

ON UNIFIED RESULTS INVOLVING PARTIAL SUMS OF A CLASS OF MEROMORPHIC FUNCTIONS

By

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Abstract. In this paper we introduce a new class $\mathcal{L}(\phi, \psi; \alpha)$ involving Hadamard product of meromorphic functions ϕ and ψ having simple poles at $z = 0$. We establish some results concerning partial sums for the functions belonging to the class $\mathcal{L}(\phi, \psi; \alpha)$. Applications of the main results are also considered.

1. Introduction

Let Σ be the class of functions $f(z)$ defined by

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k \quad (1.1)$$

which are analytic in the punctured unit disk

$$\mathcal{D} = \{z : z \in \mathbb{C}, 0 < |z| < 1\}; \mathcal{U} = \mathcal{D} \cup \{0\}.$$

We denote by $\Sigma^*(\alpha)$, $\Sigma_k(\alpha)$ and $\Sigma_c(\alpha)$ the three subclasses of the class Σ which are defined (for $\alpha \in [0, 1)$) as follows:

$$\Sigma^*(\alpha) = \left\{ f : f \in \Sigma, \Re\left(-\frac{zf'(z)}{f(z)}\right) > \alpha \right\} \quad (1.2)$$

$$\Sigma_k(\alpha) = \left\{ f : f \in \Sigma, \Re\left(-\left(1 + \frac{zf''(z)}{f'(z)}\right)\right) > \alpha \right\} \quad (1.3)$$

and

$$\Sigma_c(\alpha) = \{f : f \in \Sigma, \Re(-z^2 f'(z)) > \alpha\}, \quad (1.4)$$

where the subclasses of the class Σ denoted, respectively, by $\Sigma^*(\alpha)$, $\Sigma_k(\alpha)$ and $\Sigma_c(\alpha)$ are the well known subclasses of meromorphic starlike functions of order

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$\alpha(0 \leq \alpha < 1)$ in \mathcal{U} , meromorphic convex functions of order $\alpha(0 \leq \alpha < 1)$ in \mathcal{U} , and meromorphic close-to-convex functions of order $\alpha(0 \leq \alpha < 1)$ in \mathcal{U} . We observe that every function belonging to the class $\Sigma_c(\alpha)$ is meromorphic close-to-convex of order α in \mathcal{U} (see [2]). We also refer to Libera and Robertson [3], Miller [4], Mogra et al. [5], Mogra [6], Pommerenke [7], Raina and Srivastava [8], and Xu and Yang [10] for the related works on the subject of meromorphic functions.

If

$$h_1(z) = \frac{a}{z} + \sum_{k=1}^{\infty} a_k z^k \quad (a \in \mathbb{R} - \{0\}) \quad (1.5)$$

and

$$h_2(z) = \frac{b}{z} + \sum_{k=1}^{\infty} b_k z^k \quad (b \in \mathbb{R} - \{0\}) \quad (1.6)$$

are analytic in \mathcal{D} , then their Hadamard product (or convolution) is defined by

$$(h_1 * h_2)(z) = \frac{ab}{z} + \sum_{k=1}^{\infty} a_k b_k z^k. \quad (1.7)$$

We now introduce a class $\mathcal{L}(\phi, \psi; \alpha)$ of meromorphic functions of the form (1.1) which is defined as follows:

Suppose the functions $\phi(z)$ and $\psi(z)$ are given by

$$\phi(z) = \frac{c_1}{z} + \sum_{k=1}^{\infty} \lambda_k z^k \quad (c_1 \in \mathbb{R} - \{0\}) \quad (1.8)$$

and

$$\psi(z) = \frac{c_2}{z} + \sum_{k=1}^{\infty} \mu_k z^k \quad (c_2 \in \mathbb{R} - \{0\}) \quad (1.9)$$

then we say that $f \in \Sigma$ is in the class $\mathcal{L}(\phi, \psi; \alpha)$ if

$$\Re \left(- \frac{(f * \phi)(z)}{(f * \psi)(z)} \right) > \alpha \quad (\alpha \in [0, 1)), \quad (1.10)$$

provided that $(f * \psi)(z) \neq 0$; $\langle \lambda_k \rangle_{k=1}^{\infty}$ and $\langle \mu_k \rangle_{k=1}^{\infty}$ are increasing sequences such that $\lambda_k \geq \mu_k \geq 0$ (λ_k and μ_k are not both simultaneously equal to zero).

Silverman [9] determined sharp lower bounds for the real part of the quotients of the normalized starlike or convex functions and their sequences of partial sums.

Motivated essentially by the work in [9], Cho and Owa [1] investigated the sharp lower bounds for the real part of the quotients between the function of the form (1.1) to its sequence of partial sums

$$f_n(z) = \frac{1}{z} + \sum_{k=1}^n a_k z^k$$

when the coefficients are sufficiently small satisfying the coefficient inequalities for the classes defined by (1.2) and (1.3).

In the present paper, we establish some results concerning partial sums for the meromorphic functions belonging to the class $\mathcal{L}(\phi, \psi; \alpha)$ (defined above by (1.10)). The results not only provide unification of the various results (proved rather independently) of Cho and Owa [1], but also yield some new results. The various consequences of our main results are mentioned in the concluding section.

Special cases of the class $\mathcal{L}(\phi, \psi; \alpha)$

We mention below some known subclasses of Σ defined by (1.1) which emerge from the class $\mathcal{L}(\phi, \psi; \alpha)$ (defined by (1.10))

Let us choose

$$\phi(z) = \frac{2z-1}{z(1-z)^2} = \frac{-1}{z} + \sum_{k=1}^{\infty} k z^k,$$

and

$$\psi(z) = \frac{z^2 - z + 1}{z(1-z)} = \frac{1}{z} + \sum_{k=1}^{\infty} z^k,$$

then in view of the convolution defined by (1.7), and performing simple calculations, we observe that

$$(f * \phi)(z) = z f'(z)$$

and

$$(f * \psi)(z) = f(z).$$

Thus, the class $\mathcal{L}(\phi, \psi; \alpha)$ reduces to $\Sigma^*(\alpha)$ satisfying the relationship

$$\mathcal{L}\left(\frac{2z-1}{z(1-z)^2}, \frac{z^2-z+1}{z(1-z)}; \alpha\right) = \Sigma^*(\alpha). \quad (1.11)$$

Similarly, by putting

$$\phi(z) = \frac{1-3z+4z^2}{z(1-z)^3} = \frac{1}{z} + \sum_{k=1}^{\infty} k^2 z^k$$

and

$$\psi(z) = \frac{2z-1}{z(1-z)^2} = \frac{-1}{z} + \sum_{k=1}^{\infty} kz^k,$$

then in view of the convolution defined by (1.7), we find that

$$(f * \phi)(z) = zf'(z) + z^2 f''(z),$$

and

$$(f * \psi)(z) = zf'(z).$$

The class $\mathcal{L}(\phi, \psi; \alpha)$ then reduces to $\Sigma_k(\alpha)$ and satisfies the relation

$$\mathcal{L}\left(\frac{1-3z+4z^2}{z(1-z)^3}, \frac{2z-1}{z(1-z)^2}; \alpha\right) = \Sigma_k(\alpha). \quad (1.12)$$

Lastly, choosing

$$\phi(z) = \frac{2z-1}{z(1-z)^2} = \frac{-1}{z} + \sum_{k=1}^{\infty} kz^k,$$

and

$$\psi(z) = \frac{1}{z},$$

in (1.7), we obtain

$$(f * \phi)(z) = zf'(z),$$

and

$$(f * \psi)(z) = \frac{1}{z}.$$

The class $\mathcal{L}(\phi, \psi; \alpha)$ then reduces to $\Sigma_c(\alpha)$, satisfying the relationship

$$\mathcal{L}\left(\frac{2z-1}{z(1-z)^2}, \frac{1}{z}; \alpha\right) = \Sigma_c(\alpha). \quad (1.13)$$

2. Main Results

In this section we shall investigate the sharp bounds for $\Re\left(\frac{f(z)}{f_n(z)}\right)$, $\Re\left(\frac{f'_n(z)}{f'(z)}\right)$, $\Re\left(\frac{f'(z)}{f'_n(z)}\right)$ and $\Re\left(\frac{f''_n(z)}{f''(z)}\right)$, where $f(z)$ is defined by (1.1), and $f_n(z) = \frac{1}{z} + \sum_{k=1}^n a_k z^k$ ($n \in \mathbb{N}$) is the sequence of partial sums of (1.1).

In order to prove our main results, we require the following assertion giving the coefficient inequality of the function $f(z)$ defined by (1.1) to belong to the class $\mathcal{L}(\phi, \psi; \alpha)$.

LEMMA 1. *If $f(z) \in \Sigma$ satisfies*

$$\sum_{k=1}^{\infty} A(\alpha, k) |a_k| \leq \frac{1}{2} [|c_1 + (2\alpha - 1)c_2| - |c_1 + c_2|], \quad (2.1)$$

where $A(\alpha, k)$ is given by

$$A(\alpha, k) = \lambda_k + \alpha \mu_k, \quad (2.2)$$

then $f(z) \in \mathcal{L}(\phi, \psi; \alpha)$, provided that $\frac{c_1}{c_2} \leq -1$ ($c_1, c_2 \in \mathbb{R} - \{0\}; 0 \leq \alpha < 1$).

Proof. In order to prove the lemma, we observe that the left-hand side of (2.1) is a series of positive terms. To make the right-hand side of (2.1) also positive, we show that

$$|c_1 + (2\alpha - 1)c_2| > |c_1 + c_2|,$$

or, we show that

$$\left| \frac{c_1 + (2\alpha - 1)c_2}{c_1 + c_2} \right| > 1.$$

Elementary calculations reveal that the above inequality is true provided that $\frac{c_1}{c_2} \leq -1$ ($c_1, c_2 \in \mathbb{R} - \{0\}; 0 \leq \alpha < 1$). This condition stated in the lemma thus ensures the validity of the coefficient inequality (2.1).

Let the condition (2.1) be satisfied for the function $f(z) \in \Sigma$. By virtue of the definition (1.10), it is sufficient to show that

$$\left| \frac{(f * \phi) + (f * \psi)}{(f * \phi) + (2\alpha - 1)(f * \psi)} \right| < 1$$

Using (1.1), (1.8) and (1.9), we obtain

$$\begin{aligned} \left| \frac{(f * \phi) + (f * \psi)}{(f * \phi) + (2\alpha - 1)(f * \psi)} \right| &= \left| \frac{\frac{c_1}{z} + \sum_{k=1}^{\infty} \lambda_k a_k z^k + \frac{c_2}{z} + \sum_{k=1}^{\infty} \mu_k a_k z^k}{\frac{c_1}{z} + \sum_{k=1}^{\infty} \lambda_k a_k z^k + (2\alpha - 1) \left\{ \frac{c_2}{z} + \sum_{k=1}^{\infty} \mu_k a_k z^k \right\}} \right| \\ &= \left| \frac{(c_1 + c_2) + \sum_{k=1}^{\infty} (\lambda_k + \mu_k) a_k z^{k+1}}{c_1 + (2\alpha - 1)c_2 + \sum_{k=1}^{\infty} \{\lambda_k + (2\alpha - 1)\mu_k\} a_k z^{k+1}} \right| \\ &\leq \frac{|c_1 + c_2| + \sum_{k=1}^{\infty} (\lambda_k + \mu_k) |a_k|}{|c_1 + (2\alpha - 1)c_2| - \sum_{k=1}^{\infty} \{\lambda_k + (2\alpha - 1)\mu_k\} |a_k|}. \end{aligned}$$

The last expression is bounded by 1 if which is true by virtue of (2.1), and the proof is complete.

If we choose arbitrary functions ϕ and ψ in (1.10) according to (1.11), (1.12) and (1.13), we get the following assertions:

COROLLARY 1. *Let the function $f(z) \in \Sigma$ satisfy the coefficient inequality*

$$\sum_{k=1}^{\infty} (k + \alpha) |a_k| \leq 1 - \alpha, \quad (2.3)$$

then $f(z) \in \Sigma^(\alpha)$.*

COROLLARY 2. *Let the function $f(z) \in \Sigma$ satisfy the coefficient inequality*

$$\sum_{k=1}^{\infty} k(k + \alpha) |a_k| \leq 1 - \alpha, \quad (2.4)$$

then $f(z) \in \Sigma_k(\alpha)$.

COROLLARY 3. *Let the function $f(z) \in \Sigma$ satisfy the coefficient inequality*

$$\sum_{k=1}^{\infty} k |a_k| \leq 1 - \alpha, \quad (2.5)$$

then $f(z) \in \Sigma_c(\alpha)$.

REMARK 1. If $c_1 + c_2 = 0$ with $|c_1| = |c_2| = 1$, then the inequality (2.1) simplifies to the form

$$\sum_{k=1}^{\infty} A(\alpha, k) |a_k| \leq 1 - \alpha. \quad (2.6)$$

To prove our main results we also need the following definition:

DEFINITION 1. For two functions f and g analytic in \mathcal{U} , we say that the function f is subordinate to g in \mathcal{U} (denoted by $f \prec g$), if there exists a function $w(z)$, analytic in \mathcal{U} with $w(0) = 0$, and $|w(z)| < 1$ ($z \in \mathcal{U}$) such that $f(z) = g(w(z))$.

THEOREM 1. *If $f(z) \in \Sigma$ satisfies the coefficient inequality (2.6), then for $\max\left\{0, \frac{1-\lambda_1}{1+\mu_1}\right\} \leq \alpha < 1$:*

$$\Re\left(\frac{f(z)}{f_n(z)}\right) \geq \frac{A(\alpha, n+1) + \alpha - 1}{A(\alpha, n+1)} \quad (2.7)$$

and

$$\Re\left(\frac{f_n(z)}{f(z)}\right) \geq \frac{A(\alpha, n+1)}{A(\alpha, n+1) - \alpha + 1}, \quad (2.8)$$

where $A(\alpha, n+1)$ is given by (2.2). The results are sharp for every n , with the extremal functions given by

$$f(z) = \frac{1}{z} + \frac{1-\alpha}{A(\alpha, n+1)} z^{n+1} \quad (n \in \mathbb{N}), \quad (2.9)$$

where the equality in (2.7) is attained when $z = re^{\pi i/n+1}$ ($r \rightarrow 1^-$), and for (2.8) equality is attained when $z = re^{2\pi i/n+2}$ ($r \rightarrow 1^-$).

Proof. To prove (2.7), we have to show that

$$\frac{A(\alpha, n+1)}{1-\alpha} \left[\frac{f(z)}{f_n(z)} - \frac{A(\alpha, n+1) + \alpha - 1}{A(\alpha, n+1)} \right] < \frac{1+z}{1-z}. \quad (2.10)$$

Using definition of subordination, and putting the values of f and f_n , we have from (2.10):

$$\frac{1 + \sum_{k=1}^n a_k z^{k+1} + \frac{A(\alpha, n+1)}{1-\alpha} \sum_{k=n+1}^{\infty} a_k z^{k+1}}{1 + \sum_{k=1}^n a_k z^{k+1}} = \frac{1+w(z)}{1-w(z)}. \quad (2.11)$$

Our assertion (2.10) is true if we show that $w(0) = 0$ and $|w(z)| < 1$; $z \in \mathcal{U}$. From (2.11), we get

$$w(z) = \frac{\frac{A(\alpha, n+1)}{1-\alpha} \sum_{k=n+1}^{\infty} a_k z^{k+1}}{2 + 2 \sum_{k=1}^n a_k z^{k+1} + \frac{A(\alpha, n+1)}{1-\alpha} \sum_{k=n+1}^{\infty} a_k z^{k+1}},$$

and obviously $w(0) = 0$, and

$$|w(z)| \leq \frac{\frac{A(\alpha, n+1)}{1-\alpha} \sum_{k=n+1}^{\infty} |a_k|}{2 - 2 \sum_{k=1}^n |a_k| - \frac{A(\alpha, n+1)}{1-\alpha} \sum_{k=n+1}^{\infty} |a_k|}.$$

Now $|w(z)| \leq 1$ if and only if

$$\sum_{k=1}^n |a_k| + \frac{A(\alpha, n+1)}{1-\alpha} \sum_{k=n+1}^{\infty} |a_k| \leq 1. \quad (2.12)$$

This will hold if we show that L.H.S. of (2.12) is bounded above by $\sum_{k=1}^{\infty} \frac{A(\alpha, k)}{1-\alpha} |a_k|$ (in view of (2.6)). This is equivalent to

$$\sum_{k=1}^n \left(\frac{A(\alpha, k)}{1-\alpha} - 1 \right) |a_k| + \sum_{k=n+1}^{\infty} \left(\frac{A(\alpha, k) - A(\alpha, n+1)}{1-\alpha} \right) |a_k| \geq 0,$$

which is true due to our condition on α , and the increasingness of the sequence $A(\alpha, k)$, $\forall k \in \mathbb{N}$.

THEOREM 2. *If $f(z) \in \Sigma$ satisfies the coefficient inequality (2.6) such that $\max\left\{0, \frac{1-\lambda_1}{1+\mu_1}\right\} \leq \alpha < 1$, and the sequence $\left\langle \frac{A(\alpha, k)}{k} \right\rangle_{k=1}^{\infty}$ is nondecreasing, then*

$$\Re\left(\frac{f'(z)}{f'_n(z)}\right) \geq \frac{A(\alpha, n+1) - (1-\alpha)(n+1)}{A(\alpha, n+1)} \quad (2.13)$$

and

$$\Re\left(\frac{f'_n(z)}{f'(z)}\right) \geq \frac{A(\alpha, n+1)}{A(\alpha, n+1) + (n+1)(1-\alpha)}. \quad (2.14)$$

The results are sharp for every n with the extremal functions given by

$$f(z) = \frac{1}{z} + \frac{1-\alpha}{A(\alpha, n+1)} z^{n+1} \quad (n \in \mathbb{N}), \quad (2.15)$$

where the equality in (2.13) is attained when $z = re^{2\pi i/n+2}$ ($r \rightarrow 1^-$), and for (2.14) equality is attained when $z = re^{\pi i/n+2}$ ($r \rightarrow 1^-$).

Proof. We prove (2.13), and the proof of (2.14) is similar and is here omitted. Proceeding as in Theorem 1, we set

$$\begin{aligned} & \frac{A(\alpha, n+1)}{(1-\alpha)(n+1)} \left[\frac{f'(z)}{f'_n(z)} - \frac{A(\alpha, n+1) - (1-\alpha)(n+1)}{A(\alpha, n+1)} \right] \\ &= \frac{1 - \sum_{k=1}^n k a_k z^{k+1} - \frac{A(\alpha, n+1)}{(1-\alpha)(n+1)} \sum_{k=n+1}^{\infty} k a_k z^{k+1}}{1 - \sum_{k=1}^n k a_k z^{k+1}} = \frac{1 + w(z)}{1 - w(z)}. \end{aligned}$$

Solving for $w(z)$, we get

$$w(z) = \frac{-\frac{A(\alpha, n+1)}{(1-\alpha)(n+1)} \sum_{k=n+1}^{\infty} k a_k z^{k+1}}{2 - 2 \sum_{k=1}^n k a_k z^{k+1} - \frac{A(\alpha, n+1)}{(1-\alpha)(n+1)} \sum_{k=n+1}^{\infty} k a_k z^{k+1}},$$

which implies that

$$|w(z)| \geq \frac{\frac{A(\alpha, n+1)}{(1-\alpha)(n+1)} \sum_{k=n+1}^{\infty} k |a_k|}{2 - 2 \sum_{k=1}^n k |a_k| - \frac{A(\alpha, n+1)}{(1-\alpha)(n+1)} \sum_{k=n+1}^{\infty} k |a_k|}.$$

Now $|w(z)| = 1$ if and only if

$$\sum_{k=1}^n k |a_k| + \sum_{k=n+1}^{\infty} \frac{A(\alpha, n+1)}{(1-\alpha)(n+1)} k |a_k| \leq 1, \quad (2.16)$$

which will hold true if we show that the L.H.S. of (2.16) is bounded above by $\sum_{k=1}^{\infty} \frac{A(\alpha, k)}{1-\alpha} |a_k|$ (in view of (2.6)). This is equivalent to

$$\sum_{k=1}^n k \left(\frac{\frac{A(\alpha, k)}{k}}{(1-\alpha)} - 1 \right) |a_k| + \sum_{k=n+1}^{\infty} k \left(\frac{(n+1) \frac{A(\alpha, k)}{k} - A(\alpha, n+1)}{(1-\alpha)(n+1)} \right) |a_k| \geq 0,$$

which is true due to the condition implied on α , and the increasingness of the sequence $\left\langle \frac{A(\alpha, k)}{k} \right\rangle_{k=1}^{\infty}$.

3. Applications of Main Results

In this section we consider some applications of the main results (Theorems 1 and 2) as worthwhile consequences of our main results, by properly specializing the arbitrary functions $\phi(z)$ and $\psi(z)$ occurring in (1.8), (1.9) and (1.10).

Thus, if we set $\phi(z) = \frac{2z-1}{z(1-z)^2}$ and $\psi(z) = \frac{z^2-z+1}{z(1-z)}$ in (1.10), then as mentioned in (1.11), the class $\mathcal{L}(\phi, \psi; \alpha)$ reduces to a known class. Also, as a consequence of the above choice of functions $\phi(z)$ and $\psi(z)$, the parameters c_1 and c_2 are chosen as $c_1 = -1$ and $c_2 = 1$, and the arbitrary sequences $\{\lambda_k\}$ and $\{\mu_k\}$ set as $\lambda_k = k (k \in \mathbb{N})$ and $\mu_k = 1 (k \in \mathbb{N})$ in (2.7) and (2.8) of Theorem 1, and

also in (2.13) and (2.14) of Theorem 2, thereby, simplifying considerably the right-hand members of these inequalities. We eventually obtain the following result emerging (in combined form) from Theorems 1 and 2. The other results (Corollaries 5-6) are deducible in similar manner from Theorems 1 and 2 (in combined forms) by suitably involving the known forms of classes of functions as mentioned in (1.12) and (1.13), and by 8 setting the arbitrary functions and sequences of parameters appropriately (as indicated just above).

COROLLARY 4. *If $f(z) \in \Sigma$ satisfies the coefficient inequality (2.3), then*

$$\Re\left(\frac{f(z)}{f_n(z)}\right) \geq \frac{n+2\alpha}{n+1+\alpha} \quad (0 \leq \alpha < 1) \quad (3.1)$$

$$\Re\left(\frac{f_n(z)}{f(z)}\right) \geq \frac{n+1+\alpha}{n+2} \quad (0 \leq \alpha < 1) \quad (3.2)$$

$$\Re\left(\frac{f'(z)}{f'_n(z)}\right) \geq 0 \quad (\text{for } \alpha = 0) \quad (3.3)$$

and

$$\Re\left(\frac{f'_n(z)}{f'(z)}\right) \geq \frac{1}{2} \quad (\text{for } \alpha = 0). \quad (3.4)$$

The results are sharp for the extremal functions given by

$$f(z) = \frac{1}{z} + \frac{1-\alpha}{n+1+\alpha} z^{n+1} \quad (n \in \mathbb{N}). \quad (3.5)$$

where the equality in (3.1) is attained when $z = re^{\pi i/n+2}$ ($r \rightarrow 1^-$), and for (3.2) equality is attained $z = re^{2\pi i/n+2}$ ($r \rightarrow 1^-$). Similarly equality in (3.3) is attained when $z = re^{2\pi i/n+2}$ ($r \rightarrow 1^-$), $\alpha = 0$; and for (3.4) equality is attained when $z = re^{\pi i/n+2}$ ($r \rightarrow 1^-$), $\alpha = 0$.

COROLLARY 5. *If $f(z) \in \Sigma$ satisfies the coefficient inequality (2.4), then*

$$\Re\left(\frac{f(z)}{f_n(z)}\right) \geq \frac{(n+2)(n+\alpha)}{(n+1)(n+1+\alpha)} \quad (0 \leq \alpha < 1) \quad (3.6)$$

$$\Re\left(\frac{f_n(z)}{f(z)}\right) \geq \frac{(n+1)(n+1+\alpha)}{(n+1)(n+2)-n(1-\alpha)} \quad (0 \leq \alpha < 1) \quad (3.7)$$

$$\Re\left(\frac{f'(z)}{f'_n(z)}\right) \geq \frac{n+2\alpha}{n+1+\alpha} \quad (0 \leq \alpha < 0) \quad (3.8)$$

and

$$\Re\left(\frac{f'_n(z)}{f'(z)}\right) \geq \frac{1}{2} \quad (0 \leq \alpha < 1). \quad (3.9)$$

The results are sharp for the extremal functions given by

$$f(z) = \frac{1}{z} + \frac{1-\alpha}{(n+1+\alpha)(n+1)} z^{n+1} \quad (n \in \mathbb{N}), \quad (3.10)$$

where the equality in (3.6) and (3.9) is attained when $z = re^{\pi i/n+2}$ ($r \rightarrow 1^-$), and for (3.7) and (3.8) equality is attained when $z = re^{2\pi i/n+2}$ ($r \rightarrow 1^-$).

COROLLARY 6. If $f(z) \in \Sigma$ satisfies the coefficient inequality (2.5), then

$$\Re\left(\frac{f(z)}{f_n(z)}\right) \geq \frac{n+\alpha}{n+1} \quad (0 \leq \alpha < 1) \quad (3.11)$$

$$\Re\left(\frac{f_n(z)}{f(z)}\right) \geq \frac{n+1}{n+2-\alpha} \quad (0 \leq \alpha < 1) \quad (3.12)$$

$$\Re\left(\frac{f'(z)}{f'_n(z)}\right) \geq \alpha \quad (0 \leq \alpha < 1) \quad (3.13)$$

and

$$\Re\left(\frac{f'_n(z)}{f'(z)}\right) \geq \frac{1}{2-\alpha} \quad (0 \leq \alpha < 1). \quad (3.14)$$

The results are sharp for the extremal functions given by

$$f(z) = \frac{1}{z} + \frac{1-\alpha}{n+1} z^{n+1} \quad (n \in \mathbb{N}), \quad (3.15)$$

where the equality in (3.11) and (3.14) is attained when $z = re^{\pi i/n+2}$ ($r \rightarrow 1^-$), and for (3.12) and (3.13) equality is attained when $z = re^{2\pi i/n+2}$ ($r \rightarrow 1^-$).

REMARK 2. Each of the results of Corollaries 4 and 5 were earlier proved independently by Cho and Owa [1, Theorems 2.1, 2.2, 2.3, 2.4 and 2.5]. The results stated in Corollary 6 are believed to be new.

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