiently small constant $\varepsilon_0 > 0$ there exist a subsequence $k_j \to \infty$ and a large number j_0 such that, if $k_j \ge k_{j_0}$,

$$\frac{m_{k_j}(x)}{\alpha_{k_j}} \le c_0 + \varepsilon_0.$$

It follows from (4. 1) that we have a contradiction:

$$\alpha_{k_{j_0}} = \int_{A'^c_0} m_{k_{j_0}}(x)d\mu(x) \le (c_0 + \varepsilon_0)\alpha_{k_{j_0}}\mu(A'^c_0) = (c_0 + \varepsilon_0)\alpha_{k_{j_0}}.$$

Thus $\mu(A') > \mu(A'_0) > 0$ holds. Since h is a homeomorphism, we have $\lambda(A') > 0$. It follows from ergodicity of f that we have $\lambda(A') = 1$.

For the set B' we can apply the same argument as above. Denote

$$B'_0 = \{x \in S^1 : \liminf_{k\to\infty} \frac{\alpha_k}{m_k(x)} > c'_0\}$$

for some small $c'_0 > 0$ and assume that $\mu(B'^c_0) = 1$. If $x \in B'^c_0$, for a sufficiently small constant $\varepsilon'_0 > 0$, there exists a large number k'_{j_0} such that

$$\alpha_{k'_{j_0}} \le (c'_0 + \varepsilon'_0)m_{k'_{j_0}}(x).$$

Thus we have a contradiction,

$$\alpha_{k'_{j_0}} = \int_{B'^c_0} m_{k'_{j_0}}(x)d\mu(x) \ge \frac{\alpha_{k'_{j_0}}}{c'_0 + \varepsilon'_0}.$$

It follows that $\mu(B') > \mu(B'_0) > 0$. Applying the previous argument, we can conclude that $\lambda(B') = 1$. □

REMARK 4.2. It is known that the circle mapping f is conjugate to an irrational rotation if and only if its minimal invariant set (a non-empty compact invariant set which is minimal) is equal to S^1. Thus we can easily show that the invariant subsets A', B' are dense in S^1.

THEOREM 4.3. *Let $f : S^1 \to S^1$ be a C^2-diffeomorphism without periodic points and its rotation number α. Then we have*

$$\overline{D}_x = 1, \quad a.e. \quad x \in S^1. \tag{4.3}$$

Proof. In [11] we have shown that

$$\overline{D}_x \geq 1$$

for $x \in A$. Thus it is sufficient to show that

$$\overline{D}_x \leq 1$$

for $x \in B'$, since $\lambda(A \cap B') = 1$.

Let $x \in B'$, then for a sufficiently small constant $c_1 > 0$, there exsts a large number k_1 such that

$$m_k(x) \leq c_1^{-1} \alpha_k$$

for all $k \geq k_1$. It follows from (3.3) that

$$|f^{p_k}(x) - x| \leq c_1^{-1} |\alpha p_k - q_k| \leq \frac{1}{c_1 p_{k+1}} := \varepsilon_{k+1}.$$

Thus we have

$$\begin{aligned}
\overline{D}_x &= \limsup_{\varepsilon \to \infty} \frac{\log M(\varepsilon)}{-\log \varepsilon} \\
&= \limsup_{k \to \infty} \sup_{\varepsilon_{k+1} \leq \varepsilon \leq \varepsilon_k} \frac{\log M(\varepsilon)}{-\log \varepsilon} \\
&\leq \limsup_{k \to \infty} \frac{\log M(\varepsilon_{k+1})}{-\log \varepsilon_k} \\
&\leq \lim_{k \to \infty} \frac{\log p_k}{\log c_1 + \log p_k} \\
&= 1.
\end{aligned}$$

□

THEOREM 4.4. *Let $f : S^1 \to S^1$ be a C^2-diffeomorphism without periodic points and its rotation number α satisfies the d_0-(D) condition for $d_0 > 0$. Then we have*

$$\underline{D}_x = \frac{1}{1+d_0}, \quad a.e. \quad x \in S^1. \tag{4.4}$$

Consequently, we can estimate the gap values by

$$G_x = \frac{d_0}{1+d_0}, \quad a.e. \quad x \in S^1.$$

Proof. Note that $\lambda(A' \cap B) = 1$. Since in [11] we have shown that $\underline{D}_x \leq 1/(1+d_0)$ for $x \in B$, it is sufficient to show that

$$\underline{D}_x \geq \frac{1}{1+d_0}$$

for $x \in A'$. Let $x \in A'$, then for a sufficiently small constant $c_2 > 0$ there exists a large number k_2 such that

$$m_k(x) \geq c_2 \alpha_k$$

for all $k \geq k_2$. By Lemma 5.2 in [11] there exists a constant $b_0 : 0 < b_0 < 1$ such that

$$|f^n(x) - x| \geq b_0 m_k(x) \tag{4.5}$$

holds for every $n < p_{k+1}$. It follows from the property of Diophantine sequence that we have

$$m_k(x) \geq c_2 \alpha_k \geq \frac{c_2}{2p_{k+1}}.$$

Since the irrational number α satisfies the d_0-(D) condition, α is a $d_0 + \varepsilon$ order Roth number for every $\varepsilon > 0$. Then we have

$$p_{k+1} \leq c p_k^{1+d_0+\varepsilon} \tag{4.6}$$

for some $c > 0$ (see [9]). Thus we have

$$|f^n(x) - x| \geq \frac{b_0 c_2}{2p_{k+1}} \geq \frac{c b_0 c_2}{2p_k^{1+d_0+\varepsilon}} := \varepsilon_k$$

for every $n < p_{k+1}$ and every small $\varepsilon > 0$. It follows from the definition of the lower recurrent dimension that we can estimate

$$\begin{aligned}
\underline{D}_x &= \liminf_{\varepsilon\to\infty} \frac{\log M(\varepsilon)}{-\log\varepsilon} \\
&= \liminf_{k\to\infty} \inf_{\varepsilon_{k+1}\le\varepsilon\le\varepsilon_k} \frac{\log M(\varepsilon)}{-\log\varepsilon} \\
&\ge \liminf_{k\to\infty} \frac{\log M(\varepsilon_k)}{-\log\varepsilon_{k+1}} \\
&\ge \lim_{k\to\infty} \frac{\log p_{k+1}}{\log 2(cb_0c_2)^{-1} + (1+d_0+\varepsilon)\log p_{k+1}} = \frac{1}{1+d_0+\varepsilon}
\end{aligned}$$

for every $\varepsilon > 0$. Let $x \in A' \cap B$. Since $\lambda(A' \cap B) = 1$, we can obtain

$$\underline{D}_x = \frac{1}{1+d_0}, \quad \text{a.e. } x \in S^1.$$

Considering x in the set $A \cap A' \cap B \cap B'$, which is also of full measure, and applying the proof of Theorem 4.3, we can conclude that

$$G_x = \frac{d_0}{1+d_0}, \quad \text{a.e. } x \in S^1.$$

□

5. Numerical Calculations

In this section we give some numerical results on the recurrent dimensions, especially, gaps of dimensions, of quasi-periodic orbits given by the rotation $R_\alpha(x) = x + \alpha \pmod 1$.

Since the upper and the lower recurrent dimensions are given by

$$\begin{aligned}
\overline{D}_x &= \limsup_{k\to\infty} \sup_{\varepsilon_{k+1}\le\varepsilon\le\varepsilon_k} \frac{\log M(\varepsilon)}{-\log\varepsilon}, \\
\underline{D}_x &= \liminf_{j\to\infty} \inf_{\varepsilon_{j+1}\le\varepsilon\le\varepsilon_j} \frac{\log M(\varepsilon)}{-\log\varepsilon}
\end{aligned}$$

(the proof of these relations was given in [8]), the asymptotic behavior of the sequence $\{D_k\}$ defined by

$$D_k = \frac{\log M(\varepsilon_k)}{-\log\varepsilon_k}$$

is most strongly related to the gap values of recurrent dimensions. We can estimate the lower bound of the gap values by using the variance of the data $\{D_k\}$ as follows.

LEMMA 5.1. *Assume that there exists a subsequence $\{D_{k_j}\}$, which converges to $D_0 : \underline{D} \le D_0 \le \overline{D}$ as $j \to \infty$, and satisfies*

$$\limsup_{n\to\infty}\Big(\frac{1}{n}\sum_{k=1}^{n}|D_0 - D_k|^2\Big)^{\frac{1}{2}} = \sigma > 0. \tag{5.1}$$

Then we have

$$\overline{D} - \underline{D} \ge \sigma. \tag{5.2}$$

Proof. It follows from (5.1) that we can choose a subsequence $\{n_j\}$ such that for a sufficiently small $\varepsilon > 0$ there exits a large number j_0, which satisfies

$$\Big(\frac{1}{n_j}\sum_{k=1}^{n_j}|D_0 - D_k|^2\Big)^{\frac{1}{2}} > \sigma - \varepsilon \tag{5.3}$$

for all $j \ge j_0$. Then we can find an infinite subsequence $\{D_{l_j}\}$ in $\{D_k\}$, which satisfies

$$|D_0 - D_{l_j}| > \sigma - 2\varepsilon.$$

In fact, if we assume that

$$|D_0 - D_k| \le \sigma - 2\varepsilon, \quad \forall k \ge k_0,$$

for a large number k_0, then we have

$$\frac{1}{n_j}\sum_{k=1}^{n_j}|D_0 - D_k|^2 \le \frac{1}{n_j}\Big(\sum_{k=1}^{k_0}|D_0 - D_k|^2 + (n_j - k_0)(\sigma - 2\varepsilon)^2\Big).$$

As $j \to \infty$, it contradicts (5.3). Since

$$|D_0 - D_{k_j}| < \varepsilon, \quad \forall j \ge j_1$$

holds for a large number j_1, we have

$$|D_{l_j} - D_{k_j}| \ge |D_{l_j} - D_0| - |D_0 - D_{k_j}| > \sigma - 3\varepsilon.$$

holds for all $j \ge j_2 := \max\{j_0, j_1\}$. Choose a sufficiently large number N such that for $l_j,\ k_j \ge N$ with $j \ge j_2$

$$\overline{D} + \varepsilon \ge \max\{D_{l_j},\ D_{k_j}\}, \quad \underline{D} - \varepsilon \le \min\{D_{l_j},\ D_{k_j}\}$$

hold, then we have

$$\overline{D} - \underline{D} \ge |D_{l_j} - D_{k_j}| - 2\varepsilon \ge \sigma - 5\varepsilon$$

for every $\varepsilon > 0$. □

Since the definitions of recurrent dimensions are quite simple, we can calculate these dimensions numerically by simple programs. In view of Lemma 5.1, using "Mathematica" with "Statistics LinearRegression" package and calculating "SE": standard deviations related to the data $\{D_k\}$, we investigate the gaps of recurrent dimensions in the following cases of the rotation number α:

$$(1)\ \frac{\sqrt{5}+1}{2} \quad (2)\ \sqrt{2} \quad (3)\ e \quad (4)\ [0,2,2^2,2^3,2^4,2^5,2^6]$$
$$(5)\ [0,2,2,2,2^5,2,2] \quad (6)\ [0,2,2^{2^2},2^{2^3},2^{2^4},2^{2^5},2^{2^6}]$$

where the notations $[\cdot,\cdot,\cdots]$ are continued fraction expansions:

$$\frac{\sqrt{5}+1}{2} = [1,1,1,1,\cdots], \quad \sqrt{2} = [1,2,2,2,2,\cdots],$$
$$e = [2,1,2,1,1,4,1,1,6,1,1,8,1,1,10,...].$$

The numbers of (1) and (2) are in the "constant type" irrational class or called "badly approximable" such that $d_0 = 0$ and also the numbers of (3) and (4) are 0-order Roth numbers: $d_0 = 0$ and the numbers of (5) and (6) are examples of an α-order weak Liouville number or an α-order Roth number, that is, $d_0 > 0$ (see [9] for details). Since the order α is given by $m_{j+1} \simeq m_j^{1+\alpha}$ (see also [9]), we can numerically calculate the values of d_0 by estimating $(\log m_{j+1}/\log m_j) - 1$: (5) $d_0 = 1.39992$ (6) $d_0 = 4.04439$. We calculate the recurrent dimensions of the orbits given by the rotation $R_\alpha(x)$ as follows:

Let

$$x[n] := n\alpha \pmod 1, \quad E[n] := |x[n] - x[1]|, \quad n = 1, ..., M_1,$$

then, define

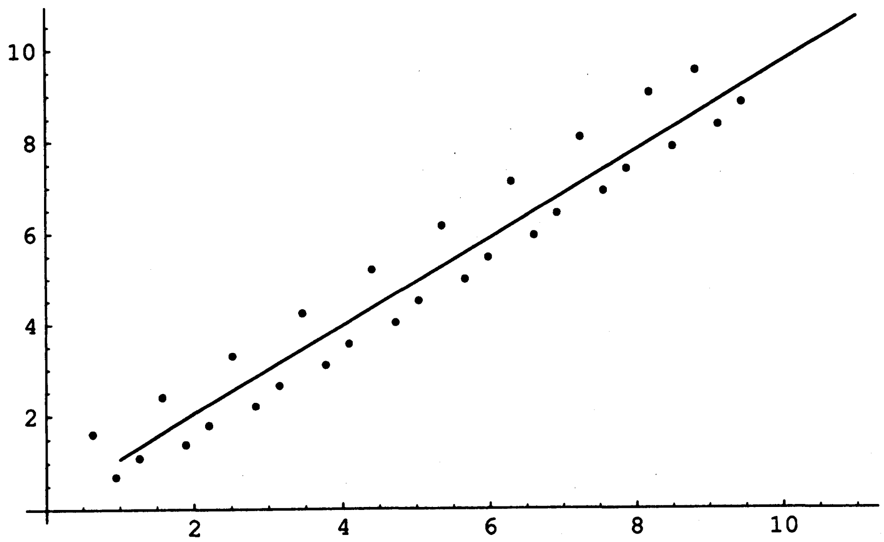
$$e[i] := c^{-i}, \quad c > 1,$$
$$m[i] := \min\{n : e[i+1] < E[n] < e[i], \quad n = 2, ..., M_1\}, \quad i = 1, ..., M_2$$

where we estimate the minimum values by using double loops such that "Do" and "If" for $n = 2, ..., M_1$ in the loop "Do" for $i = 1, ..., M_2$, not using "Min", but using "Break". Define

$$X[i] := -\log e[i], \quad Y[i] := \log m[i],$$

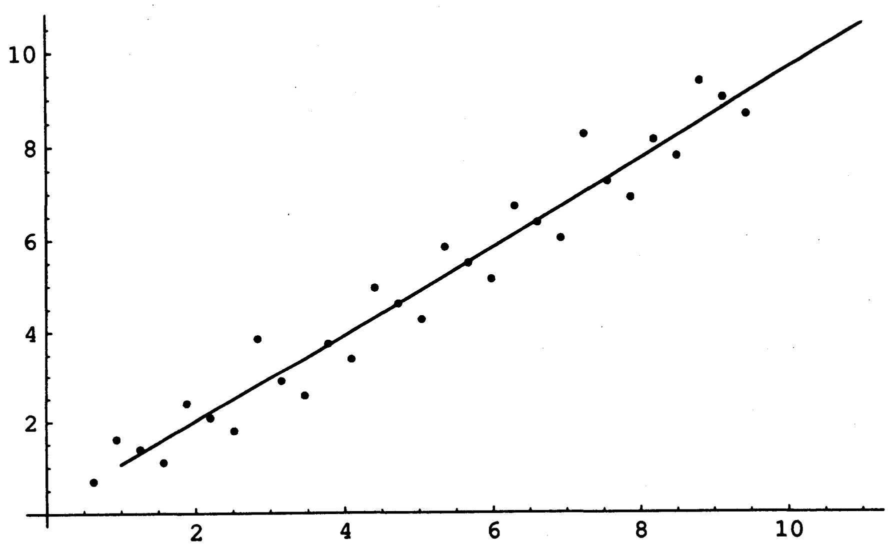
then we apply 'LinearRegression' to the data list $\{(X[i], Y[i]) : i = 1, ..., M_2\}$. Then we consider the slope of the line as one kind of mean values between upper and lower recurrent dimensions. We obtain the following results by taking the constants: $M_1 = 50000$, $M_2 = 24 \sim 30$, $c = 1.37$ for each rotation number α from the case (1) to (6).

(1) $\alpha = \dfrac{\sqrt{5}+1}{2}$



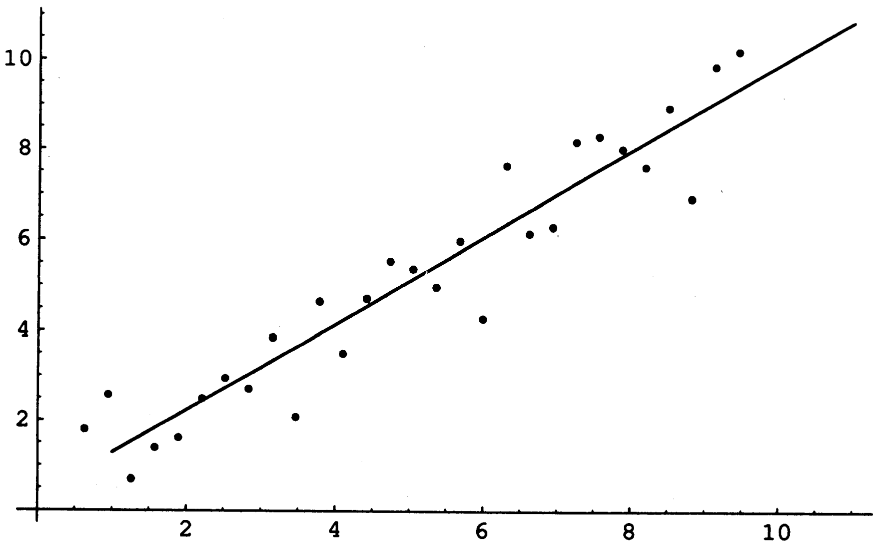
FIT: 0.124695 + 0.958591 x, SE: 0.04771

(2) $\sqrt{2}$



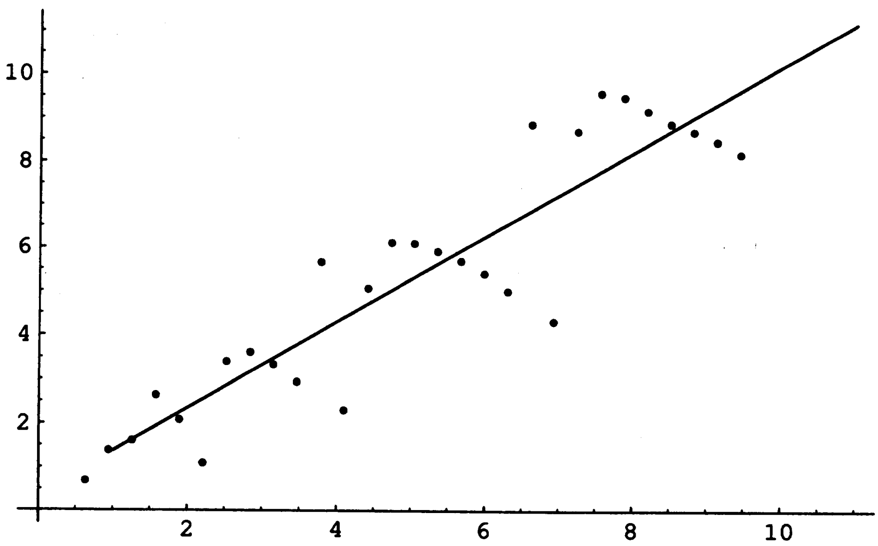
FIT: 0.104476 + 0.95213 x, SE: 0.0414112

(3) e



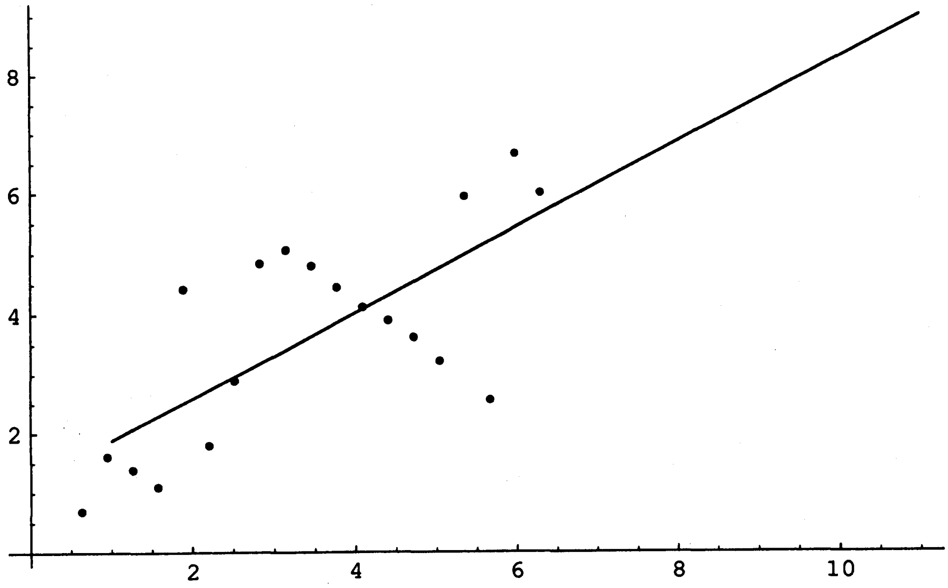
FIT: 0.319348 + 0.9574 x, SE: 0.0616199

(4) $[0, 2, 2^2, 2^3, 2^4, 2^5, 2^6]$



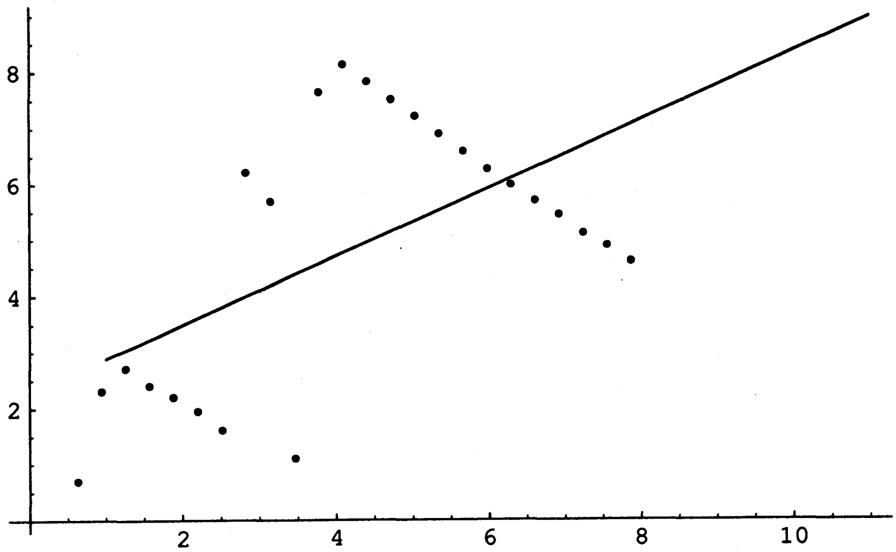
FIT: 0.396379 + 0.978274 x. SE: 0.0840199

(5) $[0, 2, 2, 2, 2^5, 2, 2]$



FIT: 1.16939 + 0.711111 x, SE: 0.168685

(6) $[0, 2, 2^{2^2}, 2^{2^3}, 2^{2^4}, 2^{2^5}, 2^{2^6}]$



FIT: 2.27988 + 0.607361 x, SE: 0.185964

According to the cases from (1) to (6) we can obtain the table of values d_0, $G_0 = d_0/(1+d_0)$, D: the slope of the line obtained by the linear regression and SE: standard deviations of D. G_1 is the gap value of recurrent dimensions given by $G_1 = \overline{D} - \underline{D} = 2(1-D)$ where we assume that $D = (\overline{D} + \underline{D})/2$ and note that $\overline{D} = 1$.

Table: Gaps of Recurrent Dimensions

Rot. num.	d_0	G_0	D	SE	G_1
$(\sqrt{5}+1)/2$	0	0	0.958591	0.04771	0.082818
$\sqrt{2}$	0	0	0.95213	0.0414112	0.09574
e	0	0	0.9574	0.0616199	0.0852
$[0, 2, 2^2, 2^3, 2^4, 2^5, 2^6]$	0	0	0.978274	0.0840199	0.043452
$[0, 2, 2, 2, 2^5, 2, 2]$	1.39992	0.583319	0.711111	0.168685	0.577778
$[0, 2, 2^{2^2}, 2^{2^3}, 2^{2^4}, 2^{2^5}, 2^{2^6}]$	4.04439	0.80176	0.607361	0.185964	0.785278

In view of Lemma 5.1 we can see that the SE values give the lower bounds of the gap values G_0 or G_1. Since $G_0 \sim G_1$ in all cases, we can say that the assumption $D = (\overline{D} + \underline{D})/2$ is considerable.

The case "Pi" contains the most mysterious problems. Our numerical results for π are

$$\mathrm{FIT} : 2.30816 + 0.640188x, \quad \mathrm{SE} : 0.14764.$$

Since we have $G_1 = 0.719624$, we can conjecture that

$$d_0 \sim \frac{G_1}{1 - G_1} = 2.56664.$$

In [1] Hata estimated the irrationality of π as follows:

$$|\pi - \frac{n}{m}| \geq m^{-8.0161}$$

holds for any integers $n, m : m \geq m_0$ where m_0 is a sufficiently large number. In our definitions he proved that π is a Roth number with its order 6.0161... The lower estimate of its order, that is, the estimate of the order as a weak Liouville number must be the most interesting and mysterious problem on π.

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