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ON STRONG SHIFT EQUIVALENCE OF HILBERT C^*-BIMODULES

By

Kengo Matsumoto

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Abstract. We will study a notion of strong shift equivalence between two Hilbert C^*-bimodules as a generalization of strong shift equivalence between two nonnegative matrices. We will prove that if two finite projective Hilbert C^*-bimodules are strong shift equivalent, the gauge actions of the C^*-algebras of the Hilbert C^*-bimodules are stably outer conjugate. Hence the K-theoretic groups of the C^*-algebras of strong shift equivalent Hilbert C^*-bimodules are invariant.

1. Introduction

Let $\mathcal{A}$ be a C^*-algebra. Let X be a Hibert C^*-right $\mathcal{A}$-module with left action of $\mathcal{A}$. It is called a Hilbert C^*-bimodule over $\mathcal{A}$. M. Pimsner constructed a C^*-algebra for a Hilbert C^*-bimodule ([19], cf. [9]). The C^*-algebra is a generalization of both Cuntz-Krieger algebras and crossed products $\mathcal{A} \rtimes \mathbb{Z}$ by the integer $\mathbb{Z}$. If $\mathcal{A}$ is finite dimensional and commutative and the bimodule X has an orthogonal finite basis, the C^*-algebra is isomorphic to a Cuntz-Krieger algebra. Let $A = [A(i,j)]_{i,j=1,\dots,n}$ be an $n \times n$ matrix with entries in nonnegative integers, that is called a nonnegative matrix for brevity. Let G_A be a finite directed graph with n vertices $\{v_1, \dots, v_n\}$ and with $A(i,j)$ directed edges whose source vertices are v_i and terminal vertices are v_j. For a directed edge e, we denote by $s(e)$ the source vertex of e and by $t(e)$ the terminal vertex of e. Let E_{G_A} be the edge set of the graph G_A. Let Λ_A be the compact set of all biinfinte sequences $(a_i)_{i\in\mathbb{Z}} \in E_{G_A}^{\mathbb{Z}}$ of edges a_i of G_A such that $t(a_i) = s(a_{i+1})$ for all $i \in \mathbb{Z}$. We denote by σ_A the shift transformation on Λ_A defined by $\sigma_A((a_i)_{i\in\mathbb{Z}}) = (a_{i+1})_{i\in\mathbb{Z}}$. The topological dynamical system (Λ_A, σ_A) is called the topological Markov shift associated with A. For the classification problem of the topological Markov shifts up to topological conjugacy, R. F. Williams in [22] proved that the topological Markov shifts (Λ_A, σ_A) and (Λ_B, σ_B) are topologically conjugate if and only if the matrices A and B are strong shift equivalent. Two nonnegative square matrices M

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and N are said to be strong shift equivalent in 1-step if there exist nonnegative rectangular matrices R, S such that $M = RS, N = SR$. If there exists a finite sequence of nonnegative matrices $A_1, A_2, \dots, A_k$ such that $A = A_1, B = A_k$ and A_i is strong shift equivalent to A_{i+1} in 1-step for $i = 1, 2, \dots, k-1$, then A and B are said to be strong shift equivalent.

Let $\mathcal{K}$ be the C^*-algebra of all compact operators on a separable infinite dimensional Hilbert space. Cuntz and Krieger proved that if two topological Markov shifts (Λ_A, σ_A) and (Λ_B, σ_B) are topologically conjugate, the gauge actions γ^A and γ^B of the Cuntz-Krieger algebras $\mathcal{O}_A$ and $\mathcal{O}_B$ are stably conjugate. That is, there exists an isomorphism ϕ from $\mathcal{O}_A \otimes \mathcal{K}$ to $\mathcal{O}_B \otimes \mathcal{K}$ such that $\phi \circ (\gamma^A \otimes \mathrm{id}) = (\gamma^B \otimes \mathrm{id}) \circ \phi$ ([5, Theorem 3.8],cf.[4]). Their proof is due to a dynamical method without using strong shift equivalence condition on the underlying nonnegative matrices.

In [15], a notion of C^*-symbolic dynamical system has been introduced as a C^*-algebraic generalization of a finite directed labeled graph. C^*-symbolic dynamical systems naturally yield Hilbert C^*-bimodules with finite bases so that they give rise to C^*-algebras of the Hilbert C^*-bimodules. The author has formulated strong shift equivalences of C^*-symbolic dynamical systems and of Hilbert C^*-bimodules and proved that if two C^*-symbolic dynamical systems are strong shift equivalent, the gauge actions of the C^*-algebras of the Hilbert C^*-bimodules are stably outer conjugate. In this short note, we will directly prove that if two Hilbert C^*-bimodules are strong shift equivalent, the gauge actions of the C^*-algebras of the Hilbert C^*-bimodules are stably outer conjugate. This result is a generalization of the similar result for C^*-symbolic dynamical systems. Hence it is a generalization of the main result of [14] for the C^*-algebras of λ-graph systems and of [5, Theorem 3.8] for the Cuntz-Krieger algebras. We will also give an exact proof for the result that two nonnegative matrices are strong shift equivalent if and only if the Hilbert C^*-bimodules associated with the matrices are strong shift equivalent.

2. Hilbert C^*-bimodules and its C^*-algebras

We review briefly Pimsner's C^*-algebras from Hilbert C^*-bimodules following [19] and [7] (cf.[9], [8]). For a C^*-algebra $\mathcal{A}$, a Hilbert C^*-right $\mathcal{A}$-module X is a $\mathbb{C}$-vector space with a right $\mathcal{A}$-module structure and an $\mathcal{A}$-valued inner product $\langle , \rangle$ satisfying the following conditions [8, Definition 1.2]:

(a) $\langle , \rangle$ is left conjugate and right linear.
(b) $\langle x, ya \rangle = \langle x, y \rangle a$ and $\langle xa, y \rangle = a^* \langle x, y \rangle$ for all $x, y \in X$ and $a \in \mathcal{A}$.
(c) $\langle x, x \rangle \geq 0$ for all $x \in X$, and $\langle x, x \rangle = 0$ if and only if $x = 0$.
(d) $\langle x, y \rangle = \langle y, x \rangle^*$ for all $x, y \in X$.

(e) X is complete with respect to the norm $\|x\| = \|\langle x, x\rangle\|^{1/2}$.

A Hilbert C^*-right $\mathcal{A}$-module X is said to be full if the closed linear span of $\{\langle x, y\rangle \mid x, y \in X\}$ is equal to $\mathcal{A}$. Let $\mathbb{L}_{\mathcal{A}}(X)$ be the algebra of bounded linear right $\mathcal{A}$-module maps on X with adjoints with respect to the $\mathcal{A}$-valued inner product on X. We denote by $\mathbb{K}_{\mathcal{A}}(X)$ the norm closure of linear combinations of rank one operators $\theta_{x,y} \in \mathbb{L}_{\mathcal{A}}(X)$ for $x, y \in X$ defined by $\theta_{x,y}(z) = x\langle y, z\rangle$ for $z \in X$. A finite subset $\{u_1, \dots, u_n\}$ of X is called a basis for X if $x = \sum_{i=1}^n u_i\langle u_i, x\rangle$ for all $x \in X$. Recall that X has a finite basis if and only if $\mathbb{K}_{\mathcal{A}}(X) = \mathbb{L}_{\mathcal{A}}(X)$ ([8]). It is equivalent to the condition that X is finite projective. Throughout this paper, we assume for simplicity that the C^*-algebras $\mathcal{A}$ are unital and Hilbert C^*-modules are full and finite projective.

Let $\phi : \mathcal{A} \to \mathbb{L}_{\mathcal{A}}(X)$ be a unital isometric $*$-homomorphism. The pair (ϕ, X) is called a Hilbert C^*-bimodule over $\mathcal{A}$ (cf.[8]). M. Pimsner [19] constructed a C^*-algebra $\mathcal{O}_{(\phi,X)}$ from Hilbert C^*-bimodule (ϕ, X). It is simply written as $\mathcal{O}_X$. The C^*-algebra is the universal C^*-algebra generated by $\{S_x \mid x \in X\}$ together with a contraction $X \ni x \to S_x \in \mathcal{O}_X$, and unital $*$-homomorphisms $\pi_{\mathcal{A}} : \mathcal{A} \to \mathcal{O}_X$ and $\pi_{\mathbb{K}} : \mathbb{K}_{\mathcal{A}}(X) \to \mathcal{O}_X$ satisfying the following relations:

$$(2.1) \qquad \begin{cases} S_{kx} = \pi_{\mathbb{K}}(k)S_x, \quad S_{xa} = S_x\pi_{\mathcal{A}}(a), \quad \pi_{\mathbb{K}}(\phi(a)) = \pi_{\mathcal{A}}(a), \\ S_xS_y^* = \pi_{\mathbb{K}}(\theta_{x,y}) \quad \text{and} \quad S_x^*S_y = \pi_{\mathcal{A}}(\langle x, y\rangle) \end{cases}$$

for $x, y \in X, k \in \mathbb{K}_{\mathcal{A}}(X), a \in \mathcal{A}$. The universality means that the algebra $\mathcal{O}_X$ is the biggest C^*-algebra in the C^*-algebras satisfying the above systems. For $z \in \mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$, the correspondence $S_x \to zS_x$ gives rise to an automorphism γ_z of $\mathcal{O}_X$. It yields an action γ of $\mathbb{T}$, that is called the gauge action.

Ideal structure and simplicity condition on the C^*-algebras have been studied in [7] and [16] (cf.[21]).

3. Strong shift equivalence of nonnegative matrices and Hilbert C^*-bimodules

Let $A = [A(i,j)]_{i,j=1,\dots,n}$ be an $n \times n$ nonnegative matrix. Assume that both every row and every column have at least one nonzero entry. Hence every vertex of the graph G_A has at least both one in-coming edge and one out-going edge. Consider the n-dimensional commutative C^*-algbera $\mathcal{A}_{G_A} = \mathbb{C}E_1 \oplus \cdots \oplus \mathbb{C}E_n$ where $E_i, i = 1, \dots, n$ are mutually orthogonal minimal projections of $\mathcal{A}$ that correspond to the vertices $v_i, i = 1, \dots, n$ of G_A. Define an $n \times n$-matrix A_e for

$e \in E_{G_A}$ with entries in $\{0,1\}$ by

$$A_e(i,j) = \begin{cases} 1 & \text{if } s(e) = v_i, t(e) = v_j, \\ 0 & \text{otherwise,} \end{cases}$$

for $i, j = 1, \dots, n$. We set

$$\rho_e^A(E_i) = \sum_{j=1}^{n} A_e(i,j) E_j, \qquad i = 1, \dots, n$$

so that ρ_e^A defines a $*$-endomorphism of $\mathcal{A}_{G_A}$. Put the projections $P_e^A = \rho_e^A(1)$ in $\mathcal{A}_{G_A}$. Let $\{\epsilon_e\}_{e \in E_{G_A}}$ denote the standard basis of the $|E_{G_A}|$-dimensional vector space $\mathbb{C}^{|E_{G_A}|}$, where $|E_{G_A}|$ denotes the cardinal number of E_{G_A}. Set

$$X_A = \sum_{e \in E_{G_A}} \mathbb{C}\epsilon_e \otimes P_e^A \mathcal{A}_{G_A}.$$

Define a right $\mathcal{A}_{G_A}$-action and an $\mathcal{A}_{G_A}$-valued inner product on X_A by setting

$$(\epsilon_e \otimes P_e^A x) y := \epsilon_e \otimes P_e^A x y,$$

$$\langle \epsilon_e \otimes P_e^A x \mid \epsilon_f \otimes P_f^A y \rangle := \begin{cases} x^* P_e^A y & \text{if } e = f, \\ 0 & \text{otherwise} \end{cases}$$

for $e, f \in E_{G_A}$ and $x, y \in \mathcal{A}_{G_A}$. Then X_A forms a Hilbert C^*-right $\mathcal{A}_{G_A}$-module. We put $u_e := \epsilon_e \otimes P_e^A$ for $e \in E_{G_A}$. The family $u_e, e \in E_{G_A}$ forms an orthogonal finite basis of X_A in the sense of [7] such that $\sum_{e \in E_{G_A}} \langle u_e \mid u_e \rangle \geq 1$. We say that a finite basis of a Hilbert C^*-module is *essential* if the basis satisfies this inequality. Define a left-action ϕ_A of $\mathcal{A}_{G_A}$ to $\mathbb{L}_{\mathcal{A}_{G_A}}(X_A)$ by setting

$$\phi_A(a) u_e x := u_e \rho_e^A(a) x, \qquad a, x \in \mathcal{A}_{G_A}, e \in E_{G_A}.$$

Hence we have a Hilbert C^*-bimodule (ϕ_A, X_A) over $\mathcal{A}_{G_A}$, that is finite projective.

Let us formulate strong shift equivalence of Hilbert C^*-bimodules. Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras. We mean by a Hilbert C^*-right $\mathcal{B}$-module $(\eta, {}_{\mathcal{A}}X_{\mathcal{B}})$ with left $\mathcal{A}$-action a Hilbert C^*-right $\mathcal{B}$-module ${}_{\mathcal{A}}X_{\mathcal{B}}$ with a unital $*$-homomorphism $\eta : \mathcal{A} \to \mathbb{L}({}_{\mathcal{A}}X_{\mathcal{B}})$. Let $(\eta, {}_{\mathcal{A}}X_{\mathcal{B}})$ be a Hilbert C^*-right $\mathcal{B}$-module with left $\mathcal{A}$-action and $(\zeta, {}_{\mathcal{B}}X_{\mathcal{C}})$ a Hilbert C^*-right $\mathcal{C}$-module with left $\mathcal{B}$-action. Define the relative tensor product

$$(\eta, {}_{\mathcal{A}}X_{\mathcal{B}}) \otimes_{\mathcal{B}} (\zeta, {}_{\mathcal{B}}X_{\mathcal{C}}) := (\eta \otimes 1, {}_{\mathcal{A}}X_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}X_{\mathcal{C}})$$

where ${}_{\mathcal{A}}X_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}X_{\mathcal{C}}$ is the Hilbert C^*-right $\mathcal{C}$-module of the tensor product relative to $\mathcal{B}$, and $\eta \otimes 1$ is the natural left $\mathcal{A}$-action on it. It is easy to check that if both $(\eta, {}_{\mathcal{A}}X_{\mathcal{B}})$ and $(\zeta, {}_{\mathcal{B}}X_{\mathcal{C}})$ are full (resp. finite projective), then the relative tensor product are full (resp. finite projective).

DEFINITION. Let $(\phi, X_{\mathcal{A}})$ be a Hilbert C^*-bimodule over $\mathcal{A}$ and $(\psi, X_{\mathcal{B}})$ a Hilbert C^*-bimodule over $\mathcal{B}$. They are said to be *strong shift equivalent in* 1*-step* if there exist a full, finte projective Hilbert C^*-right $\mathcal{B}$-module $(\eta, {}_{\mathcal{A}}X^1_{\mathcal{B}})$ with left $\mathcal{A}$-action and a full, finite projective Hilbert C^*-right $\mathcal{A}$-module $(\zeta, {}_{\mathcal{B}}X^2_{\mathcal{A}})$ with left $\mathcal{B}$-action such that

$$(\eta \otimes 1, {}_{\mathcal{A}}X^1_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}X^2_{\mathcal{A}}) = (\phi, X_{\mathcal{A}}) \quad \text{as a Hilbert } C^*\text{-bimodule over } \mathcal{A},$$
$$(\zeta \otimes 1, {}_{\mathcal{B}}X^2_{\mathcal{A}} \otimes_{\mathcal{A}} {}_{\mathcal{A}}X^1_{\mathcal{B}}) = (\psi, X_{\mathcal{B}}) \quad \text{as a Hilbert } C^*\text{-bimodule over } \mathcal{B}.$$

The above equalities of Hilbert C^*-bimodules mean unitary equivalences as Hilbert C^*-bimodules. In this situation, we say that $(\eta, {}_{\mathcal{A}}X^1_{\mathcal{B}})$ and $(\zeta, {}_{\mathcal{B}}X^2_{\mathcal{A}})$ satisfy the strong shift equivalence relations between $(\phi, X_{\mathcal{A}})$ and $(\psi, X_{\mathcal{B}})$. Consider the direct sum

$$(\eta, {}_{\mathcal{A}}X^1_{\mathcal{B}}) \oplus (\zeta, {}_{\mathcal{B}}X^2_{\mathcal{A}}) := (\eta \oplus \zeta, {}_{\mathcal{A}}X^1_{\mathcal{B}} \oplus {}_{\mathcal{B}}X^2_{\mathcal{A}})$$

as a Hilbert C^*-right $\mathcal{B} \oplus \mathcal{A}$-module with left $\mathcal{A} \oplus \mathcal{B}$-action. It is denoted by $(\xi, \widehat{X})$ and satisfies

$${}_{\mathcal{A}}X^1_{\mathcal{B}} = \xi(\mathcal{A})\widehat{X} = \widehat{X}\mathcal{B}, \qquad {}_{\mathcal{B}}X^2_{\mathcal{A}} = \xi(\mathcal{B})\widehat{X} = \widehat{X}\mathcal{A}.$$

As $\widehat{X}$ is naturally regarded as a Hilbert C^*-right $\mathcal{A} \oplus \mathcal{B}$-module, $(\xi, \widehat{X})$ is considered to be a Hilbert C^*-bimodule over $\mathcal{A} \oplus \mathcal{B}$. It is called a *bipartite* Hilbert C^*-bimodule. If there exists an N-chain of strong shift equivalences in 1-step between $(\phi, X_{\mathcal{A}})$ and $(\psi, X_{\mathcal{B}})$, they are said to be *strong shift equivalent (in* N*-step)* and written as $(\phi, X_{\mathcal{A}}) \underset{N}{\approx} (\psi, X_{\mathcal{B}})$.

We note that the two equalities of the strong shift equivalence relations above are equivalent to the equality:

$$(\xi, \widehat{X} \otimes_{\mathcal{A} \oplus \mathcal{B}} \widehat{X}) = (\phi, X_{\mathcal{A}}) \oplus (\psi, X_{\mathcal{B}}) \quad \text{as a Hilbert } C^*\text{-bimodule over } \mathcal{A} \oplus \mathcal{B}.$$

PROPOSITION 3.1. *Two nonnegative matrices A and B are strong shift equivalent if and only if the Hilbert C^*-bimodules (ϕ_A, X_A) and (ϕ_B, X_B) are strong shift equivalent.*

Proof. Let $A = [A(i,j)]_{i,j=1,\dots,n}$ and $B = [B(k,l)]_{k,l=1,\dots,m}$ be an $n \times n$ nonnegative matrix and an $m \times m$ nonnegative matrix respectively. We denote by $\mathcal{A}_A$ and $\mathcal{A}_B$ the algebras $C(V_A)$ and $C(V_B)$ of all continuous functions on the vertex sets $V_A = \{v^A_1, \dots, v^A_n\}$ and $V_B = \{v^B_1, \dots, v^B_m\}$ of the garphs G_A and G_B respectively. Let $E_1, \dots, E_n$ and $F_1, \dots, F_m$ be the minimal projections of $\mathcal{A}_A$ and of $\mathcal{A}_B$ respectively so that

$$\mathcal{A}_A = \mathbb{C}E_1 \oplus \cdots \oplus \mathbb{C}E_n, \qquad \mathcal{A}_B = \mathbb{C}F_1 \oplus \cdots \oplus \mathbb{C}F_m.$$

Suppose that A and B are strong shift equivalent in 1-step. Let R and S be the nonnegative rectangular matrices such that $A = RS$ and $B = SR$. The graph of the matrix R is the directed graph G_R with vertices $V_A \sqcup V_B$, and with $R(i,j)$ distinct edges $E_R(i,j)$ with initial vertex v_i^A and terminal vertex v_j^B for $i = 1, \dots, n,\ j = 1, \dots, m$. We denote by E_R the edge set $\cup_{\substack{i=1,\dots,n \\ j=1,\dots,m}} E_R(i,j)$. Define a $*$-homomorphism ρ_e^R for $e \in E_R$ from $\mathcal{A}_A$ to $\mathcal{A}_B$ by setting

$$\rho_e^R(E_i) = \sum_{j=1}^{m} R_e(i,j) F_j, \qquad i = 1, \dots, n$$

where $R_e(i,j) = 1$ if $s(e) = v_i^A, t(e) = v_j^B$, otherwise $R_e(i,j) = 0$. Put the projections $P_e^R = \rho_e^R(1_A)$ in $\mathcal{A}_B$. Let $\{\epsilon_e\}_{e \in E_R}$ denote the standard basis of the $|E_R|$-dimensional vector space $\mathbb{C}^{|E_R|}$. Set

$$X_R = \sum_{e \in E_R} \mathbb{C}\epsilon_e \otimes P_e^R \mathcal{A}_B.$$

Define a right $\mathcal{A}_B$-action and an $\mathcal{A}_B$-valued inner product on X_R by similar way to the situation in the begining of this section. We have a Hilbert C^*-right $\mathcal{A}_B$-module X_R. By using the $*$-homomorphism ρ_e^R from $\mathcal{A}_A$ to $\mathcal{A}_B$, we similarly define a left-action ϕ_R of $\mathcal{A}_A$ on X_R so that we have a Hilbert C^*-right $\mathcal{A}_B$-module (ϕ_R, X_R) with left $\mathcal{A}_A$-action. By using the matrix S, we similarly have a Hilbert C^*-right $\mathcal{A}_A$-module (ϕ_S, X_S) with left $\mathcal{A}_B$-action. It is straightforward to see that the tensor product Hilbert C^*-module $(\phi_R \otimes 1, X_R \otimes_{\mathcal{A}_B} X_S)$ is unitarily equivalent to the Hilbert C^*-bimodule (ϕ_{RS}, X_{RS}), and similarly the tensor product Hilbert C^*-module $(\phi_S \otimes 1, X_S \otimes_{\mathcal{A}_A} X_R)$ is unitarily equivalent to the Hilbert C^*-bimodule (ϕ_{SR}, X_{SR}). Therefore (ϕ_A, X_A) and (ϕ_B, X_B) are strong shift equivalent.

Conversely assume that (ϕ_A, X_A) and (ϕ_B, X_B) are strong shift equivalent in 1-step. Take a full, finite projective Hilbert C^*-right $\mathcal{A}_B$-module $(\eta, {}_{\mathcal{A}_A}X^1_{\mathcal{A}_B})$ with left $\mathcal{A}_A$-action and a full, finite projective Hilbert C^*-right $\mathcal{A}_A$-module $(\zeta, {}_{\mathcal{A}_B}X^2_{\mathcal{A}_A})$ with left $\mathcal{A}_B$-action that satisfy the strong shift equivalence relations between (ϕ_A, X_A) and (ψ_B, X_B). Let $\{u_c^1\}_{c \in E^1}$ be a finite basis of ${}_{\mathcal{A}_A}X^1_{\mathcal{A}_B}$ and $\{u_d^2\}_{d \in E^2}$ be a finite basis of ${}_{\mathcal{A}_B}X^2_{\mathcal{A}_A}$ respectively. For $c \in E^1$, define an $n \times m$ matrix R_c by

$$R_c(i,j)F_j = \langle u_c^1 \mid \phi_A(E_i) u_c^1 \rangle F_j, \qquad i = 1, \dots, n,\ j = 1, \dots, m.$$

Since $\langle u_c^1 \mid \phi_A(E_i) u_c^1 \rangle$ is a projection in $\mathcal{A}_B$, the matrix R_c has its entries in $\{0, 1\}$. Define the $n \times m$ matrix $R = [R(i,j)]_{i,j}$ by setting

$$R(i,j) = \sum_{c \in E^1} R_c(i,j), \qquad i = 1, \dots, n,\ j = 1, \dots, m.$$

We similarly define an $m \times n$ matrix $S = [S(j,i)]_{j,i}$ from the left $\mathcal{A}_B$-action ϕ_B to ${}_{\mathcal{A}_B}X^2_{\mathcal{A}_A}$. It is direct to see that the Hilbert C^*-bimodule $(\eta \otimes 1, {}_{\mathcal{A}_A}X^1_{\mathcal{A}_B} \otimes_{\mathcal{A}_B} {}_{\mathcal{A}_B}X^2_{\mathcal{A}_A})$ over $\mathcal{A}_A$ is unitarily equivalent to (ϕ_{RS}, X_{RS}), and the Hilbert C^*-bimodule $(\zeta \otimes 1, {}_{\mathcal{A}_B}X^2_{\mathcal{A}_A} \otimes_{\mathcal{A}_A} {}_{\mathcal{A}_A}X^1_{\mathcal{A}_B})$ over $\mathcal{A}_B$ is unitarily equivalent to (ϕ_{SR}, X_{SR}). Hence (ϕ_A, X_A) is unitarily equivalent to (ϕ_{RS}, X_{RS}), and (ϕ_B, X_B) is unitarily equivalent to (ϕ_{SR}, X_{SR}). Let $\{u^A_\alpha\}_{\alpha \in E_A}$ be the canonical basis of X_A, and $\{u^{RS}_\beta\}_{\beta \in E_{RS}}$ the canonical basis of X_{RS}. Let $\Phi : X_A \to X_{RS}$ be a unitary that intertwines between (ϕ_A, X_A) and (ϕ_{RS}, X_{RS}). We put

$$U_{\alpha,\beta} = \langle \Phi(u^A_\alpha) \mid u^{RS}_\beta \rangle, \qquad \alpha \in E_A, \beta \in E_{RS}.$$

As $\Phi(\phi_A(a)x) = \phi_{RS}(a)\Phi(x)$ for $a \in \mathcal{A}_A, x \in X_A$, one sees that for $\alpha \in E_A, \beta \in E_{RS}$ and $a \in \mathcal{A}_A (= \mathcal{A}_{RS})$

$$\rho^A_\alpha(a) = \sum_{\beta \in E_{RS}} U_{\alpha,\beta}\rho^{RS}_\beta(a)U^*_{\alpha,\beta}, \qquad \rho^{RS}_\beta(a) = \sum_{\alpha \in E_A} U^*_{\alpha,\beta}\rho^A_\alpha(a)U_{\alpha,\beta}.$$

We also have

$$\sum_{\gamma \in E_A} U^*_{\gamma,\epsilon}U_{\gamma,\beta} = \delta_{\epsilon,\beta}P^{RS}_\beta, \qquad \sum_{\beta \in E_{RS}} U_{\gamma,\beta}U^*_{\alpha,\beta} = \delta_{\gamma,\alpha}P^A_\alpha.$$

Since X_{RS} and X_A are unitarily equivalent, one sees that $|E_A| = |E_{RS}|$. By the above equalities, for $i = 1, \dots, n$ the diagonal matrices $[\rho^A_\alpha(E_i)]_{\alpha \in E_A}$ and $[\rho^{RS}_\beta(E_i)]_{\beta \in E_{RS}}$ are unitarily equivalent through $\{U_{\alpha,\beta}\}_{\alpha \in E_A, \beta \in E_{RS}}$. Thus we have

$$\sum_{\alpha \in E_A} \rho^A_\alpha(E_i) = \sum_{\beta \in E_{RS}} \rho^{RS}_\beta(E_i) \quad \text{so that} \quad \sum_{\alpha \in E_A} \rho^A_\alpha(E_i)E_j = \sum_{\beta \in E_{RS}} \rho^{RS}_\beta(E_i)E_j.$$

This means $A(i,j) = (RS)(i,j)$ for all $i, j = 1, 2, \dots, n$ so that $A = RS$. Similarly one sees $B = SR$. Therefore A is strong shift equivalent to B in 1-step. □

Shift equivalence of Hilbert C^*-bimodules are similarly defined in [15] as in the following way.

Definition. Let $(\phi, X_\mathcal{A})$ be a Hilbert C^*-bimodule over $\mathcal{A}$ and $(\psi, X_\mathcal{B})$ a Hilbert C^*-bimodule over $\mathcal{B}$. They are said to be *shift equivalent (of lag N)* if there exist a full, finite projective Hilbert C^*-right $\mathcal{B}$-module $(\eta, {}_\mathcal{A}X_\mathcal{B})$ with left $\mathcal{A}$ action and a full, finite projective Hilbert C^*-right $\mathcal{A}$-module $(\zeta, {}_\mathcal{B}X'_\mathcal{A})$ with left $\mathcal{B}$ action such that

$$(\phi \otimes 1, \underbrace{X_\mathcal{A} \otimes_\mathcal{A} \cdots \otimes_\mathcal{A} X_\mathcal{A}}_{N}) = (\eta \otimes 1, {}_\mathcal{A}X_\mathcal{B} \otimes_\mathcal{B} {}_\mathcal{B}X'_\mathcal{A}),$$

$$(\psi \otimes 1, \underbrace{X_\mathcal{B} \otimes_\mathcal{B} \cdots \otimes_\mathcal{B} X_\mathcal{B}}_{N}) = (\zeta \otimes 1, {}_\mathcal{B}X'_\mathcal{A} \otimes_\mathcal{A} {}_\mathcal{A}X_\mathcal{B}),$$

and

$$(\eta\otimes 1, {}_{\mathcal{A}}X_{\mathcal{B}}\otimes_{\mathcal{B}}X_{\mathcal{B}}) = (\phi\otimes 1, X_{\mathcal{A}}\otimes_{\mathcal{A}}{}_{\mathcal{A}}X_{\mathcal{B}}),\ (\zeta\otimes 1, {}_{\mathcal{B}}X'_{\mathcal{A}}\otimes_{\mathcal{A}}X_{\mathcal{A}}) = (\psi\otimes 1, X_{\mathcal{B}}\otimes_{\mathcal{B}}{}_{\mathcal{B}}X'_{\mathcal{A}})$$

We write this situation as $(\phi, X_{\mathcal{A}}) \underset{N}{\sim} (\psi, X_{\mathcal{B}})$.

The following proposition is shown in [15].

Proposition 3.2. *Let $(\phi, X_{\mathcal{A}})$, $(\psi, X_{\mathcal{B}})$ and $(\varphi, X_{\mathcal{C}})$ be Hilbert C^*-bimodules.*

(i) $(\phi, X_{\mathcal{A}}) \underset{N}{\approx} (\psi, X_{\mathcal{B}})$ *implies* $(\phi, X_{\mathcal{A}}) \underset{N}{\sim} (\psi, X_{\mathcal{B}})$.

(ii) $(\phi, X_{\mathcal{A}}) \underset{N}{\sim} (\psi, X_{\mathcal{B}})$ *implies* $(\phi, X_{\mathcal{A}}) \underset{N'}{\sim} (\psi, X_{\mathcal{B}})$ *for all* $N' \geq N$.

(iii) $(\phi, X_{\mathcal{A}}) \underset{N}{\sim} (\psi, X_{\mathcal{B}})$ *and* $(\psi, X_{\mathcal{B}}) \underset{L}{\sim} (\varphi, X_{\mathcal{C}})$ *imply* $(\phi, X_{\mathcal{A}}) \underset{N+L}{\sim} (\varphi, X_{\mathcal{C}})$.

Similarly to Proposition 3.1, we may straightforwardly prove that two nonnegative matrices A and B are shift equivalent if and only if the Hilbert C^*-bimodules (ϕ_A, X_A) and (ϕ_B, X_B) are shift equivalent.

4. Strong shift equivalence of Hilbert C^*-bimodules and their C^*-algebras

We will prove the following theorem.

Theorem 4.1. *Let $(\phi, X_{\mathcal{A}})$ and $(\psi, X_{\mathcal{B}})$ be finite projective Hilbert C^*-bimodules over unital C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ respectively. If $(\phi, X_{\mathcal{A}})$ and $(\psi, X_{\mathcal{B}})$ are strong shift equivalent, the C^*-algebras $\mathcal{O}_{X_{\mathcal{A}}}$ and $\mathcal{O}_{X_{\mathcal{B}}}$ with gauge actions are stably outer conjugate.*

This theorem and its proof are generalizations of [14, Theorem 3.15].

Suppose that $(\phi, X_{\mathcal{A}})$ and $(\psi, X_{\mathcal{B}})$ are strong shift equivalent in 1-step. Hence there exist a Hilbert C^*-right $\mathcal{B}$-module $(\eta, {}_{\mathcal{A}}X^1_{\mathcal{B}})$ with left $\mathcal{A}$-action and a Hilbert C^*-right $\mathcal{A}$-module $(\zeta, {}_{\mathcal{B}}X^2_{\mathcal{A}})$ with left $\mathcal{B}$-action satisfying the strong shift equivalence relations between $(\phi, X_{\mathcal{A}})$ and $(\psi, X_{\mathcal{B}})$. Let $(\xi, \widehat{X}) = (\eta, {}_{\mathcal{A}}X^1_{\mathcal{B}}) \oplus (\zeta, {}_{\mathcal{B}}X^2_{\mathcal{A}})$ be the bipartite Hilbert C^*-bimodule over $\mathcal{A}\oplus\mathcal{B}$. The C^*-algera $\mathcal{O}_{\widehat{X}}$ of $(\xi, \widehat{X})$ is a universal C^*-algebra generated by $S_x, x \in \widehat{X}$ subject to the system of the relations corresponding to (2.1). Let $\pi_{\widehat{X}} : \mathcal{A}\oplus\mathcal{B} \to \mathcal{O}_{\widehat{X}}$ be the unital $*$-homomorphism satisfying $S_{x(a,b)} = S_x\pi_{\widehat{X}}(a,b)$ for $x \in \widehat{X}, (a,b) \in \mathcal{A}\oplus\mathcal{B}$. Let $C^*(S_{12}, \mathcal{A})$ and $C^*(S_{21}, \mathcal{B})$ be the C^*-subalgebras of $\mathcal{O}_{\widehat{X}}$ defined by setting

$$C^*(S_{12}, \mathcal{A}) = C^*(S_{x^{(1)}x^{(2)}}, \pi_{\widehat{X}}(a,0) \mid x^{(1)} \in {}_{\mathcal{A}}X^1_{\mathcal{B}}, x^{(2)} \in {}_{\mathcal{B}}X^2_{\mathcal{A}}, a \in \mathcal{A}) \quad \text{and}$$
$$C^*(S_{21}, \mathcal{B}) = C^*(S_{x^{(2)}x^{(1)}}, \pi_{\widehat{X}}(0,b) \mid x^{(1)} \in {}_{\mathcal{A}}X^1_{\mathcal{B}}, x^{(2)} \in {}_{\mathcal{B}}X^2_{\mathcal{A}}, b \in \mathcal{B})$$

respectively, where $S_{x^{(1)}x^{(2)}} = S_{x^{(1)}}S_{x^{(2)}}$ and $S_{x^{(2)}x^{(1)}} = S_{x^{(2)}}S_{x^{(1)}}$. Let $1_{\mathcal{A}}$ and $1_{\mathcal{B}}$ be the units of $\mathcal{A}$ and $\mathcal{B}$ respectively. Put the projections

$$P_{\mathcal{A}} = \pi_{\widehat{X}}(1_{\mathcal{A}}, 0), \qquad P_{\mathcal{B}} = \pi_{\widehat{X}}(0, 1_{\mathcal{B}}) \qquad \text{in } \mathcal{O}_{\widehat{X}}.$$

Hence $P_{\mathcal{A}} + P_{\mathcal{B}} = 1$. Let us first prove the following proposition.

PROPOSITION 4.2. $C^*(S_{12}, \mathcal{A}) = P_{\mathcal{A}}\mathcal{O}_{\widehat{X}}P_{\mathcal{A}}, \quad C^*(S_{21}, \mathcal{B}) = P_{\mathcal{B}}\mathcal{O}_{\widehat{X}}P_{\mathcal{B}}.$

We provide some lemmas to show this proposition. The following lemma is clear from the strong shift equivalence relations.

LEMMA 4.3. *For $x^{(1)} \in {}_{\mathcal{A}}X^1_{\mathcal{B}}, x^{(2)} \in {}_{\mathcal{B}}X^2_{\mathcal{A}}$ we have*

(i) $\xi(1_{\mathcal{A}}, 0)x^{(2)} = x^{(2)}(0, 1_{\mathcal{B}}) = 0$ *and* $\xi(0, 1_{\mathcal{B}})x^{(1)} = x^{(1)}(1_{\mathcal{A}}, 0) = 0$. *Hence* $x^{(2)} = \xi(0, 1_{\mathcal{B}})x^{(2)} = x^{(2)}(1_{\mathcal{A}}, 0)$ *and* $x^{(1)} = \xi(1_{\mathcal{A}}, 0)x^{(1)} = x^{(1)}(0, 1_{\mathcal{B}})$.

(ii) $P_{\mathcal{A}}\pi_{\widehat{X}}(0, b) = \pi_{\widehat{X}}(0, b)P_{\mathcal{A}} = 0$ *for* $b \in \mathcal{B}$ *and* $P_{\mathcal{B}}\pi_{\widehat{X}}(a, 0) = \pi_{\widehat{X}}(a, 0)P_{\mathcal{B}} = 0$ *for* $a \in \mathcal{A}$.

LEMMA 4.4. *For $x^{(1)}, y^{(1)} \in {}_{\mathcal{A}}X^1_{\mathcal{B}}$ and $x^{(2)}, y^{(2)} \in {}_{\mathcal{B}}X^2_{\mathcal{A}}$, one has*

(i) $S_{x^{(1)}}S_{y^{(1)}} = S_{x^{(2)}}S_{y^{(2)}} = 0.$

(ii) $S_{x^{(1)}}S^*_{y^{(2)}} = S_{x^{(2)}}S^*_{y^{(1)}} = 0.$

Proof. (i) It follows that

$$\begin{aligned}(S_{x^{(1)}}S_{y^{(1)}})^*S_{x^{(1)}}S_{y^{(1)}} =& S^*_{y^{(1)}}\pi_{\widehat{X}}(\langle x^{(1)}, x^{(1)}\rangle)S_{y^{(1)}} \\ =& S^*_{y^{(1)}}\pi_{\mathbb{K}}(\xi(\langle x^{(1)}, x^{(1)}\rangle(0, 1_{\mathcal{B}}))S_{y^{(1)}} \\ =& S^*_{y^{(1)}}\pi_{\mathbb{K}}(\xi(\langle x^{(1)}, x^{(1)}\rangle)S_{\xi(0,1_{\mathcal{B}})y^{(1)}} = 0.\end{aligned}$$

The other is similar.

(ii) It follows that

$$\begin{aligned}S_{x^{(1)}}S^*_{y^{(2)}} =& S_{x^{(1)}}\pi_{\widehat{X}}(1_{\mathcal{A}}, 0)S^*_{y^{(2)}} + S_{x^{(1)}}\pi_{\widehat{X}}(0, 1_{\mathcal{B}})S^*_{y^{(2)}} \\ =& S_{x^{(1)}(1_{\mathcal{A}},0)}S^*_{y^{(2)}} + S_{x^{(1)}}S^*_{y^{(2)}(0,1_{\mathcal{B}})} = 0.\end{aligned}$$

The other is similar. □

LEMMA 4.5. *For $x^{(1)} \in {}_{\mathcal{A}}X^1_{\mathcal{B}}, x^{(2)}, \in {}_{\mathcal{B}}X^2_{\mathcal{A}}$ and $a \in \mathcal{A}, b \in \mathcal{B}$ one has*

(i) $S^*_{x^{(1)}}\pi_{\widehat{X}}(0, b)S_{x^{(1)}} = S^*_{x^{(2)}}\pi_{\widehat{X}}(a, 0)S_{x^{(2)}} = 0.$

(ii) $S_{x^{(1)}}\pi_{\widehat{X}}(a, 0)S^*_{x^{(1)}} = S_{x^{(2)}}\pi_{\widehat{X}}(0, b)S^*_{x^{(2)}} = 0.$

Proof. (i) It follows that

$$\begin{aligned}S^*_{x^{(1)}}\pi_{\widehat{X}}(0, b)S_{x^{(1)}} &= S^*_{x^{(1)}}\pi_{\mathbb{K}}(\xi((0, b)(0, 1_{\mathcal{B}}))S_{x^{(1)}} \\ &= S^*_{x^{(1)}}\pi_{\mathbb{K}}(\xi((0, b))S_{\xi(0,1_{\mathcal{B}})x^{(1)}} = 0.\end{aligned}$$

The other is similar.

(ii) It follows that

$$S_{x^{(1)}}\pi_{\widehat{X}}(a,0) = S_{x^{(1)}}\pi_{\widehat{X}}(1_{\mathcal{A}},0)\pi_{\widehat{X}}(a,0) = S_{x^{(1)}(1_{\mathcal{A}},0)}\pi_{\widehat{X}}(a,0) = 0.$$

We similarly have $S_{x^{(2)}}\pi_{\widehat{X}}(0,b) = 0$. Hence the assertions hold. □

Let us show the equality $C^*(S_{12},\mathcal{A}) = P_{\mathcal{A}}\mathcal{O}_{\widehat{X}}P_{\mathcal{A}}$. The other one is symmetric.

LEMMA 4.6. $C^*(S_{12},\mathcal{A}) \subset P_{\mathcal{A}}\mathcal{O}_{\widehat{X}}P_{\mathcal{A}}$.

Proof. Take an arbitrary fixed $x^{(1)} \in {}_{\mathcal{A}}X^1_{\mathcal{B}}$ and $x^{(2)} \in {}_{\mathcal{B}}X^2_{\mathcal{A}}$. As $x^{(1)} = \xi(1_{\mathcal{A}},0)x^{(1)}$, one sees $P_{\mathcal{A}}S_{x^{(1)}} = S_{\xi(1_{\mathcal{A}},0)x^{(1)}} = S_{x^{(1)}}$ and hence $P_{\mathcal{A}}S_{x^{(1)}}S_{x^{(2)}} = S_{x^{(1)}}S_{x^{(2)}}$. Similarly by the equality $x^{(2)} = x^{(2)}(1_{\mathcal{A}},0)$, one has $S_{x^{(1)}}S_{x^{(2)}} = S_{x^{(1)}}S_{x^{(2)}}P_{\mathcal{A}}$. Since the equality $P_{\mathcal{A}}\pi_{\widehat{X}}(a,0)P_{\mathcal{A}} = \pi_{\widehat{X}}(a,0)$ for $a \in \mathcal{A}$ holds, the algebra $C^*(S_{12},\mathcal{A})$ is contained in the algebra $P_{\mathcal{A}}\mathcal{O}_{\widehat{X}}P_{\mathcal{A}}$. □

LEMMA 4.7. *For $x = x_1x_2\cdots x_p, y = y_1y_2\cdots y_q$ where $x_k, y_k \in {}_{\mathcal{A}}X^1_{\mathcal{B}}$ or ${}_{\mathcal{B}}X^2_{\mathcal{A}}$ if $P_{\mathcal{A}}S_x\pi_{\widehat{X}}(a,b)S_y^*P_{\mathcal{A}} \neq 0$, one of the following two conditions holds*

(1) $x_1, y_1 \in {}_{\mathcal{A}}X^1_{\mathcal{B}}, x_p, y_q \in {}_{\mathcal{A}}X^1_{\mathcal{B}}$ *and* $S_x\pi_{\widehat{X}}(a,b)S_y^* = S_x\pi_{\widehat{X}}(0,b)S_y^*$,

(2) $x_1, y_1 \in {}_{\mathcal{A}}X^1_{\mathcal{B}}, x_p, y_q \in {}_{\mathcal{B}}X^2_{\mathcal{A}}$ *and* $S_x\pi_{\widehat{X}}(a,b)S_y^* = S_x\pi_{\widehat{X}}(a,0)S_y^*$.

In both the cases, the equality $P_{\mathcal{A}}S_x\pi_{\widehat{X}}(a,b)S_y^*P_{\mathcal{A}} = S_x\pi_{\widehat{X}}(a,b)S_y^*$ holds so that $S_x\pi_{\widehat{X}}(a,b)S_y^*$ belongs to $C^*(S_{12},\mathcal{A})$.

Proof. As $P_{\mathcal{A}}S_{x^{(1)}} = S_{x^{(1)}}$ for $x^{(1)} \in {}_{\mathcal{A}}X^1_{\mathcal{B}}$ and $P_{\mathcal{A}}S_{x^{(2)}} = 0$ for $x^{(2)} \in {}_{\mathcal{B}}X^2_{\mathcal{A}}$, one knows that $x_1, y_1 \in {}_{\mathcal{A}}X^1_{\mathcal{B}}$. Hence the equalities $P_{\mathcal{A}}S_x = S_x, S_y^*P_{\mathcal{A}} = S_y^*$ hold. Now if $x_p \in {}_{\mathcal{A}}X^1_{\mathcal{B}}, y_q \in {}_{\mathcal{B}}X^2_{\mathcal{A}}$, then we have

$$S_{x_p}\pi_{\widehat{X}}(a,b) = S_{x_p}\pi_{\widehat{X}}(0,b), \qquad \pi_{\widehat{X}}(a,b)S_{y_q}^* = \pi_{\widehat{X}}(a,0)S_{y_q}^*$$

so that $S_{x_p}\pi_{\widehat{X}}(a,b)S_{y_q}^* = 0$, a contradiction. Hence if $x_p \in {}_{\mathcal{A}}X^1_{\mathcal{B}}$, then we have $y_q \in {}_{\mathcal{A}}X^1_{\mathcal{B}}$. Similarly $x_p \in {}_{\mathcal{B}}X^2_{\mathcal{A}}$ implies $y_q \in {}_{\mathcal{B}}X^2_{\mathcal{A}}$. Thus we have $x_p, y_q \in {}_{\mathcal{A}}X^1_{\mathcal{B}}$ or $x_p, y_q \in {}_{\mathcal{B}}X^2_{\mathcal{A}}$. Suppose that $x_p, y_q \in {}_{\mathcal{A}}X^1_{\mathcal{B}}$. The equality $S_{x_p}\pi_{\widehat{X}}(a,b)S_{y_q}^* = S_{x_p}\pi_{\widehat{X}}(0,b)S_{y_q}^*$ implies $S_x\pi_{\widehat{X}}(a,b)S_y^* = S_x\pi_{\widehat{X}}(0,b)S_y^*$. Since the Hilbert C^*-modules ${}_{\mathcal{A}}X^1_{\mathcal{B}}$ and ${}_{\mathcal{B}}X^2_{\mathcal{A}}$ are finite projective, there exist finite bases $\{u_c^{(1)}\}_{c\in C}$ of ${}_{\mathcal{A}}X^1_{\mathcal{B}}$ and $\{u_d^{(2)}\}_{d\in D}$ of ${}_{\mathcal{B}}X^2_{\mathcal{A}}$. We put for $\alpha \in C \sqcup D$

$$v_\alpha = \begin{cases} (u_\alpha^{(1)},0) & \text{if } \alpha \in C, \\ (0,u_\alpha^{(2)}) & \text{if } \alpha \in D. \end{cases}$$

Then $\{v_\alpha\}_{\alpha\in C\sqcup D}$ form a finite basis of $\widehat{X}$. Hence we have

$$\sum_{\alpha\in C\sqcup D}\pi_{\widehat{X}}(\theta_{v_\alpha,v_\alpha}) = \mathrm{id} \quad \text{and} \quad \sum_{\alpha\in C\sqcup D}S_{v_\alpha}S_{v_\alpha}^* = 1.$$

As $S_{x_p}S_{v_\alpha} = 0$ and hence $S_{v_\alpha}^* S_{y_q}^* = 0$ if $\alpha \in C$, we have

$$S_x \pi_{\hat{X}}(0,b) S_y^* = \sum_{\alpha,\beta \in D} S_x S_{v_\alpha} S_{v_\alpha}^* \pi_{\hat{X}}(0,b) S_{v_\beta} S_{v_\beta}^* S_y^*.$$

For $\alpha, \beta \in D$ both the vectors v_α and $\xi(0,b)v_\beta$ belong to ${}_{\mathcal{B}}X^2_{\mathcal{A}}$ so that the inner product $\langle v_\alpha, \xi(0,b)v_\beta\rangle$ takes its value in $\mathcal{A}$. By the equality $S_{v_\alpha}^* \pi_{\hat{X}}(0,b) S_{v_\beta} = \pi_{\hat{X}}(\langle v_\alpha, \xi(0,b)v_\beta\rangle)$, the element $S_{v_\alpha}^* \pi_{\hat{X}}(0,b) S_{v_\beta}$ belongs to $\pi_{\hat{X}}(\mathcal{A},0)$. As $x_1, x_p \in {}_{\mathcal{A}}X^1_{\mathcal{B}}$, the operator $S_x S_{v_\alpha}$ is a finite product of the operators of the form: $S_{x^{(1)}} S_{x^{(2)}}$ for $x^{(1)} \in {}_{\mathcal{A}}X^1_{\mathcal{B}}, x^{(2)} \in {}_{\mathcal{B}}X^2_{\mathcal{A}}$. Similarly the operator $S_{v_\beta}^* S_y^*$ is a finite product of the operators of the form: $S_{x^{(2)}}^* S_{x^{(1)}}^*$ for $x^{(1)} \in {}_{\mathcal{A}}X^1_{\mathcal{B}}, x^{(2)} \in {}_{\mathcal{B}}X^2_{\mathcal{A}}$. Hence $\sum_{\alpha,\beta\in D} S_x S_{v_\alpha} \cdot S_{v_\alpha}^* \pi_{\hat{X}}(0,b) S_{v_\beta} \cdot S_{v_\beta}^* S_y^*$ belongs to $C^*(S_{12}, \mathcal{A})$. Therefore $S_x \pi_{\hat{X}}(0,b) S_y^*$ and hence $S_x \pi_{\hat{X}}(a,b) S_y^*$ belong to $C^*(S_{12}, \mathcal{A})$.

Suppose next that $x_p, y_q \in {}_{\mathcal{B}}X^2_{\mathcal{A}}$. We then have $S_{x_p}\pi_{\hat{X}}(0,b) S_{y_q}^* = 0$ so that

$$S_x \pi_{\hat{X}}(a,b) S_y^* = S_x \pi_{\hat{X}}(a,0) S_y^*.$$

Since the operators S_x and S_y are finite products of the operators of the form: $S_{x^{(1)}} S_{x^{(2)}}$ for $x^{(1)} \in {}_{\mathcal{A}}X^1_{\mathcal{B}}, x^{(2)} \in {}_{\mathcal{B}}X^2_{\mathcal{A}}$, the element $S_x \pi_{\hat{X}}(a,b) S_y^*$ belongs to $C^*(S_{12}, \mathcal{A})$. □

Proof of Proposition 4.2. The algebra of all finite linear combinations of elements of the form:

$$S_{x_1 \cdots x_p} \pi_{\hat{X}}(a,b) S_{y_1 \cdots y_q}^*, \quad \pi_{\hat{X}}(a,b)$$

where $a \in \mathcal{A}, b \in \mathcal{B}, x_1, \dots, x_p, y_1, \dots, y_q \in {}_{\mathcal{A}}X^1_{\mathcal{B}}$ or ${}_{\mathcal{B}}X^2_{\mathcal{A}}$, is dense in $\mathcal{O}_{\hat{X}}$. One obtains the inclusion relation $C^*(S_{12}, \mathcal{A}) \supset P_{\mathcal{A}} \mathcal{O}_{\hat{X}} P_{\mathcal{A}}$ by Lemma 4.6. Thus the equality $C^*(S_{12}, \mathcal{A}) = P_{\mathcal{A}} \mathcal{O}_{\hat{X}} P_{\mathcal{A}}$ holds by Lemma 4.5. □

We will second prove the following proposition.

PROPOSITION 4.8. *The C^*-algebras $\mathcal{O}_{{}_{\mathcal{A}}X^1_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}X^2_{\mathcal{A}}}$ and $\mathcal{O}_{{}_{\mathcal{B}}X^2_{\mathcal{A}} \otimes_{\mathcal{A}} {}_{\mathcal{A}}X^1_{\mathcal{B}}}$ of the Hilbert C^*-bimodules $(\eta \otimes 1, {}_{\mathcal{A}}X^1_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}X^2_{\mathcal{A}})$ and $(\zeta \otimes 1, {}_{\mathcal{B}}X^2_{\mathcal{A}} \otimes_{\mathcal{A}} {}_{\mathcal{A}}X^1_{\mathcal{B}})$ respectively are canonically isomorphic to the algebras $C^*(S_{12}, \mathcal{A})$ and $C^*(S_{21}, \mathcal{B})$ respectively.*

Proof. The algebra $\mathcal{O}_{{}_{\mathcal{A}}X^1_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}X^2_{\mathcal{A}}}$ is the universal C^*-algebra generated by $\{s_x \mid x \in {}_{\mathcal{A}}X^1_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}X^2_{\mathcal{A}}\}$ with a contraction ${}_{\mathcal{A}}X^1_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}X^2_{\mathcal{A}} \ni x \to s_x \in \mathcal{O}_{{}_{\mathcal{A}}X^1_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}X^2_{\mathcal{A}}}$ and unital $*$-homomorphisms $\pi_{\mathcal{A}} : \mathcal{A} \ni a \to \pi_{\mathcal{A}}(a) \in \mathcal{O}_{{}_{\mathcal{A}}X^1_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}X^2_{\mathcal{A}}}$ and $\pi_{\mathbb{K}}^{\mathcal{A}} : \mathbb{K}_{\mathcal{A}}({}_{\mathcal{A}}X^1_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}X^2_{\mathcal{A}}) \ni \theta_{x,y} \to \pi_{\mathbb{K}}^{\mathcal{A}}(\theta_{x,y}) \in \mathcal{O}_{{}_{\mathcal{A}}X^1_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}X^2_{\mathcal{A}}}$ satisfying the corresponding relations to (2.1) for the Hilbert C^*-bimodule $(\eta \otimes 1, {}_{\mathcal{A}}X^1_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}X^2_{\mathcal{A}})$. Hence there exists a surjective $*$-homomorphism Φ from $\mathcal{O}_{{}_{\mathcal{A}}X^1_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}X^2_{\mathcal{A}}}$ to $C^*(S_{12}, \mathcal{A})$ such that

$$\Phi(s_{x^{(1)} \otimes x^{(2)}}) = S_{x^{(1)}} S_{x^{(2)}}, \quad \Phi(\pi_{\mathcal{A}}(a)) = \pi_{\hat{X}}(a,0),$$
$$\Phi(\pi_{\mathbb{K}}^{\mathcal{A}}(\theta_{x^{(1)} \otimes x^{(2)}, y^{(1)} \otimes y^{(2)}})) = \pi_{\mathbb{K}}(\theta_{(x^{(1)},0),(y^{(1)},0)} \theta_{(0,x^{(2)}),(0,y^{(2)})})$$

for $x^{(1)}, y^{(1)} \in {}_{\mathcal{A}}X^1_{\mathcal{B}}, x^{(2)}, y^{(2)} \in {}_{\mathcal{B}}X^2_{\mathcal{A}}, a \in \mathcal{A}$. Let $\epsilon : \mathcal{O}_{{}_{\mathcal{A}}X^1_{\mathcal{B}}\otimes_{\mathcal{B}}{}_{\mathcal{B}}X^2_{\mathcal{A}}} \rightarrow (\mathcal{O}_{{}_{\mathcal{A}}X^1_{\mathcal{B}}\otimes_{\mathcal{B}}{}_{\mathcal{B}}X^2_{\mathcal{A}}})^{\gamma_{12}}$ be the canonical conditional expectation from $\mathcal{O}_{{}_{\mathcal{A}}X^1_{\mathcal{B}}\otimes_{\mathcal{B}}{}_{\mathcal{B}}X^2_{\mathcal{A}}}$ onto the fixed point algebra $(\mathcal{O}_{{}_{\mathcal{A}}X^1_{\mathcal{B}}\otimes_{\mathcal{B}}{}_{\mathcal{B}}X^2_{\mathcal{A}}})^{\gamma_{12}}$ under the gauge action γ_{12}. Let $\epsilon_X : \mathcal{O}_{\widehat{X}} \rightarrow (\mathcal{O}_{\widehat{X}})^{\gamma_{\widehat{X}}}$ be the similarly defined conditional expectation. As in [19] (cf. [7]), the algebras $(\mathcal{O}_{{}_{\mathcal{A}}X^1_{\mathcal{B}}\otimes_{\mathcal{B}}{}_{\mathcal{B}}X^2_{\mathcal{A}}})^{\gamma_{12}}$ and $(\mathcal{O}_{\widehat{X}})^{\gamma_{\widehat{X}}}$ are realized as the C^*-subalgebras $\mathcal{F}_{{}_{\mathcal{A}}X^1_{\mathcal{B}}\otimes_{\mathcal{B}}{}_{\mathcal{B}}X^2_{\mathcal{A}}}$ generated by $s_{x_1\otimes\cdots\otimes x_p}\pi_{\mathcal{A}}(a)s^*_{y_1\otimes\cdots\otimes y_p}$ for $a \in \mathcal{A}$, $x_1, \dots, x_p, y_1, \dots, y_p \in {}_{\mathcal{A}}X^1_{\mathcal{B}}$ or ${}_{\mathcal{B}}X^2_{\mathcal{A}}$, $p \in \mathbb{N}$ and $\mathcal{F}_{\widehat{X}}$ generated by $S_{z_1}\cdots S_{z_n}\pi_{\widehat{X}}(a,b)S^*_{w_n}\cdots S^*_{w_1}$ for $(a,b) \in \mathcal{A}\oplus\mathcal{B}$, $z_1, \dots, z_n, w_1, \dots, w_n \in \widehat{X}$, $n \in \mathbb{N}$. respectively. Let $\mathcal{F}_{C^*(S_{12},\mathcal{A})}$ be the C^*-subalgebra of $C^*(S_{12},\mathcal{A})$ generated by $S_{x_1}\cdots S_{x_n}\pi_{\widehat{X}}(a,0)S^*_{y_n}\cdots S^*_{y_1}$ where $n = 2m$ and $x_1, x_3, \dots, x_{2m-1}, y_1, y_3, \dots, y_{2m-1} \in {}_{\mathcal{A}}X^1_{\mathcal{B}}, x_2, x_4, \dots, x_{2m}, y_2, y_4, \dots, y_{2m} \in {}_{\mathcal{B}}X^2_{\mathcal{A}}$, $a \in \mathcal{A}$, It is a subalgebra of $\mathcal{F}_{\widehat{X}}$ such that $\epsilon_X(C^*(S_{12},\mathcal{A})) = \mathcal{F}_{C^*(S_{12},\mathcal{A})}$. We then have a following commutative diagram:

$$\begin{array}{ccc}
\mathcal{O}_{{}_{\mathcal{A}}X^1_{\mathcal{B}}\otimes_{\mathcal{B}}{}_{\mathcal{B}}X^2_{\mathcal{A}}} & \xrightarrow{\ \Phi\ } & C^*(S_{12},\mathcal{A}) \\
{\scriptstyle\epsilon}\big\downarrow & & \big\downarrow{\scriptstyle\epsilon_X|_{C^*(S_{12},\mathcal{A})}} \\
\mathcal{F}_{{}_{\mathcal{A}}X^1_{\mathcal{B}}\otimes_{\mathcal{B}}{}_{\mathcal{B}}X^2_{\mathcal{A}}} & \xrightarrow[\Phi|_{\mathcal{F}(\mathcal{A},\rho,\Sigma)}]{} & \mathcal{F}_{C^*(S_{12},\mathcal{A})}.
\end{array}$$

Now $\Phi|_{\mathcal{A}}(= \mathrm{id})$ is one-to-one, by [7, Theorem 4.3(i)] one knows that $\Phi|_{\mathcal{F}_{{}_{\mathcal{A}}X^1_{\mathcal{B}}\otimes_{\mathcal{B}}{}_{\mathcal{B}}X^2_{\mathcal{A}}}} : \mathcal{F}_{{}_{\mathcal{A}}X^1_{\mathcal{B}}\otimes_{\mathcal{B}}{}_{\mathcal{B}}X^2_{\mathcal{A}}} \rightarrow \mathcal{F}_{C^*(S_{12},\mathcal{A})}$ is one-to-one. For $x \in \mathcal{O}_{{}_{\mathcal{A}}X^1_{\mathcal{B}}\otimes_{\mathcal{B}}{}_{\mathcal{B}}X^2_{\mathcal{A}}}$, if $\Phi(x^*x) = 0$, then one has $\Phi\circ\epsilon(x^*x) = \epsilon_X\circ\Phi(x^*x) = 0$. Hence $\epsilon(x^*x) = 0$ so that $x = 0$ because ϵ is faithful. Therefore Φ is isomorphic. The other isomorphism between $\mathcal{O}_{{}_{\mathcal{B}}X^2_{\mathcal{A}}\otimes_{\mathcal{A}}{}_{\mathcal{A}}X^1_{\mathcal{B}}}$ and $C^*(S_{21},\mathcal{B})$ is similarly shown. □

We will next prove that the subalgebras $P_{\mathcal{A}}\mathcal{O}_{\widehat{X}}P_{\mathcal{A}}$ and $P_{\mathcal{B}}\mathcal{O}_{\widehat{X}}P_{\mathcal{B}}$ are complementary full corners in $\mathcal{O}_{\widehat{X}}$.

LEMMA 4.9. *Assume that both the finite bases $\{u^{(1)}_\alpha\}_{\alpha\in C}$ of ${}_{\mathcal{A}}X^1_{\mathcal{B}}$ and $\{u^{(2)}_\beta\}_{\beta\in D}$ of ${}_{\mathcal{B}}X^2_{\mathcal{A}}$ are essential. Then the projections $P_{\mathcal{A}}$ and $P_{\mathcal{B}}$ are both full in the algebra $\mathcal{O}_{\widehat{X}}$.*

Proof. Let $\{v_\alpha\}_{\alpha\in C\sqcup D}$ be the finite basis of $\widehat{X}$ defined in the proof of Lemma 4.7. We will prove that $P_{\mathcal{A}}$ is full in $\mathcal{O}_{\widehat{X}}$. By Lemma 4.4, one sees that for $c \in C$

$$S^*_{v_c}P_{\mathcal{A}}S_{v_c} = S^*_{v_c}S_{v_c} = \pi_{\widehat{X}}(\langle v_c, v_c\rangle) = \pi_{\mathcal{B}}(\langle u^{(1)}_c, u^{(1)}_c\rangle).$$

As $\{u^{(1)}_c\}_{c\in C}$ is essential, one has $\sum_{c\in C} S^*_{v_c}P_{\mathcal{A}}S_{v_c} \geq \pi_{\widehat{X}}(0, 1_{\mathcal{B}})$ and hence

$$\sum_{c\in C} S^*_{v_c}P_{\mathcal{A}}S_{v_c} + P_{\mathcal{A}} \geq \pi_{\widehat{X}}(0, 1_{\mathcal{B}}) + \pi_{\widehat{X}}(1_{\mathcal{A}}, 0).$$

This means that the two sided ideal of $\mathcal{O}_{\widehat{X}}$ generated by $P_{\mathcal{A}}$ coincides with $\mathcal{O}_{\widehat{X}}$. That is, $P_{\mathcal{A}}$ and similarly $P_{\mathcal{B}}$ are full projections in $\mathcal{O}_{\widehat{X}}$ (cf.[1]). □

Proof of Theorem 4.1. Put

$$ {}_{\mathcal{A}}\mathfrak{X}_{\mathcal{B}} = P_{\mathcal{A}} \mathcal{O}_{\widehat{X}} P_{\mathcal{B}}. $$

It is well-known that ${}_{\mathcal{A}}\mathfrak{X}_{\mathcal{B}}$ has a Hilbert C^*-left $P_{\mathcal{A}}\mathcal{O}_{\widehat{X}}P_{\mathcal{A}}$-module structure with left $P_{\mathcal{A}}\mathcal{O}_{\widehat{X}}P_{\mathcal{A}}$-valued inner product ${}_{\mathcal{A}}\langle , \rangle$ and a right $P_{\mathcal{B}}\mathcal{O}_{\widehat{X}}P_{\mathcal{B}}$-module structure with right $P_{\mathcal{B}}\mathcal{O}_{\widehat{X}}P_{\mathcal{B}}$-valued inner product $\langle , \rangle_{\mathcal{B}}$ as in the following way:

$$ a \cdot x \cdot b := axb, \qquad {}_{\mathcal{A}}\langle x, y\rangle := xy^*, \quad \langle x, y\rangle_{\mathcal{B}} := x^* y $$

for $a \in P_{\mathcal{A}}\mathcal{O}_{\widehat{X}}P_{\mathcal{A}}, b \in P_{\mathcal{B}}\mathcal{O}_{\widehat{X}}P_{\mathcal{B}}$ and $x, y \in {}_{\mathcal{A}}\mathfrak{X}_{\mathcal{B}}$. Regard the algebras $C^*(S_{12}, \mathcal{A})$ and $C^*(S_{21}, \mathcal{B})$ as the C^*-subalgebras $P_{\mathcal{A}}\mathcal{O}_{\widehat{X}}P_{\mathcal{A}}$ and $P_{\mathcal{B}}\mathcal{O}_{\widehat{X}}P_{\mathcal{B}}$ of $\mathcal{O}_{\widehat{X}}$ respectively. Let α^{12} be the gauge action of $\mathbb{T}$ on $C^*(S_{12}, \mathcal{A})$ that satisfies

$$ \alpha^{12}_z(S_{x^{(1)}x^{(2)}}) = zS_{x^{(1)}x^{(2)}}, \qquad \alpha^{12}_z(\pi_{\widehat{X}}(a, 0)) = \pi_{\widehat{X}}(a, 0), \qquad z \in \mathbb{T} $$

for $x^{(1)} \in {}_{\mathcal{A}}X^1_{\mathcal{B}}, x^{(2)} \in {}_{\mathcal{B}}X^2_{\mathcal{A}}, a \in \mathcal{A}$. The gauge action α^{21} on $C^*(S_{21}, \mathcal{B})$ is similarly defined. The two gauge actions α^{12} on $C^*(S_{12}, \mathcal{A})$ and α^{21} on $C^*(S_{21}, \mathcal{B})$ are canonically isomorphic to γ^{12} on $\mathcal{O}_{{}_{\mathcal{A}}X^1_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}X^2_{\mathcal{A}}}$ and γ^{21} on $\mathcal{O}_{{}_{\mathcal{B}}X^2_{\mathcal{A}} \otimes_{\mathcal{A}} {}_{\mathcal{A}}X^1_{\mathcal{B}}}$ respectively. Let us consider an action γ^1 of $\mathbb{T}$ on the algebra $\mathcal{O}_{\widehat{X}}$ defined by

$$ \gamma^1_z(S_{x^{(1)}}) = S_{x^{(1)}}, \quad \gamma^1_z(S_{x^{(2)}}) = zS_{x^{(2)}}, \quad \gamma^1_z(\pi_{\widehat{X}}(a, b)) = \pi_{\widehat{X}}(a, b), \qquad z \in \mathbb{T} $$

for $x^{(1)} \in {}_{\mathcal{A}}X^1_{\mathcal{B}}, x^{(2)} \in {}_{\mathcal{B}}X^2_{\mathcal{A}}, (a, b) \in \mathcal{A} \oplus \mathcal{B}$. The restrictions of γ^1 to the subalgebras $C^*(S_{12}, \mathcal{A})$ and $C^*(S_{21}, \mathcal{B})$ of $\mathcal{O}_{\widehat{X}}$ coincide with the actions α^{12} and α^{21} respectively. Since the projections $P_{\mathcal{A}}$ and $P_{\mathcal{B}}$ are both fixed under γ^1, the action γ^1 induces a continuous action of $\mathbb{T}$ on the Hilbert C^*-bimodule ${}_{\mathcal{A}}\mathfrak{X}_{\mathcal{B}}$. We denote it by u. It satisfies the equalities:

$$ \alpha^{12}_z({}_{\mathcal{A}}\langle x, y\rangle) = {}_{\mathcal{A}}\langle u_z(x), u_z(y)\rangle, \qquad \alpha^{21}_z(\langle x, y\rangle_{\mathcal{B}}) = \langle u_z(x), u_z(y)\rangle_{\mathcal{B}} $$

for $x, y \in {}_{\mathcal{A}}\mathfrak{X}_{\mathcal{B}}$ and $z \in \mathbb{T}$. Then by Lemma 4.9, two C^*-dynamical systems $(C^*(S_{12}, \mathcal{A}), \alpha^{12}, \mathbb{T})$ and $(C^*(S_{21}, \mathcal{B}), \alpha^{21}, \mathbb{T})$ are Morita equivalent via $({}_{\mathcal{A}}\mathfrak{X}_{\mathcal{B}}, u, \mathbb{T})$ in the sense of F. Combes [3] and Curto-Muhly-Williams [6]. Hence by an equivariant version of the Brown-Green-Rieffel Theorem [2] proved by F. Combes in [3] their stabilizations $(C^*(S_{12}, \mathcal{A}) \otimes \mathcal{K}, \alpha^{12} \otimes \mathrm{id}, \mathbb{T})$ and $(C^*(S_{21}, \mathcal{B}) \otimes \mathcal{K}, \alpha^{21} \otimes \mathrm{id}, \mathbb{T})$ are cocycle conjugate. This means that $(\mathcal{O}_{X_{\mathcal{A}}} \otimes \mathcal{K}, \gamma^{\mathcal{A}} \otimes \mathrm{id}, \mathbb{T})$ and $(\mathcal{O}_{X_{\mathcal{B}}} \otimes \mathcal{K}, \gamma^{\mathcal{B}} \otimes \mathrm{id}, \mathbb{T})$ are cocycle conjugate. □

Remark. Similar results to the main result of this paper have been shown in the following two papers in more general setting:

P. S. Muhly, D. Pask and M. Tomforde: Strong shift equivalence of C^*-correspondences, 2005, Aug.

M. Tomforde : Strong shift equivalence in the C^*-algebra setting: Graphs and C^*-correspondences, 2005, Aug.

References

[1] L. G. Brown, Stable isomorphism of hereditary subalgebras of C^*-algebras, *Pacific. J. Math.* **71** (1977) 335–348.

[2] L. G. Brown, P. Green and M. A. Rieffel, Stable isomorphism and strong Morita equivalence of C^*-algebras, *Pacific. J. Math.* **71** (1977) 349–363.

[3] F. Combes, Crossed products and Morita equivalence, *Proc. London Math. Soc.* **49** (1984) 289–306.

[4] J. Cuntz, A class of C^*-algebras and topological Markov chains II: reducible chains and the Ext-functor for C^*-algebras, *Invent. Math.* **63** (1980) 25–40.

[5] J. Cuntz and W. Krieger, A class of C^*-algebras and topological Markov chains, *Invent. Math.* **56** (1980) 251–268.

[6] R. E. Curto, P. Muhly and D. P. Williams, Crossed products of strong Morita equivalent C^*-algebras, *Proc. Amer. Math. Soc.* **90** (1984) 528–530.

[7] T. Kajiwara, C. Pinzari and Y. Watatani, Ideal structure and simplicity of the C^*-algebras generated by Hilbert modules, *J. Funct. Anal.* **159** (1998) 295–322.

[8] T. Kajiwara and Y. Watatani, Jones index theory by Hilbert C^*-bimodules and K-theory, *Trans. Amer. Math. Soc.* **352** (2000) 3429–3472.

[9] Y. Katayama, Generalized Cuntz algebras $\mathcal{O}_N^M$, *RIMS kokyuroku* **858** (1994) 87–90.

[10] D. Lind and B. Marcus, *An introduction to symbolic dynamics and coding*, Cambridge University Press, 1995.

[11] K. Matsumoto, Presentations of subshifts and their topological conjugacy invariants, *Doc. Math.* **4** (1999) 285–340.

[12] K. Matsumoto, Stabilized C^*-algebra constructed from symbolic dynamical systems, *Ergodic Theory Dynam. Systems* **20** (2000) 821–841.

[13] K. Matsumoto, C^*-algebras associated with presentations of subshifts, *Doc. Math.* **7** (2002) 1–30.

[14] K. Matsumoto, Strong shift equivalence of symbolic matrix systems and Morita equivalence of C^*-algebras, *Ergodic Theory Dynam. Systems* **24** (2004) 199–215.

[15] K. Matsumoto, Actions of symbolic dynamical systems on C^*-algebras, *to appear in J. Reine Angew. Math.*

[16] P.S. Muhly and B. Solel, On the simplicity of some Cuntz-Pimsner algebras, *Math. Scand.* **83** (1998) 53–73.

[17] M. Nasu, Topological conjugacy for sofic shifts, *Ergodic Theory Dynam. Systems* **6** (1986) 265–280.

[18] M. Nasu, Textile systems for endomorphisms and automorphisms of the shift, *Mem. Amer. Math. Soc.* No.546, **114** (1995).

[19] M. V. Pimsner, A class of C^*-algebras generalizing both Cuntz-Krieger algebras and crossed product by $\mathbb{Z}$, in *Free Probability Theory, Fields Institute Communications* **12** (1996) 189–212.

[20] M. A. Rieffel, Induced representations of C^*-algebras, *Adv. in Math.* **13** (1974) 176–257.

[21] J. Schweizer, Dilatiions of C^*-correspondences and the simplicity of Cuntz-Pimsner algebras, *J. Funct. Anal.* **180** (2001) 404–425.

[22] R.F. Williams, Classification of subshifts of finite type, *Ann. Math.* **98** (1973) 120–153. erratum, *Ann. Math.* **99** (1974), 380–381.

Department of Mathematical Sciences
Yokohama City University
Seto 22-2, Kanazawa-ku, Yokohama 236-0027,
Japan
E-mail: kengo@yokohama-cu.ac.jp