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EXPLICIT CLASSIFICATIONS OF ORBITS IN CAYLEY ALGEBRAS OVER THE GROUPS OF TYPE G_2

By

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Abstract. We give the explicit classifications of orbits in Cayley algebras $\mathfrak{C}^C$, $\mathfrak{C}$ and $\mathfrak{C}'$ by the natural actions of the exceptional Lie groups of type G_2.

1. Introduction

In [2], [3], [4], [6], [7], [8], [9] and [13], we studied the structure of orbits of representation spaces by the natural actions of the exceptional Lie groups.

It is known that the compact group G_2 acts transitively on the space of all elements having the same norm in the space $\mathfrak{C}_0$ of pure imaginary Cayley numbers. In this paper, we shall give another proof for the compact case, and give the explicit classification of orbits in the space ${\mathfrak{C}_0}^C$ (resp. ${\mathfrak{C}_0}'$) of pure imaginary complex (resp. split) Cayley numbers over the non-compact group ${G_2}^C$ (resp. $G_{2(2)}$). As a result, we obtain that their orbits in the non-compact case have similar properties to the compact case, that is, their orbits are characterized by their norms. In detail, the following three theorems hold:

THEOREM. *Any non-zero element $x \in {\mathfrak{C}_0}^C$ can be transformed to the following canonical form by some element of ${G_2}^C$:*

(1) *In the case of $N(x) \neq 0$:*

$$(\xi + i\eta)e_1 \quad \left(\xi > 0 \ \text{or} \ \left\{ \begin{array}{l} \xi = 0 \\ \eta > 0 \end{array} \right. \right),$$

where $(\xi + i\eta)^2 = N(x)$.

**Dedicated to Professor Ichiro Yokota on his eightieth birthday*

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(2) *In the case of* $N(x) = 0$:

$$e_1 + ie_4.$$

Moreover, all their orbits in $\mathfrak{C}_0{}^C$ *over* $G_2{}^C$ *are distinct, and the union of all their orbits and* $\{0\}$ *is the whole space* $\mathfrak{C}_0{}^C$.

THEOREM. *Any element* $x \in \mathfrak{C}_0$ *can be transformed to the following canonical form by some element of* G_2 :

$$\xi e_1 \quad (\xi = \sqrt{N(x)} \geq 0).$$

Moreover, all their orbits in $\mathfrak{C}_0$ *over* G_2 *are distinct, and the union of all their orbits is the whole space* $\mathfrak{C}_0$.

THEOREM. *Any non-zero element* $x \in \mathfrak{C}_0{}'$ *can be transformed to the following canonical form by some element of* $G_{2(2)}$:

(1) *In the case of* $N(x) > 0$:

$$\xi e_1 \quad (\xi = \sqrt{N(x)} > 0).$$

(2) *In the case of* $N(x) < 0$:

$$\xi e_4{}' \quad (\xi = \sqrt{-N(x)} > 0).$$

(3) *In the case of* $N(x) = 0$:

$$e_1 + e_4{}'.$$

Moreover, all their orbits in $\mathfrak{C}_0{}'$ *over* $G_{2(2)}$ *are distinct, and the union of all their orbits and* $\{0\}$ *is the whole space* $\mathfrak{C}_0{}'$.

Finally, in [5], we have already proved that the group $G_2{}^C$ (resp. $G_2, G_{2(2)}$) acts transitively on the sphere $(S^C)^6$ (resp. $S^6, (S')^6$). This means that the orbit of an element with a non-zero norm is determined by its norm. We can thus omit the proof in the case of the non-zero norm. However, in this paper, we shall write again without omitting its proof in order to make the paper self-contained.

2. Cayley algebras and exceptional Lie groups of type G_2

Let $\boldsymbol{R}$ and $C = \boldsymbol{R}^C = \boldsymbol{R} \oplus \boldsymbol{R}i$ ($i^2 = -1$) be the fields of real and complex numbers, respectively. We denote by $\mathfrak{C}^C$ the complex Cayley algebra. $\mathfrak{C}^C$ is an eight-dimensional vector space over C with basis $\{e_0(=1), e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ and a (non-commutative non-associative) algebra over C with multiplication that

$$\begin{cases} e_0(=1) \text{ is a unit element;} \\ e_1e_2 = e_3,\ e_1e_4 = e_5,\ e_1e_6 = e_7,\ e_2e_5 = e_7,\ e_2e_6 = e_4,\ e_3e_4 = e_7,\ e_3e_5 = e_6; \\ {e_k}^2 = -1\ (k \neq 0); \quad e_ke_l = -e_le_k\ (k, l \text{ are distinct, and non-zero}); \\ e_ke_l = e_m \text{ implies } e_le_m = e_k\ (k, l, m \text{ are distinct, and non-zero}). \end{cases}$$

In $\mathfrak{C}^C$, the conjugation $\overline{x}$, the complex conjugation τx, the inner product (x, y), the norm $N(x)$ and the C-linear transformations γ, γ_1 of $\mathfrak{C}^C$ are defined respectively by

$$\overline{x_0 + \sum_{k=1}^{7} x_ke_k} = x_0 - \sum_{k=1}^{7} x_ke_k, \quad x_k \in C,$$

$$\tau\Big(\sum_{k=0}^{7}(x_k + iy_k)e_k\Big) = \sum_{k=0}^{7}(x_k - iy_k)e_k, \quad x_k, y_k \in \boldsymbol{R},$$

$$(x, y) = \frac{1}{2}(x\overline{y} + y\overline{x}), \qquad N(x) = x\overline{x},$$

$$\gamma\Big(\sum_{k=0}^{7} x_ke_k\Big) = \sum_{k=0}^{3} x_ke_k - \sum_{k=4}^{7} x_ke_k, \quad x_k \in C,$$

$$\gamma_1\Big(\sum_{k=0}^{7} x_ke_k\Big) = x_0 - x_1e_1 + x_2e_2 - x_3e_3 + x_4e_4 - x_5e_5 + x_6e_6 - x_7e_7.$$

Then, for $x, y \in \mathfrak{C}^C$, we have

$$\overline{xy} = \overline{y}\,\overline{x}, \quad \tau(xy) = (\tau x)(\tau y), \quad \gamma(xy) = (\gamma x)(\gamma y), \quad \gamma_1(xy) = (\gamma_1 x)(\gamma_1 y).$$

Next, we define the Cayley division algebra $\mathfrak{C}$ over $\boldsymbol{R}$ by

$$\mathfrak{C} = \{x \in \mathfrak{C}^C \mid \tau x = x\} = \Big\{\sum_{k=0}^{7} x_ke_k \,\Big|\, x_k \in \boldsymbol{R}\Big\},$$

and define the split Cayley algebra $\mathfrak{C}'$ over $\boldsymbol{R}$ by

$$\mathfrak{C}' = \{x \in \mathfrak{C}^C \mid \tau\gamma x = x\} = \Big\{\sum_{k=0}^{3} x_ke_k + \sum_{k=4}^{7} x_k(ie_k) \,\Big|\, x_k \in \boldsymbol{R}\Big\}.$$

In the case of $\mathfrak{C}'$, for convenience, we write down

$$e_0 = 1 = {e_0}',\ e_1 = {e_1}',\ e_2 = {e_2}',\ e_3 = {e_3}',$$
$$ie_4 = {e_4}',\ ie_5 = {e_5}',\ ie_6 = {e_6}',\ ie_7 = {e_7}'.$$

$\mathfrak{C}^C$ contains the algebra $\boldsymbol{C}^C$ of complex numbers over C and the algebra $\boldsymbol{H}^C$ of quaternions over C as

$$\boldsymbol{C}^C = \{x_0 + x_1e_4 \,|\, x_k \in C\}, \quad \boldsymbol{H}^C = \{x_0 + x_1e_1 + x_2e_2 + x_3e_3 \,|\, x_k \in C\}.$$

$\mathfrak{C}$ contains the field $\boldsymbol{C}$ of complex numbers and the field $\boldsymbol{H}$ of quaternions as

$$\boldsymbol{C} = \{x_0 + x_1e_4 \,|\, x_k \in \boldsymbol{R}\}, \quad \boldsymbol{H} = \{x_0 + x_1e_1 + x_2e_2 + x_3e_3 \,|\, x_k \in \boldsymbol{R}\}.$$

$\mathfrak{C}'$ contains the algebra $\boldsymbol{C}'$ of split complex numbers and the field $\boldsymbol{H}$ of quaternions as

$$\boldsymbol{C}' = \{x_0 + x_1{e_4}' \,|\, x_k \in \boldsymbol{R}\}, \quad \boldsymbol{H} = \{x_0 + x_1e_1 + x_2e_2 + x_3e_3 \,|\, x_k \in \boldsymbol{R}\}.$$

Any $x \in \mathfrak{C}^C$ (resp. $\mathfrak{C}$) is expressed as

$$\begin{aligned} x &= x_0 + x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 + x_5e_5 + x_6e_6 + x_7e_7 \quad x_k \in C \text{ (resp. } \boldsymbol{R}) \\ &= (x_0 + x_1e_1 + x_2e_2 + x_3e_3) + (x_4 + x_5e_1 - x_6e_2 + x_7e_3)e_4 \\ &= (x_0 + x_4e_4) + (x_1 - x_5e_4)e_1 + (x_2 + x_6e_4)e_2 + (x_3 - x_7e_4)e_3, \end{aligned}$$

and any $x \in \mathfrak{C}'$ is expressed as

$$\begin{aligned} x &= x_0 + x_1e_1 + x_2e_2 + x_3e_3 + x_4{e_4}' + x_5{e_5}' + x_6{e_6}' + x_7{e_7}' \quad x_k \in \boldsymbol{R} \\ &= (x_0 + x_1e_1 + x_2e_2 + x_3e_3) + (x_4 + x_5e_1 - x_6e_2 + x_7e_3){e_4}' \\ &= (x_0 + x_4{e_4}') + (x_1 - x_5{e_4}')e_1 + (x_2 + x_6{e_4}')e_2 + (x_3 - x_7{e_4}')e_3. \end{aligned}$$

Hence any $x \in \mathfrak{C}^C$ (resp. $\mathfrak{C}$) is expressed as

$$x = a + be_4, \quad a, b \in \boldsymbol{H}^C \text{ (resp. } \boldsymbol{H}).$$

In $\boldsymbol{H}^C \oplus \boldsymbol{H}^C e_4$ ($\boldsymbol{H} \oplus \boldsymbol{H}e_4$), we define the multiplication, the inner product and the conjugation respectively by

$$\begin{aligned} (a + be_4)(c + de_4) &= (ac - \overline{d}b) + (b\overline{c} + da)e_4, \\ (a + be_4, c + de_4) &= (a, c) + (b, d), \\ \overline{a + be_4} &= \overline{a} - be_4. \end{aligned}$$

Then, since these operations correspond to their respective operations in $\mathfrak{C}^C$ (resp. $\mathfrak{C}$), we can identify $\boldsymbol{H}^C \oplus \boldsymbol{H}^C e_4$ (resp. $\boldsymbol{H} \oplus \boldsymbol{H}e_4$) with $\mathfrak{C}^C$ (resp. $\mathfrak{C}$): $\boldsymbol{H}^C \oplus \boldsymbol{H}^C e_4 = \mathfrak{C}^C$ (resp. $\boldsymbol{H} \oplus \boldsymbol{H}e_4 = \mathfrak{C}$). Similarly, any $x \in \mathfrak{C}'$ is expressed as

$$x = a + b{e_4}', \quad a, b \in \boldsymbol{H}.$$

In $\boldsymbol{H}\oplus\boldsymbol{H}e_4{}'$, we define the multiplication, the inner product and the conjugation respectively by

$$\begin{aligned}(a+be_4{}')(c+de_4{}') &= (ac+\overline{d}b)+(b\overline{c}+da)e_4{}',\\ (a+be_4{}',c+de_4{}') &= (a,c)-(b,d),\\ \overline{a+be_4{}'} &= \overline{a}-be_4{}'.\end{aligned}$$

Then, since these operations correspond to their respective operations in $\mathfrak{C}'$, we can identify $\boldsymbol{H}\oplus\boldsymbol{H}e_4{}'$ with $\mathfrak{C}'$: $\boldsymbol{H}\oplus\boldsymbol{H}e_4{}'=\mathfrak{C}'$.
On the other hand, any $x\in\mathfrak{C}^C$ (resp. $\mathfrak{C},\mathfrak{C}'$) is also expressed as

$$x=a+m_1e_1+m_2e_2+m_3e_3,\quad a,m_k\in\boldsymbol{C}^C\ (\text{resp.}\,\boldsymbol{C},\boldsymbol{C}').$$

We associate such x with

$$a+\begin{pmatrix}m_1\\ m_2\\ m_3\end{pmatrix}\in\boldsymbol{C}^C\oplus(\boldsymbol{C}^C)^3\ (\text{resp.}\,\boldsymbol{C}\oplus\boldsymbol{C}^3,\boldsymbol{C}'\oplus(\boldsymbol{C}')^3).$$

In $\boldsymbol{C}^C\oplus(\boldsymbol{C}^C)^3$ (resp. $\boldsymbol{C}\oplus\boldsymbol{C}^3,\boldsymbol{C}'\oplus(\boldsymbol{C}')^3$), we define the multiplication, the inner product and the conjugation by

$$\begin{aligned}(a+\boldsymbol{m})(b+\boldsymbol{n}) &= (ab-{}^t\boldsymbol{m}\overline{\boldsymbol{n}})+(a\boldsymbol{n}+\overline{b}\boldsymbol{m}+\overline{\boldsymbol{m}\times\boldsymbol{n}}),\\ (a+\boldsymbol{m},b+\boldsymbol{n}) &= (a,b)+(\boldsymbol{m},\boldsymbol{n}),\\ \overline{a+\boldsymbol{m}} &= \overline{a}-\boldsymbol{m},\end{aligned}$$

where $(a,b),(\boldsymbol{m},\boldsymbol{n}),\boldsymbol{m}\times\boldsymbol{n}$ are respectively defined by

$$(a,b)=\frac{1}{2}(a\overline{b}+b\overline{a}),\quad (\boldsymbol{m},\boldsymbol{n})=\frac{1}{2}({}^t\boldsymbol{m}\overline{\boldsymbol{n}}+{}^t\boldsymbol{n}\overline{\boldsymbol{m}}),\quad \boldsymbol{m}\times\boldsymbol{n}=\begin{pmatrix}m_2n_3-n_2m_3\\ m_3n_1-n_3m_1\\ m_1n_2-n_1m_2\end{pmatrix},$$

for $\boldsymbol{m}=\begin{pmatrix}m_1\\ m_2\\ m_3\end{pmatrix},\boldsymbol{n}=\begin{pmatrix}n_1\\ n_2\\ n_3\end{pmatrix}\in(\boldsymbol{C}^C)^3$ (resp. $\boldsymbol{C}^3,(\boldsymbol{C}')^3$). Then $\boldsymbol{C}^C\oplus(\boldsymbol{C}^C)^3$ (resp. $\boldsymbol{C}\oplus\boldsymbol{C}^3,\boldsymbol{C}'\oplus(\boldsymbol{C}')^3$) is isomorphic to $\mathfrak{C}^C$ (resp. $\mathfrak{C},\mathfrak{C}'$) as algebra.

The connected linear Lie groups of type G_2 are obtained as the automorphism groups of the Cayley algebras:

$$\begin{aligned}{G_2}^C &= \mathrm{Aut}(\mathfrak{C}^C)=\{\alpha\in\mathrm{Iso}_C(\mathfrak{C}^C)\,|\,\alpha(xy)=(\alpha x)(\alpha y)\},\\ G_2 &= \mathrm{Aut}(\mathfrak{C})=\{\alpha\in\mathrm{Iso}_R(\mathfrak{C})\,|\,\alpha(xy)=(\alpha x)(\alpha y)\},\\ G_{2(2)} &= \mathrm{Aut}(\mathfrak{C}')=\{\alpha\in\mathrm{Iso}_R(\mathfrak{C}')\,|\,\alpha(xy)=(\alpha x)(\alpha y)\}.\end{aligned}$$

It is known that ${G_2}^C$ and G_2 are simply connected, and that G_2 is compact, ${G_2}^C$ and $G_{2(2)}$ are non-compact([1], [10], [12], [14]).

PROPOSITION 2.1 ([10, 14]). *$\alpha \in {G_2}^C$ (resp. $G_2, G_{2(2)}$) leaves the inner product (x, y) of $\mathfrak{C}^C$ (resp. $\mathfrak{C}$, $\mathfrak{C}'$), in particular, it leaves invariant the norm $N(x)$:*

$$(\alpha x, \alpha y) = (x, y), \quad N(\alpha x) = N(x), \quad x, y \in \mathfrak{C}^C \, (resp. \, \mathfrak{C}, \mathfrak{C}').$$

3. Elements of ${G_2}^C, G_2$ and $G_{2(2)}$

Let a complex six-dimensional unit sphere $(S^C)^6$, a six-dimensional unit sphere S^6 and a Minkowski six-dimensional unit sphere $(S')^6$ be respectively

$$(S^C)^6 = \{u \in \mathfrak{C}^C \,|\, \overline{u} = -u, u\overline{u} = 1\} = \Big\{\sum_{k=1}^{7} a_k e_k \,\Big|\, \sum_{k=1}^{7} {a_k}^2 = 1, a_k \in C\Big\},$$

$$S^6 = \{u \in \mathfrak{C} \,|\, \overline{u} = -u, u\overline{u} = 1\} = \Big\{\sum_{k=1}^{7} a_k e_k \,\Big|\, \sum_{k=1}^{7} {a_k}^2 = 1, a_k \in \boldsymbol{R}\Big\},$$

$$(S')^6 = \{u \in \mathfrak{C}' \,|\, \overline{u} = -u, u\overline{u} = 1\} = \Big\{\sum_{k=1}^{7} a_k {e_k}' \,\Big|\, \sum_{k=1}^{3} {a_k}^2 - \sum_{k=4}^{7} {a_k}^2 = 1, a_k \in \boldsymbol{R}\Big\}.$$

PROPOSITION 3.1. *For $a = -\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}u \in \mathfrak{C}^C$ (resp. $\mathfrak{C}$, $\mathfrak{C}'$), $u \in (S^C)^6$ (resp. S^6, $(S')^6$), define $\alpha_a : \mathfrak{C}^C \to \mathfrak{C}^C$ (resp. $\mathfrak{C} \to \mathfrak{C}, \mathfrak{C}' \to \mathfrak{C}'$) by*

$$\alpha_a x = a x \overline{a}, \quad x \in \mathfrak{C}^C \, (resp. \, \mathfrak{C}, \mathfrak{C}').$$

Then, α_a belongs to the group ${G_2}^C$ (resp. $G_2, G_{2(2)}$).

Proof. We know that formulas

$$a(x\overline{a}) = (ax)\overline{a}, \quad x(\overline{a}a) = (x\overline{a})a, \quad x(aa) = (xa)a$$

and Moufang's formulas

$$\text{(i)} \quad (ax)(ya) = a(xy)a, \qquad \text{(ii)} \quad x(aya) = ((xa)y)a,$$

for $a, x, y \in \mathfrak{C}^C$, are valid. Note that for $a = -\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}u, u \in (S^C)^6$, it holds that $a^3 = 1, a\overline{a} = 1$ and $a^2 = \overline{a}$. Putting $ax\overline{a}$ instead of x and ya instead of y in Moufang's formula (ii), we have

$$(ax\overline{a})(aya^2) = (((ax\overline{a})a)(ya))a.$$

Hence,

$$(ax\overline{a})(ay\overline{a}) = ((ax)(ya))a = (a(xy)a)a = a(xy)a^2 = a(xy)\overline{a}.$$

Therefore, we obtain that α_a belongs to ${G_2}^C$, since α_a obviously is a C-linear isomorphism of $\mathfrak{C}^C$. Both cases of G_2 and $G_{2(2)}$ are easily seen by the analogous argument above, because the same formulas of $\mathfrak{C}^C$ are valid in $\mathfrak{C}$ and $\mathfrak{C}'$. □

PROPOSITION 3.2. *For* $a = -\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}u \in \mathfrak{C}^C$ *(resp.* $\mathfrak{C}$, $\mathfrak{C}'$*),* $u \in (S^C)^6$ *(resp.* S^6, $(S')^6$*), it holds that*

(1) $\alpha_a a = a, \quad \alpha_a u = u,$

(2) $(\alpha_a)^3 = 1, \quad (\alpha_a)^{-1} = \alpha_{\overline{a}} = \alpha_{a^2} = (\alpha_a)^2,$

(3) $\beta\alpha_a\beta^{-1} = \alpha_{\beta a}, \quad \beta \in {G_2}^C$ *(resp.* $G_2, G_{2(2)}$*).*

Proof. (1),(2) are easily obtained by direct calculation.

(3) follows immediately, since $\beta\overline{x} = \overline{\beta x}, \beta \in {G_2}^C$ (resp. $G_2, G_{2(2)}$), $x \in \mathfrak{C}^C$ (resp. $\mathfrak{C}, \mathfrak{C}'$). □

LEMMA 3.3. (1) *For* $a = -\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}\sum_{k=1}^{7} a_k e_k \in \mathfrak{C}^C$, $\sum_{k=1}^{7} {a_k}^2 = 1, a_k \in C$, *the images of* $\alpha_a e_k$ $(k = 0, 1, \ldots, 7)$ *are expressed as follows:*

$$\alpha_a e_0 = e_0,$$

$$\begin{aligned}
\alpha_a e_1 = 0 &+ \frac{1}{2}(-1 + 3{a_1}^2)e_1 + \frac{\sqrt{3}}{2}(-a_3 + \sqrt{3}a_1a_2)e_2 + \frac{\sqrt{3}}{2}(a_2 + \sqrt{3}a_1a_3)e_3 \\
&+ \frac{\sqrt{3}}{2}(-a_5 + \sqrt{3}a_1a_4)e_4 + \frac{\sqrt{3}}{2}(a_4 + \sqrt{3}a_1a_5)e_5 \\
&+ \frac{\sqrt{3}}{2}(-a_7 + \sqrt{3}a_1a_6)e_6 + \frac{\sqrt{3}}{2}(a_6 + \sqrt{3}a_1a_7)e_7,
\end{aligned}$$

$$\begin{aligned}
\alpha_a e_2 = 0 &+ \frac{\sqrt{3}}{2}(a_3 + \sqrt{3}a_1a_2)e_1 + \frac{1}{2}(-1 + 3{a_2}^2)e_2 + \frac{\sqrt{3}}{2}(-a_1 + \sqrt{3}a_2a_3)e_3 \\
&+ \frac{\sqrt{3}}{2}(a_6 + \sqrt{3}a_2a_4)e_4 + \frac{\sqrt{3}}{2}(-a_7 + \sqrt{3}a_2a_5)e_5 \\
&+ \frac{\sqrt{3}}{2}(-a_4 + \sqrt{3}a_2a_6)e_6 + \frac{\sqrt{3}}{2}(a_5 + \sqrt{3}a_2a_7)e_7,
\end{aligned}$$

$$\begin{aligned}
\alpha_a e_3 = 0 &+ \frac{\sqrt{3}}{2}(-a_2 + \sqrt{3}a_1a_3)e_1 + \frac{\sqrt{3}}{2}(a_1 + \sqrt{3}a_2a_3)e_2 + \frac{1}{2}(-1 + 3{a_3}^2)e_3 \\
&+ \frac{\sqrt{3}}{2}(-a_7 + \sqrt{3}a_3a_4)e_4 + \frac{\sqrt{3}}{2}(-a_6 + \sqrt{3}a_3a_5)e_5
\end{aligned}$$

$$+\frac{\sqrt{3}}{2}(a_5+\sqrt{3}a_3a_6)e_6+\frac{\sqrt{3}}{2}(a_4+\sqrt{3}a_3a_7)e_7,$$

$$\alpha_a e_4 = 0+\frac{\sqrt{3}}{2}(a_5+\sqrt{3}a_1a_4)e_1+\frac{\sqrt{3}}{2}(-a_6+\sqrt{3}a_2a_4)e_2+\frac{\sqrt{3}}{2}(a_7+\sqrt{3}a_3a_4)e_3$$
$$+\frac{1}{2}(-1+3{a_4}^2)e_4+\frac{\sqrt{3}}{2}(-a_1+\sqrt{3}a_4a_5)e_5$$
$$+\frac{\sqrt{3}}{2}(a_2+\sqrt{3}a_4a_6)e_6+\frac{\sqrt{3}}{2}(-a_3+\sqrt{3}a_4a_7)e_7,$$

$$\alpha_a e_5 = 0+\frac{\sqrt{3}}{2}(-a_4+\sqrt{3}a_1a_5)e_1+\frac{\sqrt{3}}{2}(a_7+\sqrt{3}a_2a_5)e_2+\frac{\sqrt{3}}{2}(a_6+\sqrt{3}a_3a_5)e_3$$
$$+\frac{\sqrt{3}}{2}(a_1+\sqrt{3}a_4a_5)e_4+\frac{1}{2}(-1+3{a_5}^2)e_5$$
$$+\frac{\sqrt{3}}{2}(-a_3+\sqrt{3}a_5a_6)e_6+\frac{\sqrt{3}}{2}(-a_2+\sqrt{3}a_5a_7)e_7,$$

$$\alpha_a e_6 = 0+\frac{\sqrt{3}}{2}(a_7+\sqrt{3}a_1a_6)e_1+\frac{\sqrt{3}}{2}(a_4+\sqrt{3}a_2a_6)e_2+\frac{\sqrt{3}}{2}(-a_5+\sqrt{3}a_3a_6)e_3$$
$$+\frac{\sqrt{3}}{2}(-a_2+\sqrt{3}a_4a_6)e_4+\frac{\sqrt{3}}{2}(a_3+\sqrt{3}a_5a_6)e_5$$
$$+\frac{1}{2}(-1+3{a_6}^2)e_6+\frac{\sqrt{3}}{2}(-a_1+\sqrt{3}a_6a_7)e_7,$$

$$\alpha_a e_7 = 0+\frac{\sqrt{3}}{2}(-a_6+\sqrt{3}a_1a_7)e_1+\frac{\sqrt{3}}{2}(-a_5+\sqrt{3}a_2a_7)e_2$$
$$+\frac{\sqrt{3}}{2}(-a_4+\sqrt{3}a_3a_7)e_3+\frac{\sqrt{3}}{2}(a_3+\sqrt{3}a_4a_7)e_4+\frac{\sqrt{3}}{2}(a_2+\sqrt{3}a_5a_7)e_5$$
$$+\frac{\sqrt{3}}{2}(a_1+\sqrt{3}a_6a_7)e_6+\frac{1}{2}(-1+3{a_7}^2)e_7.$$

(2) *For* $a = -\frac{1}{2}+\frac{\sqrt{3}}{2}\sum_{k=1}^{7} a_k e_k \in \mathfrak{C}$, $\sum_{k=1}^{7} {a_k}^2 = 1, a_k \in \boldsymbol{R}$, *the images of* $\alpha_a e_k$ $(k = 0, 1, \ldots, 7)$ *are expressed as the same form in* (1) (*replace* C *by* $\boldsymbol{R}$ *in* (1)).

(3) *For* $a = -\frac{1}{2}+\frac{\sqrt{3}}{2}\sum_{k=1}^{7} a_k e_k' \in \mathfrak{C}'$, $\sum_{k=1}^{3} {a_k}^2 - \sum_{k=4}^{7} {a_k}^2 = 1, a_k \in \boldsymbol{R}$, *the*

images of $\alpha_a e_k{}'$ $(k = 0, 1, \dots, 7)$ *are expressed as follows*:

$$\alpha_a e_0{}' = e_0{}',$$

$$\begin{aligned}\alpha_a e_1{}' = 0 &+ \frac{1}{2}(-1 + 3a_1{}^2)e_1{}' + \frac{\sqrt{3}}{2}(-a_3 + \sqrt{3}a_1a_2)e_2{}' + \frac{\sqrt{3}}{2}(a_2 + \sqrt{3}a_1a_3)e_3{}' \\ &+ \frac{\sqrt{3}}{2}(-a_5 + \sqrt{3}a_1a_4)e_4{}' + \frac{\sqrt{3}}{2}(a_4 + \sqrt{3}a_1a_5)e_5{}' \\ &+ \frac{\sqrt{3}}{2}(-a_7 + \sqrt{3}a_1a_6)e_6{}' + \frac{\sqrt{3}}{2}(a_6 + \sqrt{3}a_1a_7)e_7{}',\end{aligned}$$

$$\begin{aligned}\alpha_a e_2{}' = 0 &+ \frac{\sqrt{3}}{2}(a_3 + \sqrt{3}a_1a_2)e_1{}' + \frac{1}{2}(-1 + 3a_2{}^2)e_2{}' + \frac{\sqrt{3}}{2}(-a_1 + \sqrt{3}a_2a_3)e_3{}' \\ &+ \frac{\sqrt{3}}{2}(a_6 + \sqrt{3}a_2a_4)e_4{}' + \frac{\sqrt{3}}{2}(-a_7 + \sqrt{3}a_2a_5)e_5{}' \\ &+ \frac{\sqrt{3}}{2}(-a_4 + \sqrt{3}a_2a_6)e_6{}' + \frac{\sqrt{3}}{2}(a_5 + \sqrt{3}a_2a_7)e_7{}',\end{aligned}$$

$$\begin{aligned}\alpha_a e_3{}' = 0 &+ \frac{\sqrt{3}}{2}(-a_2 + \sqrt{3}a_1a_3)e_1{}' + \frac{\sqrt{3}}{2}(a_1 + \sqrt{3}a_2a_3)e_2{}' + \frac{1}{2}(-1 + 3a_3{}^2)e_3{}' \\ &+ \frac{\sqrt{3}}{2}(-a_7 + \sqrt{3}a_3a_4)e_4{}' + \frac{\sqrt{3}}{2}(-a_6 + \sqrt{3}a_3a_5)e_5{}' \\ &+ \frac{\sqrt{3}}{2}(a_5 + \sqrt{3}a_3a_6)e_6{}' + \frac{\sqrt{3}}{2}(a_4 + \sqrt{3}a_3a_7)e_7{}',\end{aligned}$$

$$\begin{aligned}\alpha_a e_4{}' = 0 &- \frac{\sqrt{3}}{2}(a_5 + \sqrt{3}a_1a_4)e_1{}' + \frac{\sqrt{3}}{2}(a_6 - \sqrt{3}a_2a_4)e_2{}' - \frac{\sqrt{3}}{2}(a_7 + \sqrt{3}a_3a_4)e_3{}' \\ &- \frac{1}{2}(1 + 3a_4{}^2)e_4{}' - \frac{\sqrt{3}}{2}(a_1 + \sqrt{3}a_4a_5)e_5{}' \\ &+ \frac{\sqrt{3}}{2}(a_2 - \sqrt{3}a_4a_6)e_6{}' + \frac{\sqrt{3}}{2}(a_3 + \sqrt{3}a_4a_7)e_7{}',\end{aligned}$$

$$\begin{aligned}\alpha_a e_5{}' = 0 &+ \frac{\sqrt{3}}{2}(a_4 - \sqrt{3}a_1a_5)e_1{}' - \frac{\sqrt{3}}{2}(a_7 + \sqrt{3}a_2a_5)e_2{}' - \frac{\sqrt{3}}{2}(a_6 + \sqrt{3}a_3a_5)e_3{}' \\ &+ \frac{\sqrt{3}}{2}(a_1 - \sqrt{3}a_4a_5)e_4{}' - \frac{1}{2}(1 + 3a_5{}^2)e_5{}' \\ &- \frac{\sqrt{3}}{2}(a_3 + \sqrt{3}a_5a_6)e_6{}' - \frac{\sqrt{3}}{2}(a_2 + \sqrt{3}a_5a_7)e_7{}',\end{aligned}$$

$$\begin{aligned}\alpha_a e_6{}' = 0 &- \frac{\sqrt{3}}{2}(a_7 + \sqrt{3}a_1a_6)e_1{}' - \frac{\sqrt{3}}{2}(a_4 + \sqrt{3}a_2a_6)e_2{}' + \frac{\sqrt{3}}{2}(a_5 - \sqrt{3}a_3a_6)e_3{}' \\ &- \frac{\sqrt{3}}{2}(a_2 + \sqrt{3}a_4a_6)e_4{}' + \frac{\sqrt{3}}{2}(a_3 - \sqrt{3}a_5a_6)e_5{}' \\ &- \frac{1}{2}(1 + 3{a_6}^2)e_6{}' - \frac{\sqrt{3}}{2}(a_1 + \sqrt{3}a_6a_7)e_7{}',\end{aligned}$$

$$\begin{aligned}\alpha_a e_7{}' = 0 &+ \frac{\sqrt{3}}{2}(a_6 - \sqrt{3}a_1a_7)e_1{}' + \frac{\sqrt{3}}{2}(a_5 - \sqrt{3}a_2a_7)e_2{}' + \frac{\sqrt{3}}{2}(a_4 - \sqrt{3}a_3a_7)e_3{}' \\ &+ \frac{\sqrt{3}}{2}(a_3 - \sqrt{3}a_4a_7)e_4{}' + \frac{\sqrt{3}}{2}(a_2 - \sqrt{3}a_5a_7)e_5{}' \\ &+ \frac{\sqrt{3}}{2}(a_1 - \sqrt{3}a_6a_7)e_6{}' - \frac{1}{2}(1 + 3{a_7}^2)e_7{}'.\end{aligned}$$

Proof. These are obtained by direct calculation of the definition of α_a. □

We arrange here some groups used later.

$$\begin{aligned}SU(n,K) &= \{A \in M(n,K) \mid AA^* = E, \det A = 1\}, \quad K = \boldsymbol{C}^C, \boldsymbol{C}, \boldsymbol{C}', \\ Sp(n,K) &= \{A \in M(n,K) \mid AA^* = E\}, \quad K = \boldsymbol{H}^C, \boldsymbol{H},\end{aligned}$$

where E is $n \times n$ unit matrix: $E = \mathrm{diag}(1, 1, \ldots, 1)$, and $\det A$ is the determinant of A defined as usual. Usually the following symbols are used.

$$SU(n) = SU(n, \boldsymbol{C}), \qquad Sp(n) = Sp(n, \boldsymbol{H}).$$

We have the following isomorphisms as groups ([12]).

$$SU(n, \boldsymbol{C}^C) \cong SL(n, C), \ SU(n, \boldsymbol{C}') \cong SL(n, \boldsymbol{R}), \ Sp(n, \boldsymbol{H}^C) \cong Sp(n, C),$$

where $SL(n,K) = \{A \in M(n,K) \mid \det A = 1\}$, $K = \boldsymbol{R}, C$ and $Sp(n, C) = \{A \in M(2n, C) \mid {}^tAJ_nA = J_n\}$, $J_n = \mathrm{diag}(J_1, \ldots, J_1)$, $J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Lemma 3.4 ([11, 14]). *For $A \in SU(3, \boldsymbol{C}^C)$ (resp. $SU(3)$, $SU(3, \boldsymbol{C}')$), we define a mapping $\varphi(A) : \mathfrak{C}^C \to \mathfrak{C}^C$ (resp. $\mathfrak{C} \to \mathfrak{C}$, $\mathfrak{C}' \to \mathfrak{C}'$) by*

$$\varphi(A)(a + \boldsymbol{m}) = a + A\boldsymbol{m}, \quad a + \boldsymbol{m} \in \boldsymbol{C}^C \oplus (\boldsymbol{C}^C)^3 = \mathfrak{C}^C$$
$$(\text{resp. } \boldsymbol{C} \oplus \boldsymbol{C}^3 = \mathfrak{C},\ \boldsymbol{C}' \oplus (\boldsymbol{C}')^3 = \mathfrak{C}'),$$

then $\varphi(A) \in {G_2}^C$ (resp. G_2, $G_{2(2)}$).

Proof. First, we have that for $A \in M(n, K)$, $\boldsymbol{m}, \boldsymbol{n} \in K^3, K = \boldsymbol{C}^C, \boldsymbol{C}, \boldsymbol{C}'$, it holds that

(i) ${}^t(A\boldsymbol{m})\overline{\boldsymbol{n}} = {}^t\boldsymbol{m}\,\overline{({}^t\overline{A}\boldsymbol{n})}\,(= {}^t\boldsymbol{m}\,({}^tA\overline{\boldsymbol{n}}))$,

(ii) $A\boldsymbol{m} \times A\boldsymbol{n} = {}^t\widetilde{A}\,(\boldsymbol{m} \times \boldsymbol{n})$,

where $\widetilde{A}$ is the cofactor matrix of $A : \widetilde{A}A = A\widetilde{A} = (\det A)E$. Indeed, these follow from direct calculation similar to the case $K = \boldsymbol{C}$ ([11], [14]), since K is commutative. Using these properties (i) and (ii), we can show

$$\varphi(A)((a + \boldsymbol{m})(b + \boldsymbol{n})) = \varphi(A)(a + \boldsymbol{m})\,\varphi(A)(b + \boldsymbol{n})$$

in exactly the same way as in the proof of the case $K = \boldsymbol{C}$. Thus the lemma follows. □

LEMMA 3.5 ([12, 14]). (1) *For $p, q \in Sp(1, \boldsymbol{H}^C)$ (resp. $Sp(1)$), we define a mapping $\varphi(p, q) : \mathfrak{C}^C \to \mathfrak{C}^C$ (resp. $\mathfrak{C} \to \mathfrak{C}$) by*

$$\varphi(p, q)(a+be_4) = qa\overline{q}+(pb\overline{q})e_4,\ a+be_4 \in \boldsymbol{H}^C \oplus \boldsymbol{H}^C e_4 = \mathfrak{C}^C\ (\textit{resp.}\ \boldsymbol{H} \oplus \boldsymbol{H}e_4 = \mathfrak{C}),$$

then $\varphi(p, q) \in {G_2}^C$ (resp. G_2).

(2) *For $p, q \in Sp(1)$, we define a mapping $\varphi(p, q) : \mathfrak{C}' \to \mathfrak{C}'$ by*

$$\varphi(p, q)(a + be_4{}') = qa\overline{q} + (pb\overline{q})e_4{}',\quad a + be_4{}' \in \boldsymbol{H} \oplus \boldsymbol{H}e_4{}' = \mathfrak{C}',$$

then $\varphi(p, q) \in G_{2(2)}$.

LEMMA 3.6. *Let $\delta_k : \mathfrak{C} \to \mathfrak{C}, k = 1, 2, 3, 4, 5$ be $\boldsymbol{R}$-linear mappings satisfying*

$$\delta_1 : \begin{cases} e_0 \to e_0,\ e_1 \to e_1,\ e_2 \to e_4,\ e_3 \to e_5, \\ e_4 \to e_2,\ e_5 \to e_3,\ e_6 \to -e_6,\ e_7 \to -e_7, \end{cases}$$
$$\delta_2 : \begin{cases} e_0 \to e_0,\ e_1 \to e_1,\ e_2 \to e_6,\ e_3 \to e_7, \\ e_4 \to -e_4,\ e_5 \to -e_5,\ e_6 \to e_2,\ e_7 \to e_3, \end{cases}$$
$$\delta_3 : \begin{cases} e_0 \to e_0,\ e_1 \to e_4,\ e_2 \to e_2,\ e_3 \to e_6, \\ e_4 \to e_1,\ e_5 \to -e_5,\ e_6 \to e_3,\ e_7 \to -e_7, \end{cases}$$
$$\delta_4 : \begin{cases} e_0 \to e_0,\ e_1 \to e_5,\ e_2 \to e_2,\ e_3 \to -e_7, \\ e_4 \to -e_4,\ e_5 \to e_1,\ e_6 \to -e_6,\ e_7 \to -e_3, \end{cases}$$
$$\delta_5 : \begin{cases} e_0 \to e_0,\ e_1 \to e_4,\ e_2 \to e_7,\ e_3 \to e_3, \\ e_4 \to e_1,\ e_5 \to -e_5,\ e_6 \to -e_6,\ e_7 \to e_2. \end{cases}$$

Then ${\delta_k}^2 = 1$ and $\delta_k \in G_2 \subset {G_2}^C$.

Proof. It needs to check $\delta_k(e_l e_m) = \delta_k(e_l)\delta_k(e_m), l, m = 1, 2, \ldots, 7$, however we can easily check by direct calculation. □

4. Canonical form of an element of $\mathfrak{C}_0{}^C$ and explicit classification of orbits in $\mathfrak{C}_0{}^C$ over $G_2{}^C$

Since it holds that $\alpha 1 = 1$ for any $\alpha \in G_2{}^C$ (resp. G_2, $G_{2(2)}$), to determine the canonical form of an element of $\mathfrak{C}^C$ (resp. $\mathfrak{C}$, $\mathfrak{C}'$) and the classification of orbits in $\mathfrak{C}^C$ (resp. $\mathfrak{C}$, $\mathfrak{C}'$) by the action of the group $G_2{}^C$ (resp. G_2, $G_{2(2)}$), it only has to consider the space $\mathfrak{C}_0{}^C$ (resp. $\mathfrak{C}_0$, $\mathfrak{C}_0{}'$) of pure imaginary Cayley numbers, where the space $K_0 (= \mathfrak{C}_0{}^C, \mathfrak{C}_0, \mathfrak{C}_0{}')$ of pure imaginary Cayley numbers is given by $K_0 = \{x \in K \,|\, \overline{x} = -x\}$. For $K = \boldsymbol{C}, \boldsymbol{H}$, the space K_0 of pure imaginary (complex, quaternion) numbers is also defined as that of Cayley algebra.

THEOREM 4.1. *Any non-zero element $x \in \mathfrak{C}_0{}^C$ can be transformed to the following canonical form by some element of $G_2{}^C$:*

(1) *In the case of $N(x) \neq 0$:*

$$(\xi + i\eta)e_1 \qquad \left(\xi > 0 \;\; or \;\; \left\{ \begin{array}{l} \xi = 0 \\ \eta > 0 \end{array} \right. \right),$$

where $(\xi + i\eta)^2 = N(x)$.

(2) *In the case of $N(x) = 0$:*

$$e_1 + ie_4.$$

Moreover, all their orbits in $\mathfrak{C}_0{}^C$ over $G_2{}^C$ are distinct, and the union of all their orbits and $\{0\}$ is the whole space $\mathfrak{C}_0{}^C$.

Proof. Let $x = \sum_{k=1}^{7} x_k e_k \in \mathfrak{C}_0{}^C$.

(1) Since $N(x)$ can be uniquely expressed as $N(x) = (\xi + i\eta)^2$, $\xi > 0$ or $\left\{ \begin{array}{l} \xi = 0 \\ \eta > 0 \end{array} \right.$, put

$$z = \sum_{k=1}^{7} z_k e_k = \frac{1}{\xi + i\eta} \sum_{k=1}^{7} x_k e_k \in \mathfrak{C}_0{}^C.$$

Then $N(z) = 1$. Now, applying $\gamma_1 \in G_2{}^C$ to z if necessary, we may assume $z_1 \neq -1$. We solve the equation $\alpha_a e_1 = z = \sum_{k=1}^{7} z_k e_k$ of Lemma 3.3 (1), then the solution $a = -\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}\sum_{k=1}^{7} a_k e_k \in \mathfrak{C}^C, \sum_{k=1}^{7} {a_k}^2 = 1$ is given by

$$(*)\begin{cases} a_1 = \sqrt{\dfrac{2z_1+1}{3}}, & a_2 = \dfrac{z_2\sqrt{2z_1+1}+z_3}{\sqrt{3}(z_1+1)}, \quad a_3 = \dfrac{z_3\sqrt{2z_1+1}-z_2}{\sqrt{3}(z_1+1)}, \\ a_4 = \dfrac{z_4\sqrt{2z_1+1}+z_5}{\sqrt{3}(z_1+1)}, & a_5 = \dfrac{z_5\sqrt{2z_1+1}-z_4}{\sqrt{3}(z_1+1)}, \quad a_6 = \dfrac{z_6\sqrt{2z_1+1}+z_7}{\sqrt{3}(z_1+1)}, \\ a_7 = \dfrac{z_7\sqrt{2z_1+1}-z_6}{\sqrt{3}(z_1+1)}. \end{cases}$$

Then $\alpha_a e_1 = z$, that is, $\alpha_{\overline{a}} z = e_1, \alpha_{\overline{a}} \in {G_2}^C$. Thus we obtain

$$\alpha_{\overline{a}} x = (\xi + i\eta)\alpha_{\overline{a}} z = (\xi + i\eta)e_1.$$

The uniqueness is obvious from $N(\alpha x) = N(x), x \in {\mathfrak{C}_0}^C, \alpha \in {G_2}^C$.

(2) For $x = \sum_{k=1}^{7} x_k e_k \in {\mathfrak{C}_0}^C$, we define $N_{\boldsymbol{H}_0}(x) = \sum_{k=1}^{3} {x_k}^2$. Assume that $N_{\boldsymbol{H}_0}(x) = 0$ and further $N_{\boldsymbol{H}_0}(\delta_k x) = 0$ for all $\delta_k \in {G_2}^C (k = 1,2,3,4,5)$ of Lemma 3.6. Then from

$$\begin{cases} {x_1}^2 + {x_2}^2 + {x_3}^2 = 0 \\ {x_1}^2 + {x_4}^2 + {x_5}^2 = 0 \\ {x_1}^2 + {x_6}^2 + {x_7}^2 = 0 \\ {x_2}^2 + {x_4}^2 + {x_6}^2 = 0 \\ {x_2}^2 + {x_5}^2 + {x_7}^2 = 0 \\ {x_3}^2 + {x_4}^2 + {x_7}^2 = 0 \\ {x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2 + {x_5}^2 + {x_6}^2 + {x_7}^2 = 0 \end{cases}$$

we have $x_1 = x_2 = \cdots = x_7 = 0$, which contradicts $x \neq 0$. Hence, applying some $\delta_k \in {G_2}^C$ to x, we may assume $N_{\boldsymbol{H}_0}(x) \neq 0$. Now, let

$$x = \sum_{k=1}^{7} x_k e_k = a + be_4 \in {\boldsymbol{H}_0}^C \oplus \boldsymbol{H}^C e_4 = {\mathfrak{C}_0}^C.$$

Then since $N(x) = 0$ and $N_{\boldsymbol{H}_0}(x) \neq 0$, we have $0 \neq a\overline{a} = -b\overline{b} \left(\underset{\text{put}}{=} \lambda^2, \lambda \in C\right)$, so we have

$$\frac{a^2}{\lambda^2} = -\frac{a}{\lambda}\frac{\overline{a}}{\lambda} = -1 = {e_1}^2.$$

Put $\mu^2 = N\Big(\dfrac{a}{\lambda} + e_1\Big), \mu \in C$. Then for $\dfrac{a}{\lambda} = \sum_{k=1}^{3} a_k e_k$, since $N\Big(\dfrac{a}{\lambda}\Big) = 1$, that is, ${a_1}^2 + {a_2}^2 + {a_3}^2 = 1$, we have

$$\mu^2 = N\Big(\frac{a}{\lambda} + e_1\Big) = (a_1 + 1)^2 + {a_2}^2 + {a_3}^2 = 2(a_1 + 1).$$

Therefore we have $\mu = 0$ only in case $a_1 = -1$. In this case, applying $\gamma_1 \in {G_2}^C$, we may assume $\mu \neq 0$. Put $q = \dfrac{1}{\mu}\Big(\dfrac{a}{\lambda} + e_1\Big)$. Then we have $q \in Sp(1, \boldsymbol{H}^C)$ and

$$\begin{aligned} q\frac{a}{\lambda}\overline{q} &= \frac{1}{\mu}\Big(\frac{a}{\lambda} + e_1\Big)\frac{a}{\lambda}\overline{\frac{1}{\mu}\Big(\frac{a}{\lambda} + e_1\Big)} = -\frac{1}{\mu^2}\Big(\frac{a}{\lambda} + e_1\Big)\frac{a}{\lambda}\Big(\frac{a}{\lambda} + e_1\Big) \\ &= -\frac{1}{\mu^2}\Big(\frac{a^2}{\lambda^2} + \frac{e_1 a}{\lambda}\Big)\Big(\frac{a}{\lambda} + e_1\Big) = -\frac{1}{\mu^2}\Big({e_1}^2 + \frac{e_1 a}{\lambda}\Big)\Big(\frac{a}{\lambda} + e_1\Big) \\ &= -\frac{1}{\mu^2}e_1\Big(e_1 + \frac{a}{\lambda}\Big)\Big(\frac{a}{\lambda} + e_1\Big) = -\frac{1}{\mu^2}e_1(-\mu^2) = e_1. \end{aligned}$$

Further, put $p = \dfrac{q\overline{b}}{i\lambda}$. Then we have $p \in Sp(1, \boldsymbol{H}^C), \varphi(p, q) \in {G_2}^C$ (Lemma 3.5 (1)) and

$$\varphi(p,q)(a + be_4) = \lambda\varphi(p,q)\Big(\frac{a}{\lambda} + i\frac{b}{i\lambda}e_4\Big) = \lambda\Big(q\frac{a}{\lambda}\overline{q} + \Big(ip\frac{b}{i\lambda}\overline{q}\Big)e_4\Big) = \lambda(e_1 + ie_4).$$

On the other hand, for any $\nu \in C$, $\nu \neq 0$, let

$$A(\nu) = \frac{1}{2\nu}\mathrm{diag}(\nu^2 + 1 - (\nu^2 - 1)ie_4, 2\nu, \nu^2 + 1 + (\nu^2 - 1)ie_4) \in SU(3, \boldsymbol{C}^C).$$

Then we have $\varphi(A(\nu)) \in {G_2}^C$ (Lemma 3.4) and

$$\varphi(-e_1, 1)\varphi(A(\nu))\varphi(e_1, 1)(e_1 + ie_4) = \nu(e_1 + ie_4).$$

Indeed,

$$\begin{aligned} \varphi(-e_1,1)\varphi(A(\nu))\varphi(e_1,1)(e_1 + ie_4) &= \varphi(-e_1,1)\varphi(A(\nu))(e_1 + (e_1 i)e_4) \\ &= \varphi(-e_1,1)\varphi(A(\nu))(e_1 - ie_4 e_1) = \varphi(-e_1,1)\varphi(A(\nu))((1 - ie_4)e_1) \\ &= \varphi(-e_1,1)\varphi(A(\nu))\left(0 + \begin{pmatrix} 1 - ie_4 \\ 0 \\ 0 \end{pmatrix}\right) = \varphi(-e_1,1)\left(A(\nu)\begin{pmatrix} 1 - ie_4 \\ 0 \\ 0 \end{pmatrix}\right) \\ &= \varphi(-e_1,1)\left(\nu\begin{pmatrix} 1 - ie_4 \\ 0 \\ 0 \end{pmatrix}\right) = \varphi(-e_1,1)(\nu(1 - ie_4)e_1) \\ &= \nu\varphi(-e_1,1)(e_1 + (ie_1)e_4) = \nu(e_1 + ((-e_1)ie_1)e_4) = \nu(e_1 + ie_4). \end{aligned}$$

Therefore, the element $e_1 + ie_4$ is transformed to the form $\nu(e_1 + ie_4)$ for any multiple ν $(\neq 0)$ by some element of ${G_2}^C$. Thus the theorem is proved. □

5. Canonical form of an element of $\mathfrak{C}_0{}^C$ and explicit classification of orbits in $\mathfrak{C}_0{}^C$ over the maximal compact subgroup of $G_2{}^C$

A maximal compact subgroup $(G_2{}^C)_K$ of $G_2{}^C$:

$$(G_2{}^C)_K = \{\alpha \in G_2{}^C \mid \langle \alpha x, \alpha y\rangle = \langle x, y\rangle,\ x, y \in \mathfrak{C}^C\} = \{\alpha \in G_2{}^C \mid \tau\alpha\tau = \alpha\}$$

is isomorphic to the compact group G_2, where $\langle x, y\rangle := (x, \tau y)$ is a positive definite inner product in $\mathfrak{C}^C$. It is clear that this isomorphism is given by corresponding $\alpha \in G_2$ to $\alpha^C \in (G_2{}^C)_K (\subset G_2{}^C)$, where $\alpha^C \in (G_2{}^C)_K$ is defined by

$$\alpha^C(u + iv) = \alpha u + i\alpha v, \quad u + iv \in \mathfrak{C} \oplus i\mathfrak{C} = \mathfrak{C}^C.$$

THEOREM 5.1. *Any element $x = u + iv \in \mathfrak{C}_0 \oplus i\mathfrak{C}_0 = \mathfrak{C}_0{}^C$ can be transformed to the following canonical form by some element of $(G_2{}^C)_K$:*

(1) *In the case of $v \neq 0$:*

$$\xi e_1 + (\eta + i\zeta)e_4 \quad (\xi \geq 0,\ \zeta > 0,\ \eta \in \boldsymbol{R}).$$

(2) *In the case of $v = 0$:*

$$\xi e_1 \quad (\xi = \sqrt{N(u)} \geq 0).$$

Moreover, all their orbits in $\mathfrak{C}_0{}^C$ over $(G_2{}^C)_K$ are distinct, and the union of all their orbits is the whole space $\mathfrak{C}_0{}^C$.

Proof. (1) Let $x = u + iv \in \mathfrak{C}_0{}^C (v \neq 0)$. Since G_2 acts transitively on the subset of elements of $\mathfrak{C}_0$ with the same norm, there exists some element $\alpha \in G_2$ such that $\alpha v = \zeta e_4$ ($\zeta = \sqrt{N(v)} > 0$), that is,

$$\alpha^C x = u' + i\zeta e_4, \quad u' \in \mathfrak{C}_0,\ \zeta > 0$$

by Theorem 6.1 below. Let further $u' = \eta e_4 + \boldsymbol{m} \in \boldsymbol{C}_0 \oplus \boldsymbol{C}^3$. Since u' is already a canonical form if $\boldsymbol{m} = \boldsymbol{0}$, let $\boldsymbol{m} \neq \boldsymbol{0}$. Then, since $SU(3)$ acts transitively on $(S_{\boldsymbol{C}})^2 (:= \{\boldsymbol{n} \in \boldsymbol{C}^3 \mid (\boldsymbol{n}, \boldsymbol{n}) = 1\})$, there exists some element $A \in SU(3)$ such that $A\Big(\dfrac{1}{\xi}\boldsymbol{m}\Big) = {}^t(1, 0, 0)$, $\xi = \sqrt{(\boldsymbol{m}, \boldsymbol{m})} > 0$, that is,

$$\begin{aligned}\varphi(A)^C(\alpha^C x) &= \varphi(A)^C(u' + i\zeta e_4) = \varphi(A)u' + i\varphi(A)(\zeta e_4)\\ &= \varphi(A)\Big(\eta e_4 + \xi\Big(\frac{1}{\xi}\boldsymbol{m}\Big)\Big) + i\varphi(A)(\zeta e_4 + \boldsymbol{0}) = \eta e_4 + \xi A\Big(\frac{1}{\xi}\boldsymbol{m}\Big) + i\zeta e_4\\ &= \eta e_4 + \xi e_1 + i\zeta e_4 = \xi e_1 + (\eta + i\zeta)e_4\end{aligned}$$

by Lemma 3.4. The uniqueness is obvious from $(\alpha x, \alpha x) = (x, x)$, $\langle \alpha x, \alpha x \rangle = \langle x, x \rangle, \alpha \in ({G_2}^C)_K$, $x \in {\mathfrak{C}_0}^C$ and $\xi \geq 0$, $\zeta > 0$, $\eta \in \boldsymbol{R}$.

(2) In this case, it is exactly Theorem 6.1 below. □

6. Canonical form of an element of $\mathfrak{C}_0$ by G_2 and explicit classification of orbits in $\mathfrak{C}_0$ over G_2

THEOREM 6.1. *Any element of $x \in \mathfrak{C}_0$ can be transformed to the following canonical form by some element of G_2 :*

$$\xi e_1 \quad (\xi = \sqrt{N(x)} \geq 0).$$

Moreover, all their orbits in $\mathfrak{C}_0$ over G_2 are distinct, and the union of all their orbits is the whole space $\mathfrak{C}_0$.

Proof. Let $x = \sum_{k=1}^{7} x_k e_k \in \mathfrak{C}_0$. Since it is trivial when $x = 0$, we assume $x \neq 0$. Let $\xi = \sqrt{N(x)} > 0$ and put

$$z = \sum_{k=1}^{7} z_k e_k = \frac{1}{\xi} \sum_{k=1}^{7} x_k e_k \in \mathfrak{C}_0.$$

Then $N(z) = 1$. Now, applying $\gamma_1 \in G_2$ to z if necessary, we may assume $z_1 \geq 0$. We solve the equation $\alpha_a e_1 = z = \sum_{k=1}^{7} z_k e_k$ of Lemma 3.3 (2), then the solution $a = -\frac{1}{2} + \frac{\sqrt{3}}{2} \sum_{k=1}^{7} a_k e_k \in \mathfrak{C}, \sum_{k=1}^{7} {a_k}^2 = 1$ is given by the same formula as $(*)$ of the proof of Theorem 4.1 (1). Then we have $\alpha_a e_1 = z$, that is, $\alpha_{\overline{a}} z = e_1, \alpha_{\overline{a}} \in G_2$. Thus we obtain

$$\alpha_{\overline{a}} x = \xi \alpha_{\overline{a}} z = \xi e_1.$$

□

7. Canonical form of an element of ${\mathfrak{C}_0}'$ by $G_{2(2)}$ and explicit classification of orbits in ${\mathfrak{C}_0}'$ over $G_{2(2)}$

THEOREM 7.1. *Any non-zero element $x \in {\mathfrak{C}_0}'$ can be transformed to the following canonical form by some element of $G_{2(2)}$:*

(1) *In the case of* $N(x) > 0$:

$$\xi e_1 \qquad (\xi = \sqrt{N(x)} > 0).$$

(2) *In the case of* $N(x) < 0$:

$$\xi e_4{}' \qquad (\xi = \sqrt{-N(x)} > 0).$$

(3) *In the case of* $N(x) = 0$:

$$e_1 + e_4{}'.$$

Moreover, all their orbits in $\mathfrak{C}_0{}'$ *over* $G_{2(2)}$ *are distinct, and the union of all their orbits and* $\{0\}$ *is the whole space* $\mathfrak{C}_0{}'$.

Proof. Let $x = \sum_{k=1}^{7} x_k e_k{}' \in \mathfrak{C}'$. In case $N(x) \left(= \sum_{k=1}^{3} x_k{}^2 - \sum_{k=4}^{7} x_k{}^2\right) \neq 0$, let $\xi = \sqrt{|N(x)|} > 0$ and put

$$z = \sum_{k=1}^{7} z_k e_k{}' = \frac{1}{\xi} \sum_{k=1}^{7} x_k e_k{}' \in \mathfrak{C}'_0.$$

(1) In this case, $N(z) = 1$. Now, applying $\gamma_1 \in G_{2(2)}$ to z if necessary, we may assume $z_1 \geq 0$. We solve the equation $\alpha_a e_1{}' = z = \sum_{k=1}^{7} z_k e_k{}'$ of Lemma 3.3 (3), then the solution $a = -\frac{1}{2} + \frac{\sqrt{3}}{2} \sum_{k=1}^{7} a_k e_k{}' \in \mathfrak{C}', \sum_{k=1}^{3} a_k{}^2 - \sum_{k=4}^{7} a_k{}^2 = 1$ is given by the same formula as $(*)$ of the proof of Theorem 4.1 (1). Then we have $\alpha_a e_1{}' = z$, that is, $\alpha_{\overline{a}} z = e_1{}', \alpha_{\overline{a}} \in G_{2(2)}$. Thus we obtain

$$\alpha_{\overline{a}} x = \xi \alpha_{\overline{a}} z = \xi e_1.$$

(2) In this case, $N(z) = -1$. We may assume $z_4 \leq -\frac{1}{2}$. Indeed, since $\sum_{k=1}^{3} z_k{}^2 - \sum_{k=4}^{7} z_k{}^2 = -1$, that is, $\sum_{k=4}^{7} z_k{}^2 = 1 + \sum_{k=1}^{3} z_k{}^2 \geq 1$, at least one of z_k $(k = 4,5,6,7)$ satisfies $|z_k| \geq \frac{1}{2}$. Hence, applying $\gamma, \varphi(e_k, 1)$ or $\varphi(-e_k, 1) \in G_{2(2)}$ $(k = 1,2,3)$ to z if necessary, we may assume $z_4 \leq -\frac{1}{2}$. (Note that $\varphi(-e_1, 1)e_5{}' = e_4{}'$, $\varphi(-e_1, 1)e_k{}' \neq \pm e_4{}' \, (k \neq 5)$; $\varphi(e_2, 1)e_6{}' = e_4{}'$, $\varphi(e_2, 1)e_k{}' \neq \pm e_4{}' \, (k \neq 6)$; $\varphi(-e_3, 1)e_7{}' = e_4{}', \varphi(-e_3, 1)e_k{}' \neq \pm e_4{}' \, (k \neq 7)$.) Here we consider

the following two cases.

1° Case $-1 \neq z_4 \leq -\dfrac{1}{2}$. We solve the equation $\alpha_a e_4{}' = z = \sum_{k=1}^{7} z_k e_k{}'$ of Lemma 3.3 (3), then the solution $a = -\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}\sum_{k=1}^{7} a_k e_k{}' \in \mathfrak{C}', \sum_{k=1}^{3} a_k{}^2 - \sum_{k=4}^{7} a_k{}^2 =$ 1 is given by

$$\begin{cases} a_1 = \dfrac{z_1\sqrt{-(2z_4+1)} - z_5}{\sqrt{3}(z_4+1)}, & a_2 = \dfrac{z_2\sqrt{-(2z_4+1)} + z_6}{\sqrt{3}(z_4+1)}, \\ a_3 = \dfrac{z_3\sqrt{-(2z_4+1)} - z_7}{\sqrt{3}(z_4+1)}, & a_4 = \sqrt{-\dfrac{2z_4+1}{3}}, \\ a_5 = \dfrac{z_5\sqrt{-(2z_4+1)} - z_1}{\sqrt{3}(z_4+1)}, & a_6 = \dfrac{z_6\sqrt{-(2z_4+1)} + z_2}{\sqrt{3}(z_4+1)}, \\ a_7 = \dfrac{z_7\sqrt{-(2z_4+1)} - z_3}{\sqrt{3}(z_4+1)}. \end{cases}$$

Then $\alpha_a e_4{}' = z$, that is, $\alpha_{\overline{a}} z = e_4{}', \alpha_{\overline{a}} \in G_{2(2)}$. Thus we obtain

$$\alpha_{\overline{a}} x = \xi \alpha_{\overline{a}} z = \xi e_4{}'.$$

2° Case $z_4 = -1$. If $z_5 = z_6 = z_7 = 0$, then since $\sum_{k=1}^{3} z_k{}^2 = 0$ from $N(z) = -1$, we have $z_1 = z_2 = z_3 = 0$, so $z = -e_4{}'$ hence $\gamma z = e_4{}'$. Next, assume $z_k \neq 0$ $(k = 5, 6, 7)$, for instance, assume $z_5 \neq 0$. Now, for any $\theta \in \boldsymbol{R}$, put $p = \cos\theta + e_1 \sin\theta$ $(= e^{e_1\theta})$. Then we have $p \in Sp(1)$ and

$$\begin{aligned} z_4{}' &:= (\text{the coefficient of } e_4{}' \text{ in } \varphi(p,1)z) \\ &= (\text{the coefficient of } e_4{}' \text{ in } \varphi(p,1)(a + be_4{}')) \quad \text{where } z = a + be_4{}' \\ &= (\text{the coefficient of } e_0 \text{ in } pb1) \\ &= (\text{the coefficient of } e_0 \text{ in } e^{e_1\theta}(-1 + z_5e_1 - z_6e_2 + z_7e_3)) \\ &= (\text{the coefficient of } e_0 \text{ in } e^{e_1\theta}(-1 + z_5e_1)) \\ &= (\text{the coefficient of } e_0 \text{ in } e^{e_1\theta}\sqrt{1+z_5{}^2}e^{e_1\alpha})) \quad \alpha \in \boldsymbol{R} \\ &= (\text{the coefficient of } e_0 \text{ in } \sqrt{1+z_5{}^2}e^{e_1(\theta+\alpha)}) \\ &= \sqrt{1+z_5{}^2}\cos(\theta+\alpha). \end{aligned}$$

Hence, if we choose a suitable $\theta \in \boldsymbol{R}$, then $z_4{}'$ can attain any value in the range $-\sqrt{1+z_5{}^2} \leq z_4{}' \leq \sqrt{1+z_5{}^2}$, so we may assume $-1 \neq z_4 \leq -\dfrac{1}{2}$. Therefore, this is reduced to the case 1°. The case $z_6 \neq 0$ or $z_7 \neq 0$ is also shown in exactly the same way.

(3) In this case, if we put $x = a + be_4{}' \in \boldsymbol{H}_0 \oplus \boldsymbol{H}e_4{}' = \mathfrak{C}_0{}'$ then it holds that $a\bar{a} = b\bar{b} > 0$ from $N(x) = 0$. Let $\lambda = \sqrt{N(a)}$. Then, applying $\gamma_1 \in G_{2(2)}$ to x if necessary, we may assume $\dfrac{a}{\lambda} \neq -e_1$, that is, $\dfrac{a}{\lambda} + e_1 \neq 0$. Put $\mu = \sqrt{N\Big(\dfrac{a}{\lambda} + e_1\Big)}$ and $q = \dfrac{1}{\mu}\Big(\dfrac{a}{\lambda} + e_1\Big)$. Then we have $q \in Sp(1)$ and $q\dfrac{a}{\lambda}\bar{q} = e_1$ by direct calculation similar to the proof of Theorem 4.1 (2). Further, put $p = \dfrac{q\bar{b}}{\lambda}$. Then we have $p \in Sp(1)$, $\varphi(p, q) \in G_{2(2)}$ (Lemma 3.5 (2)) and

$$\varphi(p,q)(a + be_4{}') = \lambda\varphi(p,q)\Big(\frac{a}{\lambda} + \frac{b}{\lambda}e_4{}'\Big) = \lambda\Big(q\frac{a}{\lambda}\bar{q} + \Big(p\frac{b}{\lambda}\bar{q}\Big)e_4{}'\Big) = \lambda(e_1 + e_4{}').$$

On the other hand, for any $\nu \in \boldsymbol{R}, \nu \neq 0$, let

$$A(\nu) = \frac{1}{2\nu}\mathrm{diag}(\nu^2 + 1 - (\nu^2 - 1)e_4{}', 2\nu, \nu^2 + 1 + (\nu^2 - 1)e_4{}') \in SU(3, \boldsymbol{C}').$$

Then we have $\varphi(A(\nu)) \in G_{2(2)}$ (Lemma 3.4) and

$$\varphi(-e_1, 1)\varphi(A(\nu))\varphi(e_1, 1)(e_1 + e_4{}') = \nu(e_1 + e_4{}')$$

by direct calculation similar to the proof of Theorem 4.1 (2). Therefore, the element $e_1 + e_4{}'$ is transformed to the form $\nu(e_1 + e_4{}')$ for any multiple ν ($\neq 0$) by some element of $G_{2(2)}$. Thus the theorem is proved. □

8. Canonical form of an element of $\mathfrak{C}_0{}'$ and explicit classification of orbits in $\mathfrak{C}_0{}'$ over the maximal compact subgroup of $G_{2(2)}$

A maximal compact subgroup $(G_{2(2)})_K$ of $G_{2(2)}$:

$$(G_{2(2)})_K = \{\alpha \in G_{2(2)} \,|\, (\alpha x, \alpha y)_\gamma = (x, y)_\gamma,\ x, y \in \mathfrak{C}'\} = \{\alpha \in G_{2(2)} \,|\, \gamma\alpha\gamma = \alpha\}$$

is isomorphic to the compact group $(Sp(1) \times Sp(1))/\boldsymbol{Z}_2$, where $(x, y)_\gamma := (x, \gamma y)$ is a positive definite inner product in $\mathfrak{C}'$. This isomorphism is induced by the homomorphism $\varphi : Sp(1) \times Sp(1) \to G_{2(2)}$ in Lemma 3.5 (2). (For detail, see [10], [12].)

THEOREM 8.1. *Any element of $\mathfrak{C}_0{}'$ can be transformed to the following canonical form by some element of $(G_{2(2)})_K$:*

$$\xi e_1 + \eta e_4{}' \quad (\xi \geq 0,\ \eta \geq 0).$$

Moreover, all their orbits in $\mathfrak{C}_0{}'$ *over* $(G_{2(2)})_K$ *are distinct, and the union of all their orbits is the whole space* $\mathfrak{C}_0{}'$.

Proof. Let $x = a + be_4{}' \in \boldsymbol{H}_0 \oplus \boldsymbol{H}e_4{}' = \mathfrak{C}_0{}'$. Let further $a \neq 0$ and $a' = \dfrac{1}{\xi}a$, $\xi = \sqrt{N(a)} > 0$.

1° Case $a' = -e_1$. Put $q = e_2 \in Sp(1)$. Then we have

$$qa\overline{q} = \xi(qa'\overline{q}) = \xi(e_2(-e_1)\overline{e}_2) = \xi(e_1e_2\overline{e}_2) = \xi e_1.$$

2° Case $a' \neq -e_1$. Put $q = \dfrac{1}{\eta}(a' + e_1) \in Sp(1) \cap \boldsymbol{H}_0, \eta = \sqrt{N(a' + e_1)} > 0$. Then we have

$$\begin{aligned} qa\overline{q} &= \xi(qa'\overline{q}) = \frac{\xi}{\eta^2}(a' + e_1)a'\overline{(a' + e_1)} = \frac{\xi}{\eta^2}(a'^2 + e_1a')\overline{(a' + e_1)} \\ &= \frac{\xi}{\eta^2}(-1 + e_1a')\overline{(a' + e_1)} = \frac{\xi}{\eta^2}(e_1{}^2 + e_1a')\overline{(a' + e_1)} \\ &= \frac{\xi}{\eta^2}e_1(e_1 + a')\overline{(a' + e_1)} = \xi e_1. \end{aligned}$$

Therefore, including the case of $a = 0$, we can obtain that for a given $x = a + be_4{}' \in \mathfrak{C}_0{}'$, there exists some $q \in Sp(1)$ such that

$$\varphi(1, q)(a + be_4{}') = \xi e_1 + b'e_4{}', \quad \xi \geq 0, \, b' \in \boldsymbol{H}.$$

Let next $b' \neq 0$, since it is trivial when $b' = 0$. Put $p = \dfrac{1}{\zeta}\overline{b'} \in Sp(1)$, $\zeta = \sqrt{N(b')} > 0$. Then we have

$$\varphi(p, 1)(\xi e_1 + b'e_4{}') = \xi e_1 + \zeta e_4{}'.$$

Consequently we obtain that for a given $x \in \mathfrak{C}_0{}'$, there exists some $\alpha \in (G_{2(2)})_K$ such that

$$\alpha x = \xi e_1 + \eta e_4{}', \qquad \xi \geq 0, \, \eta \geq 0.$$

The uniqueness is obvious from $(\alpha x, \alpha x) = (x, x)$, $(\alpha x, \alpha x)_\gamma = (x, x)_\gamma$, $\alpha \in (G_{2(2)})_K, x \in \mathfrak{C}_0{}^C$ and $\xi \geq 0$, $\eta \geq 0$. □

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