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ON $\mathbb{Q}$-SIMPLE FACTORS OF JACOBIAN VARIETIES OF MODULAR CURVES

By

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Abstract. Let N be a positive integer and $J_0(N)$ be the Jacobian variety of the modular curve $X_0(N)$. In this paper we gives all positive integers N such that all $\mathbb{Q}$-simple factors of $J_0(N)$ are elliptic curves.

1. Introduction

Let N be a positive integer, and $X_0(N)$ be the coarse moduli space that classifies a pair (E, C) where E is an elliptic curve and C is a cyclic subgroup of order N. The space $X_0(N)$ has a canonical structure of algebraic curve over $\mathbb{Q}$. Let $J_0(N)$ be the Jacobian variety of $X_0(N)$. The number of $\mathbb{Q}$-simple factors in $J_0(N)$ relates to the conjecture that the Mordell-Weil rank of elliptic curves over $\mathbb{Q}$ is unbounded when curves vary arbitrarily (see [4]). So it seems to be interesting to investigate $\mathbb{Q}$-simple factors of $J_0(N)$.

Our purpose in this paper is to determine all levels N for which all $\mathbb{Q}$-simple factors of $J_0(N)$ are elliptic curves. As a result, we have the following Theorem.

THEOREM 1.1. *All $\mathbb{Q}$-simple factors of $J_0(N)$ are elliptic curves and $X_0(N)$ is not of genus 0 if and only if*

N=11, 15, 17, 19, 21, 27, 33, 37, 45, 49, 57, 75, 99, 121,
14, 20, 22, 24, 26, 30, 32, 34, 36, 38, 40, 42, 44, 48, 50, 52, 54, 56, 60, 64, 66, 72,
76, 80, 84, 90, 96, 100, 108, 112, 114, 120, 128, 132, 144, 150, 168, 180, 192, 198,
200, 216, 240, 288, 300, 336, 360, 384, 396, 400, 432, 576, 600, 672, 720, 1152, 1200.

It is known experientially that $J_0(N)$ has a $\mathbb{Q}$-simple factor of high dimension for arbitrarily large N [3]. Note that H.Cohen has also studied the same of above theorem for odd N [8]. His result and proof overlap with ours for odd N. He gave bound of such N using an element of the Hecke algebra (see equation (1)). But we can not apply his method in the case arbitrarily level. To overcome this

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problem we will devide our problem into the case where the level is the power of a prime number and join to the case arbitrarily level.

This paper is organized as follows. In Section 2 and Section 3, we recall basic facts for modular forms, the Jacobian varieties of modular curves, and the trace formula of Hecke operators. We will prove Theorem 1.1 by estimating some elements in the Hecke algebra (see equation (1)). In Section 4, we will estimate the trace formula of Hecke operators explicitly and give the proof of Theorem 1.1. Finally, we introduce some related problems.

NOTATION.

- N: a positive integer
- $\mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}\tau > 0\}$
- $\mathbb{H}^* := \mathbb{H} \cup \mathbb{Q} \cup \{i\infty\}$
- $\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \middle| c \equiv 0 \bmod N \right\}$
- $\overline{i\infty}$: the image of $i\infty$ under the natural projection from $\mathbb{H}^*$ to $\Gamma_0(N)\backslash\mathbb{H}^*$
- $q = e^{2\pi\sqrt{-1}\tau}$: an uniformizer at $i\infty$
- $S_2(\Gamma_0(N))$: the space of cusp forms of weight 2 with respect to $\Gamma_0(N)$
- $X_0(N) \simeq \Gamma_0(N)\backslash\mathbb{H}^*$: the modular curve of level N. We regard it as an algebraic curve over $\mathbb{Q}$ via an isomorphism between analytic spaces.

2. Preliminary

Let N be a positive integer. We denote the space of cusp forms with respect to $\Gamma_0(N)$ by $S_2(\Gamma_0(N))$. First, we recall an important property of $S_2(\Gamma_0(N))$ which is a key of our problem. For a positive divisor M of N, let New_M be the set of normalized Hecke eigen new forms with respect to $\Gamma_0(M)$. Then the space $S_2(\Gamma_0(N))$ is decomposed into new forms:

THEOREM 2.1 ([1]). *Let $G_{\mathbb{Q}}$ be the absolute galois group of $\mathbb{Q}$ and $\mathbb{Q}_f := \mathbb{Q}(\{a_n\}_{n\geq 1})$ for a modular form $f = \sum_{n=1}^{\infty} a_n q^n$. Then we have a decomposition*

$$S_2(\Gamma_0(N)) = \bigoplus_{\substack{M|N \\ d|\frac{N}{M}}} \bigoplus_{f \in G_{\mathbb{Q}}\backslash \mathrm{New}_M} \bigoplus_{\tau:\mathbb{Q}_f \hookrightarrow \mathbb{C}} {}^{\tau}f(dz),$$

where ${}^{\tau}f = \sum_{n=1}^{\infty} a_n^{\tau} q^n$ for $f = \sum_{n=1}^{\infty} a_n q^n \in \mathrm{New}_M$.

Next we recall some properties of $J_0(N)$. Let A_f be the abelian variety associated to a new form f constructed by G. Shimura ([14]). He proved that A_f is an abelian variety (K. Ribet [12] for $\mathbb{Q}$-simplicity) which is isogenous to a $\mathbb{Q}$-simple factor of $J_0(N)$ and $\dim A_f = [\mathbb{Q}_f : \mathbb{Q}]$ [13]. It is also well known that A_f is of conductor $N^{[\mathbb{Q}_f:\mathbb{Q}]}$ (see [5]). Furthermore, in parallel with Theorem 2.1, the following theorem is well known (see [10]).

THEOREM 2.2.

$$J_0(N) \overset{\mathbb{Q}}{\sim} \bigoplus_{M|N} \bigoplus_{f \in G_{\mathbb{Q}} \backslash \mathrm{New}_M} A_f^{n_f},$$

where n_f is the number of positive divisors of $\frac{N}{M}$.

By Theorem 2.1 and Theorem 2.2, we can find a necessary condition that all $\mathbb{Q}$-simple factors of $J_0(N)$ are elliptic curves. We now explain this below. Let p be a prime number with $(p, N) = 1$, T_p the Hecke operator on $S_2(\Gamma_0(N))$ and $\chi_{T_p}(x)$ the characteristic polynomial of T_p. If all $\mathbb{Q}$-simple factors of $J_0(N)$ are elliptic curves, then $\chi_{T_p}(x)$ must split completely over $\mathbb{Q}$. Furthermore, since $\chi_{T_p}(x) \in \mathbb{Z}[x]$ is monic, then any root α of χ_{T_p} is an integer. It is well known as the Hasse-Weil bound that

$$a_p \le 2\sqrt{p}, \text{ for } f = \sum_{n=1}^{\infty} a_n q^n \in \mathrm{New}_M$$

(see [2]). Therefore, α is one of $0, \pm 1, \ldots, \pm[2\sqrt{p}]$, where $[k]$ is the greatest integer $\le k$.

First of all, we consider in the case where N is odd. Then we can use T_2. Any root of $\chi_{T_2}(x)$ is one of $0, \pm 1, \pm 2$. Then if all $\mathbb{Q}$-simple factors of $J_0(N)$ are elliptic curves, the element

$$f[T_2] := T_2^2(T_2^2 - T_1)(T_2^2 - 4T_1) = T_{64} + 5T_{16} + 10T_4 + 8T_1 \tag{1}$$

must be zero in $\mathrm{End}(S_2(\Gamma_0(N)))$. This is an idea of Cohen in [8]. From this fact, we have a necessary condition for odd N.

PROPOSITION 2.3. *Let N be an odd positive integer. If all $\mathbb{Q}$-simple factors of $J_0(N)$ are elliptic curves, then the trace of $f[T_2]$ must be zero.*

In Section 4, we will obtain an upper bound of odd N in Proposition 2.3 by estimating the right-hand of equation (1).

REMARK 2.4. We adopted $f[T_2]$ instead of $T_2(T_2^2-T_1)(T_2^2-4T_1)$ by a technical reason for trace formula of Hecke operators.

REMARK 2.5. By Theorem 2.2, we see that our problem are closed in the divisors of N. Namely, if $J_0(N)$ is isogenous to a product of one dimensional $\mathbb{Q}$-simple factors, then $J_0(M)$ is so, or zero for any positive divisor M of N.

3. The trace formula of Hecke operators for $\Gamma_0(N)$

We recall the trace formula according to [7]. Let n be the square of a positive integer which is co prime to N and let T_n be the Hecke operator on $S_2(\Gamma_0(N))$.

THEOREM 3.1 ([7]).

$$\begin{aligned}\mathrm{Trace}(T_n) = \frac{\Psi(N)}{12} &- \sum_{\substack{t^2-4n<0\\ t\in\mathbb{Z}}} \sum_{f^2|t^2-4n} \frac{h(\frac{t^2-4n}{f^2})}{w(\frac{t^2-4n}{f^2})}\mu(t,f,n) \\ &- \sum_{\substack{d|n\\ d\leq\sqrt{n}}} \delta(d)d \sum_{\substack{c|N\\ (c,\frac{N}{c})|(\frac{n}{d}-d)}} \phi\left(\left(c,\frac{N}{c}\right)\right) \\ &+ \sum_{\substack{d|n\\ (N,\frac{n}{d})=1}} d.\end{aligned} \tag{2}$$

Notation is: $\mu(t,f,n) = (N,f)\prod_{\substack{p|N\\ p\nmid\frac{N}{(N,f)}}}\left(1+\frac{1}{p}\right)\sum_{\substack{x \bmod N\\ x^2-tx+n\equiv 0 \bmod (fN,N^2)}} 1,$

$$\Psi(N) = N\prod_{p|N}\left(1+\frac{1}{p}\right).$$

ϕ is the Euler function. $h(d)$ denotes the class number of positive definite primitive quadratic forms of discriminant d and $w(d)$ denotes the order of the unit group of $\mathbb{Q}(\sqrt{d})$;
$\delta(d)$ is 1 if $d<\sqrt{n}$, $\frac{1}{2}$ if the term $d=\sqrt{n}$ exists.

In the next section, we will estimate four main terms of $\mathrm{Trace}(T_n)$.

4. An estimation of the trace formula

Let r be a positive integer , $N = p^r$ and $n = 2^{2\alpha}$ for $l = 0,1,2,3$. We assume that p is greater than 5. We will use the following two facts to estimate the trace formula in our case.

PROPOSITION 4.1 (Hurwitz, Eichler [6]). *Let*

$$H(n) = \sum_{f^2|n} \frac{2h(\frac{-n}{f^2})}{w(\frac{-n}{f^2})}.$$

Then, we have the following formula.

$$\sum_{\substack{t^2-4n<0\\ t\in\mathbb{Z}}} H(4n-t^2) = \frac{1}{6} + 2\sigma(n) - \sum_{d|n} \min\Big(d, \frac{n}{d}\Big),$$

where $\sigma(n) = \sum_{d|n} d$.

REMARK 4.2. We have only considered $n = 2^{2l}, l = 0, 1, 2, 3$. In this case we have

$$\sum_{\substack{t^2-4n<0\\ t\in\mathbb{Z}}} H(4n-t^2) = \frac{1}{6} + 4n - 3\sqrt{n}.$$

LEMMA 4.3. *Let N and n be as above. Then, we have*

$$\sum_{\substack{x \bmod N\\ x^2-tx+n\equiv 0 \bmod N}} 1 = \begin{cases} p^{[\frac{\mathrm{ord}_p(t^2-4n)}{2}]} & \text{if } \mathrm{ord}_p(t^2-4n) \geq r \\ 0 & \text{if } \mathrm{ord}_p(t^2-4n) < r, \\ & \quad \mathrm{ord}_p(t^2-4n) \text{ is odd} \\ (1 + (\frac{b}{p}))p^{\frac{\mathrm{ord}_p(t^2-4n)}{2}} & \text{if } \mathrm{ord}_p(t^2-4n) < r, \\ & \quad \mathrm{ord}_p(t^2-4n) \text{ is even} \end{cases}$$

where $b = \frac{t^2-4n}{p^{\mathrm{ord}_p(t^2-4n)}}$ and $[k]$ is the greatest integer $\leq k$ and $(\frac{b}{p})$ is the Jacobi symbol.

Proof. Since p is coprime to 2, we have

$$S := \sum_{\substack{x \bmod p^r\\ x^2-tx+n\equiv 0 \bmod p^r}} 1 = \sum_{\substack{x \bmod p^r\\ x^2\equiv t^2-4n \bmod p^r}} 1.$$

Put $b = t^2 - 4n = p^k b'$, where $k = \mathrm{ord}_p(t^2-4n)$ and take x which satisfies the condition in the summation of the right side in the equation defining S. If $k < r$, then

$$2\mathrm{ord}_p(x) = k.$$

Therefore, k must be even ($S = 0$, if k is odd). Note that b' is square in modulo p if and only if b' is square in modulo p^r. We assume that there exists α such that

$$b' \equiv \alpha^2 \bmod p^r.$$

We consider $p^r\alpha + a$, where a is an integer such that $\mathrm{ord}_p(a) > \dfrac{k}{2}$. Then we have

$$(p^{\frac{k}{2}}\alpha + a)^2 \equiv b + 2p^{\frac{k}{2}}a + a^2 \bmod p^r.$$

If $\mathrm{ord}_p(2p^{\frac{k}{2}}a) = \frac{k}{2} + \mathrm{ord}_p(a) \geq r$, then we have

$$\mathrm{ord}_p(a) \geq r - \frac{k}{2}. \tag{3}$$

From (3), we have

$$\mathrm{ord}_p(a^2) \geq 2r - k > r.$$

Therefore, If a satisfies inequality (4), then we have

$$(p^{\frac{k}{2}}\alpha + a)^2 \equiv b \bmod p^r.$$

The number of such a must be $p^{\frac{k}{2}}$. Therefore , we obtain

$$S = \left(1 + \left(\frac{b'}{p}\right)\right)p^{\frac{k}{2}}.$$

If $k \geq r$, then

$$S = \sum_{\substack{x \bmod p^r \\ x^2 \equiv 0 \bmod p^r}} 1.$$

If $\mathrm{ord}_p(x) \geq \frac{r}{2}$, then $x^2 \equiv 0 \bmod p^r$. Therefore, the number of such x must be $p^{[\frac{r}{2}]}$. □

We now estimate the trace formula explicitly. First we consider the case where $N = p$ is a prime number. Since $n = 2^{2l}, l = 0, 1, 2, 3$, we have $f \leq 5$ by simple calculations (i.e. If $p > 5$, then $\mathrm{ord}_p(t^2 - 4n) \leq 1$). Therefore, if $p > 5$, then $(p, f) = 1$. By Theorem 3.1 and Lemma 4.3, we have

$$\begin{aligned}
\mathrm{Trace}(T_n) &= \frac{p+1}{12} - \frac{1}{2}\sum_{\substack{t^2-4n<0 \\ t\in\mathbf{Z}}} \left(1 + \left(\frac{t^2-4n}{p}\right)\right) H(4n - t^2) + 2 - 3\sqrt{n} + 2n - 1 \\
&\geq \frac{p+1}{12} - \sum_{\substack{t^2-4n<0 \\ t\in\mathbf{Z}}} H(4n - t^2) + 1 - 3\sqrt{n} + 2n
\end{aligned} \tag{4}$$

$$= \frac{p+1}{12} - \left(\frac{1}{6} + 4n - 3\sqrt{n}\right) + 1 - 3\sqrt{n} + 2n$$
$$= \frac{p+11}{12} - 2n.$$

By (1) and (4), we then have

$$\begin{aligned} 0 &= \mathrm{Trace}(f[T_2]) \\ &\geq \frac{p+11}{12} - 128 + 5\Big(\frac{p+11}{12} - 32\Big) + 10\Big(\frac{p+11}{12} - 8\Big) + 8\Big(\frac{p+11}{12} - 2\Big) \\ &= 2(p-181). \end{aligned}$$

Therefore we obtain $p \leq 181$. More precisely, only 7,11,13,17,19,37 among such p's satisfy $\mathrm{Trace}(f[T_2]) = 0$. From this we have the following Theorem.

THEOREM 4.4. *Let p be a prime number. All $\mathbb{Q}$-simple factors of $J_0(p)$ are elliptic curves if and only if $p = 11, 17, 19, 37$.*

Proof. The modular curve $X_0(7)$ and $X_0(13)$ are of genus zero and the modular curves $X_0(11)$, $X_0(19)$ are elliptic curves. So there is nothing to do in these case. So we have only to consider $p = 37$. Since the Atkin-Lehner involution w_{37} is bielliptic involution of $X_0(37)$, $J_0(37)$ is isogenous to the product of two elliptic curves. This completes the proof. □

Next we consider the case where $N = p^r$ is a power of a prime number. We may assume that $p \geq 7$ and r is even. Let us write

$$A_1^{(n)} = \frac{\Psi(N)}{12},$$

$$A_2^{(n)} = \frac{1}{2} \sum_{\substack{t^2-4n<0 \\ t\in\mathbb{Z}}} \sum_{f^2 | t^2-4n} \frac{2h(\frac{t^2-4n}{f^2})}{w(\frac{t^2-4n}{f^2})} \mu(t,f,n),$$

$$A_3^{(n)} = \sum_{\substack{d|n \\ d\leq\sqrt{n}}} \delta(d) d \sum_{\substack{c|N \\ (c,\frac{N}{c})|(\frac{n}{d}-d)}} \phi\Big(\Big(c, \frac{N}{c}\Big)\Big)$$

and

$$A_4^{(n)} = \sum_{\substack{d|n \\ (N,\frac{n}{d})=1}} d,$$

where $n = 2^{2l}, l = 0, 1, 2, 3$.

Put $r = 2m$ where m is a positive integer. If $p > 7$, then p is coprime to $\frac{n}{d} - d$ for any $d|n$ such that $n^2 \neq d$. If $p = 7$, only $(n, d) = (64, 1)$ dose not satisfy such a condition. Then, we have

$$A_1^{(n)} = \frac{p^{2m} + p^{2m-1}}{12}, \quad A_4^{(n)} = 2n - 1, \tag{5}$$

$$A_3^{(n)} = 2\sqrt{n} - 2 + \frac{\sqrt{n}}{2}(p^m + p^{m-1}) + \epsilon_n, \tag{6}$$

where ϵ_n is 12 if $n = 64$ and $p = 7$, 0 otherwise.

Finally, we estimate $A_2^{(n)}$. By simple calculations, it is easy to see that $\mathrm{ord}_p(t^2 - 4n) \leq 1$ $n = 2^{2l}$, $p > 5$. Then, by Lemma 4.3, we have

$$\mu(t, f, n) \leq 2.$$

Therefore, we have an estimation of $A_2^{(n)}$:

$$\begin{aligned} A_2^{(n)} &= \frac{1}{2} \sum_{\substack{t^2-4n<0 \\ t\in\mathbb{Z}}} \sum_{f^2|t^2-4n} \frac{2h(\frac{t^2-4n}{f^2})}{w(\frac{t^2-4n}{f^2})} \mu(t, f, n) \\ &\leq \sum_{\substack{t^2-4n<0 \\ t\in\mathbb{Z}}} H(t^2 - 4n) \\ &= \frac{1}{6} + 4n - 3\sqrt{n}. \end{aligned} \tag{7}$$

By the equations (5), (6) and (7), we have

$$\begin{aligned} \mathrm{Trace}(T_n) &= A_1^{(n)} - A_2^{(n)} - A_3^{(n)} + A_4^{(n)} \\ &\geq \frac{p^{2m} + p^{2m-1}}{12} - \left(\frac{1}{6} + 4n - 3\sqrt{n}\right) \\ &\quad - \left(2\sqrt{n} - 2 + \frac{\sqrt{n}}{2}(p^m + p^{m-1}) + \epsilon_n\right) + 2n - 1 \\ &= \frac{p^{2m} + p^{2m-1}}{12} - \frac{\sqrt{n}}{2}(p^m + p^{m-1}) - 2n + \sqrt{n} \\ &\quad + \frac{5}{6} - \epsilon_n. \end{aligned} \tag{8}$$

By Theorem 4.4, we have only to consider for $p \leq 37$. We will discuss later the case where $N = 2^r, 3^r$ and 5^r.

By (1) and (8), we have

$$\begin{aligned} 0 = \mathrm{Trace}(f(T_n)) &= T_{64} + 5T_{16} + 10T_4 + 8T_1 \\ &\geq 2\left(1 + \frac{1}{p}\right)p^{2m} - 28\left(1 + \frac{1}{p}\right)p^m - 320 \\ &\geq 2(p^{2m} - 28p^m - 160). \end{aligned}$$

Then we obtain

$$p^m \leq 32.$$

Therefore, m must be 1 if $p = 7, 11, 13, 17, 19$ and there dose not exist such m if $p = 37$. Finally, we discuss the case where $N = 2^r, 3^r, 5^r$. One will be able to estimate the trace formula in case $N = 2^r, 3^r, 5^r$ (we will apply T_3 in case $N = 2^r$) and get a bound of r as a result. But this bound will not be small. Therefore, we will apply the direct computation of eigenpolynomials of the Hecke operators and Cremona's table in the three cases above. Then we obtain the following.

THEOREM 4.5. *Let p be a prime number and r a positive integer. Then, all $\mathbb{Q}$-simple factors of $J_0(p^r)$ are elliptic curves if and only if*

$$N = 2^5, 2^6, 2^7, 3^3, 7^2, 11, 11^2, 17, 19, 37.$$

Proof. By the above observation, we see that

$$r \leq 2 \text{ for } 5 \leq p \leq 19 \text{ and } r = 1 \text{ for } p = 37.$$

If $N = 2, 2^2, 2^3, 2^4, 3, 3^2, 5, 5^2, 7, 13$, $X_0(N)$ is of genus 0. By computation of Hecke operators, when $N = 2^8, 3^4, 5^3, 13^2, 17^2, 19^2$, $J_0(N)$ has a $\mathbb{Q}$-factor which has dimension greater than one (see TABLE 1). Therefore we have only to prove the statement for $N = 2^5, 2^6, 2^7, 3^3, 7^2, 11^2$. In the six cases above, applying Cremona's table [9], we see that the number of New_N is equal to the number of $\mathbb{Q}$ isogeny classes with conductor N. Here the number of New_N are computed by using Theorem 2.1 and the Möbius' inverse formula:

$$\sharp\mathrm{New}_N = \sum_{n|N} \mu\left(\frac{N}{n}\right) \cdot \sigma(n) \cdot \dim_{\mathbb{C}} S_2(\Gamma_0(n)).$$

We can read off the number of $\mathbb{Q}$ isogeny classes with conductor N from [9]. This completes the proof. □

Proof of Theorem 1.1. We consider the following set:

$$P = \{2, 2^2, 2^3, 2^4, 2^5, 2^6, 2^7\} \cup \{3, 3^2, 3^3\} \cup \{5, 5^2\} \cup \{7, 7^2\} \cup \{11, 11^2\} \cup \\ \{13, 17, 19, 37\}.$$

By Theorem 4.5 and Remark 2.5, we have only to check that for any r elements $M_1, \dots, M_r$ in P, whether $J_0(M_1 \cdots M_r)$ is isogenous to the product of one dimensional factors or not. To avoid tautological computation, we proceed as follows. First we determine the integers $M = M_1 M_2$, $M_1, M_2 \in \{11, 11^2, 13, 17, 19, 37\}$ such that $J_0(M)$ is isogenous to the product of one dimensional factors. Then we can check easily that such M is only 11^2. Let $P_1 = \{1, 11, 11^2, 13, 17, 19, 37\}$. Let $N = 2^{r_2} \cdot 3^{r_3} \cdot 5^{r_5} \cdot 7^{r_7} \cdot M$, $0 \leq r_2 \leq 7$, $0 \leq r_3 \leq 3$, $0 \leq r_5, r_7 \leq 2$, and M in P_1. Then we check whether $J_0(N)$ is isogenous to the product of one dimensional factors or not in the lexicographic order for (r_2, r_3, r_5, r_7) for given M in P_1. Starting from low order, if we find some law (r_2, r_3, r_5, r_7) such that $J_0(N)$ has a factor of dimension greater than 2, we may stop the procedure by Remark 2.5. After some computations, we easily get the claim.

5. Some related problems

Finally, we will collect some related problems and a conjecture. First we extend our problem to the case of quotient modular curves. Let $W(N)$ be the group generated by Atkin-Lehner involutions and H it's subgroup [1]. Since $W(N) \subset \mathrm{Aut}_{\mathbb{Q}} X_0(N)$, the quotient modular curve $X_0(N)_H$ is defined over $\mathbb{Q}$. Then $J_0(N)_H = \mathrm{Pic}^0(X_0(N)_H)$ is also defined over $\mathbb{Q}$. Therefore, we can consider the following problem.

PROBLEM 5.1. How many pairs (N, H) such that all $\mathbb{Q}$-simple factors of $J_0(N)_H$ are elliptic curves are there?

The author of this paper gave a partial answer to this problem in [16].

Next we consider a higher dimensional generalization.

PROBLEM 5.2. Fix a positive integer n. Then, find upper bound B(n) of N for which all $\mathbb{Q}$-simple factors of $J_0(N)$ are of dimension less than or equal to n.

In the case $n = 1$, our result claims $B(1) = 1200$. The author has computational results for this problem [17]. But it seems that the case of n greater than 2 is much harder than the case of $n = 1$.

Finally we give a conjecture for the new part of $J_0(N)$ (see [10] for the definition of "new part").

CONJECTURE 5.3. Let N be a positive integer. The level N such that all $\mathbb{Q}$-simple factors of $J_0^{\mathrm{new}}(N)$ are elliptic curves is bounded. In particular, if so, N is less than or equal to 1800.

By the results in [4], if Conjecture 5.3 does not hold, the Mordell-Weil rank of elliptic curves over $\mathbb{Q}$ is unbounded (say "the rank conjecture") when curves vary arbitrarily. From the difficulty of the rank conjecture, it seems to be natural to expect that Conjecture 5.3 is true. In the last statement, one can observe the bound of such N from the table in [9]

6. Tables

The following table is a list for the characteristic polynomials of T_p. The second column of Table 1 denotes the Hecke operators which we chose suitably. The third column of Table 1 gives the characteristic polynomials $\chi_{T_p}[x] \in \mathbb{Z}[x]$ of Hecke operators T_p.

Table 1

$S_2(\Gamma_0(N))$	T_p	χ_{T_p}
$S_2(\Gamma_0(2^8))$	T_3	$x^9(x^2-8)(x+2)^5(x-2)^5$
$S_2(\Gamma_0(3^4))$	T_2	$x^2(x^2-3)$
$S_2(\Gamma_0(5^3))$	T_2	$(x^2+x-1)(x^2-x-1)(x^4-8x^2+11)$
$S_2(\Gamma_0(13^2))$	T_2	$(x^2-3)(x^3+2x^2-x-1)(x^3-2x^2-x+1)$
$S_2(\Gamma_0(17^2))$	T_2	$(x^2-2x-1)^2(x^2+x-3)^2(x^3-3x+1)^2(x+1)^3$
$S_2(\Gamma_0(19^2))$	T_2	$x^4(x^2+x-1)(x^2-x-1)(x^3-3x^2+3)$ $\times(x^3+3x^2-3)(x^4-5x^2+5)(x^2-5)^2$

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