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PIVOTAL INVERSIONS OF A FINITE POINT-SET

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Abstract. For two distinct points P, Q in the plane, let Q^P denote the point on the ray $\overrightarrow{PQ}$ such that $PQ \cdot PQ^P = 1$, and let $P^P = P$. For a point-set τ in the plane and $P \in \tau$, define $\tau^P = \{Q^P \mid Q \in \tau\}$. The transformation $\tau \to \tau^P$ is called the pivotal inversion at $P \in \tau$. We show that if $n \geq 4$ then starting from any n-point-set, it is possible, by applying a sequence of pivotal inversions, to produce an n-point-set whose diameter exceeds any prescribed value, but it is impossible to produce more than $n+1$ mutually non-similar n-point-sets. The latter part is proved by showing a group induced by pivotal inversions of ordered n-point-sets is isomorphic to the symmetric group of degree $n+1$.

1. Introduction

For two points P, Q in the plane, denote by Q^P the inversion of Q with respect to the unit circle centered at P. Thus, Q^P is the point on the ray $\overrightarrow{PQ}$ satisfying $PQ \cdot PQ^P = 1$, where PQ denotes the length of the line segment connecting P and Q. For $Q = P$, usually P^P is either not defined, or defined to be the point ∞ at 'infinity'. For inversions, see Coxeter [1].

Suppose $n \geq 3$, and let σ be a set of n points in the plane in general position in the sense that no three points are collinear and no four points are concyclic. Then σ determines a set of $\binom{n}{2}$ lines and $\binom{n}{3}$ circles, which is called the line-circle-system on σ, and σ itself is called the pivot set of the line-circle-system. Figure 1 shows a line-circle-system on $\{P, Q, R\}$. Since an inversion of the plane transforms a circle or a line into a circle or line, an inversion with respect to a unit circle centered at a point of σ, say P, transforms the line-circle-system on σ into another line-circle-system, whose pivot set is denoted by σ^P. Then σ^P is also an n-point-set. The transformation $\sigma \to \sigma^P$ is called the pivotal transformation of σ at P. For example, the line-circle-system on $\{P, Q, R\}$ is transformed by the inversion with center P into the line-circle-system on $\{P, Q^P, R^P\}$. Thus, $\{P, Q, R\}^P = \{P, Q^P, R^P\}$.

Formally, for every finite point-set τ in the plane, and a point $P \in \tau$, we

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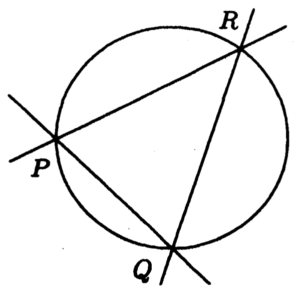

Figure 1 A line-circle-system on $\{P, Q, R\}$

define a new point-set τ^P by

$$\tau^P = \{Q^P \mid Q \in \tau\} \text{ with } P^P = P,$$

and call this transformation $\tau \to \tau^P$ the *pivotal inversion* of τ at $P \in \tau$. The point P is called the center of the pivotal inversion. Note that the center of every pivotal inversion of a point-set τ is supposed to be a point of τ.

Now, from an n-point-set τ in the plane, by a pivotal inversion of τ, a new n-point-set is produced. Next, by a pivotal inversion of this new n-point-set, another n-point-set is produced. Proceeding in this way, many n-point-sets will be produced. These newly produced point-sets are called relatives of τ. More precisely, an n-point-set σ is a *relative* of τ if σ is produced from τ by applying a sequence of pivotal inversions.

Then, how many, mutually *non-congruent* relatives can be produced from an n-point-set? Here, two point set τ and σ are *congruent* if there is a bijection form τ to σ that preserves the distances. Two point-sets τ, σ are *similar* if there is a *similarity* $f : \tau \to \sigma$, that is, a bijection f such that for every pair $P, Q \in \tau$ the distance between $f(P)$ and $f(Q)$ is equal to λ times the distance between P and Q for a common constant $\lambda > 0$. The *diameter* of an n-point-set is the longest distance between the n points in the set.

Since $(\tau^P)^P = \tau$ holds for every τ and for every $P \in \tau$, the following result may be slightly curious.

THEOREM 1. *Let ΔPQR be a non-equilateral (possibly degenerate) triangle. Then, for any prescribed value d, the 3-point-set $\{P, Q, R\}$ has a relative that is similar to $\{P, Q, R\}$ and whose diameter is greater than d.*

Thus $\{P, Q, R\}$ has infinitely many mutually non-congruent relatives unless ΔPQR is an equilateral triangle. Since every 4-point-set in the plane contains three points that do not span an equilateral triangle, we have the following.

Corollary 1. *For $n \geq 4$, every n-point-set has a relative with diameter greater than any prescribed value.* □

Now, how many mutually *non-similar* relatives can be produced from an n-point-set? The case $n = 3$ is easy: Since

$$\Delta PQR \sim \Delta P^P R^P Q^P \tag{1}$$

holds (Lemma 1), where $\sim$ implies 'be similar to', all relatives of a 3-point-set are mutually similar. Generally we have the following.

Theorem 2. *No n-point-set has more than $n+1$ mutually non-similar relatives. If $n \geq 4$, then there is an n-point-set that has exactly $n+1$ mutually non-similar relatives.*

An *ordered* point-set is a point-set whose points are ordered. For every ordered point-set, we denote by φ_i the pivotal inversion with center at the ith point. Then, for every ordered n-point-set τ and for every i, $1 \leq i \leq n$, $\varphi_i(\tau)$ can be defined, and it becomes an ordered n-point-set by the ordering induced from τ via φ_i. Thus, we can regard φ_i $(1 \leq i \leq n)$ as a transformation of the set of all ordered n-point-sets.

Two ordered n-point-sets τ and σ are called *order-preservingly similar* (written $\tau \sim \sigma$) if the bijection $f : \tau \to \sigma$ that preserves the order of the points is a similarity. The set of all ordered n-point-sets can be partitioned into equivalence classes by the relation $\sim$. Let Σ_n be a family of complete representatives of these equivalence classes, that is, Σ_n is a set of ordered n-point-sets composed by taking one ordered n-point-set from each equivalence class. It follows easily from the fact (1) that $\tau \sim \sigma$ implies $\varphi_i(\tau) \sim \varphi_i(\sigma)$ for every $1 \leq i \leq n$. Therefore, each φ_i induces naturally a transformation $\phi_i : \Sigma_n \to \Sigma_n$. Let G_n denote the transformation group of Σ_n generated by $\phi_1, \phi_2, \phi_3, \ldots, \phi_n$. We prove the following.

Theorem 3. *If $n > 3$, then the transformation group G_n of Σ_n is isomorphic to the symmetric group S_{n+1}.*

Remark 1. Clearly G_2 is the identity group, and it is also easy to see that G_3 is isomorphic to the symmetric group S_3 (see Example in Section 3).

Remark 2. Pivotal inversions for a finite point-set in higher dimensional space are defined similarly, and Theorems 1, 2, 3 also hold.

2. The cone of Euclidean metrics

Let $n \geq 3$. An *ordered* n-point-set is determined, up to order-preserving congruence, by the $\binom{n}{2}$ distances x_{ij} $(1 \leq i < j \leq n)$ between the ith point and the jth point in the set, and hence represented by a point $(x_{12}, x_{13}, \dots)$ of $R^{\binom{n}{2}}$ with coordinates x_{ij} in lexicographic order of the suffixes. The set of such points in $R^{\binom{n}{2}}$ forms a cone Γ_n, called the *cone of Euclidean metrics* on n points in the plane. For general metric cone, see Deza and Laurent [2]. Thus, each congruence class of ordered n-point-sets is represented by a point of Γ_n. For each $1 \leq i \leq n$ and two ordered n-point-sets τ, σ, if τ is order-preservingly congruent to τ, then $\varphi_i(\tau)$ is order-preservingly congruent to $\varphi_i(\sigma)$. Hence φ_i induces a transformation $f_i : \Gamma_n \to \Gamma_n$.

We need the following lemma.

LEMMA 1. *Let ΔPQR be a triangle with $PQ = x, PR = y, QR = z$. Then $\Delta PQR \sim \Delta P^P R^P Q^P$ and $P^P Q^P = \frac{1}{x}, P^P R^P = \frac{1}{y}, Q^P R^P = \frac{z}{xy}$.*

Proof. $PQ \cdot P^P Q^P = PR \cdot P^P R^P = 1$ implies $PQ : PR = P^P R^P : P^P Q^P$. Hence $\Delta PQR \sim \Delta P^P R^P Q^P$. □

Let $f_i : \mathit{\Gamma}_3 \to \mathit{\Gamma}_3$ $(i = 1, 2, 3)$ be the transformation induced by φ_i $(i = 1, 2, 3)$. Then by Lemma 1, f_1, f_2, f_3 transform Γ_3 in the following way:

$$f_1(x, y, z) = \left(\frac{1}{x}, \frac{1}{y}, \frac{z}{xy}\right)$$
$$f_2(x, y, z) = \left(\frac{1}{x}, \frac{y}{xz}, \frac{1}{z}\right)$$
$$f_3(x, y, z) = \left(\frac{x}{yz}, \frac{1}{y}, \frac{1}{z}\right)$$

where $(x, y, z) := (x_{12}, x_{13}, x_{23})$.

Proof of Theorem 1. Let $PQ = x, PR = y, QR = z$ with $x > z$. Then the ordered 3-point-set $\{P, Q, R\}$ corresponds to $(x, y, z) \in \Gamma_3$. By the formulae for f_1, f_2, f_3, we have

$$f_1(x, y, z) = \left(\frac{1}{x}, \frac{1}{y}, \frac{z}{xy}\right)$$
$$f_2\left(\frac{1}{x}, \frac{1}{y}, \frac{z}{xy}\right) = \left(x, \frac{x^2}{z}, \frac{xy}{z}\right)$$
$$f_3\left(x, \frac{x^2}{z}, \frac{xy}{z}\right) = \left(\frac{z^2}{x^2 y}, \frac{z}{x^2}, \frac{z}{xy}\right) \quad .$$

Hence, if $g = f_3 f_2 f_1$, the composition of f_1, f_2, f_3, then

$$g(x,y,z) = \left(\frac{z^2}{x^2 y}, \frac{z}{x^2}, \frac{z}{xy}\right),$$

and

$$g^2(x,y,z) = \left(\frac{x^4}{z^3}, \frac{x^3 y}{z^3}, \frac{x^3}{z^2}\right) = \left(\frac{x}{z}\right)^3 (x,y,z).$$

Hence, $g^2(\lambda x, \lambda y, \lambda z) = (\frac{x}{z})^3(\lambda x, \lambda y, \lambda z) = \lambda g^2(x,y,z)$. Therefore

$$g^{2k}(x,y,z) = \left(\frac{x}{z}\right)^{3k} (x,y,z), \quad k = 1,2,3,\ldots.$$

Since $x/z > 1$, $(x/z)^{3k} \to \infty$ as $k \to \infty$. Thus $(\varphi_3\varphi_2\varphi_1)^{2k}(\{P,Q,R\})$ is similar to $\{P,Q,R\}$ and its diameter tends to infinity as $k \to \infty$. □

Remark 3. If $n \geq 4$ then for every ordered n-point-set τ with $x_{12} > x_{23}$, each $(\phi_1\phi_2\phi_3)^{4k}(\tau)$ becomes similar to τ and its diameter tends to ∞ as $k \to \infty$.

3. Homogeneous coordinates

Two ordered n-point-sets represented by $\mathbf{x}, \mathbf{y} \in \Gamma_n$ are order-preservingly similar, if and only if there is a $\lambda > 0$ such that $\mathbf{x} = \lambda \mathbf{y}$. Therefore, *every member of Σ_n can be represented by a set of homogeneous coordinates* $[x_{12}, x_{13}, \ldots]$ in lexicographic order of the suffixes. The word *homogeneous* implies that $[\lambda x_{12}, \lambda x_{13}, \ldots] = [x_{12}, x_{13}, \ldots]$ for all $\lambda \neq 0$.

The transformation $f_i : \Gamma_n \to \Gamma_n$ naturally induces the transformation $\phi_i : \Sigma_n \to \Sigma_n$, and G_n is the group generated by the transformations $\phi_1, \phi_2, \ldots, \phi_n$ of Σ_n. By Lemma 1, ϕ_i transforms the coordinates of $[x_{12}, x_{13}, \ldots]$ in the following way: For $i < j < k$,

$$x_{ij} \to \frac{1}{x_{ij}} \quad \text{and} \quad x_{jk} \to \frac{x_{jk}}{x_{ij}x_{ik}}.$$

Example. Let $n = 3$, and put $[x_{12}, x_{13}, x_{23}] = [x,y,z]$. Then

$$\phi_1[x,y,z] = \left[\frac{1}{x}, \frac{1}{y}, \frac{z}{xy}\right] = [y,x,z]$$

$$\phi_2[x,y,z] = \left[\frac{1}{x}, \frac{y}{xz}, \frac{1}{z}\right] = [z,y,x]$$

$$\phi_3[x,y,z] = \left[\frac{x}{yz}, \frac{1}{y}, \frac{1}{z}\right] = [x,z,y].$$

Hence ϕ_i is a transposition of a pair of coordinates in $[x,y,z]$, and hence, ϕ_1, ϕ_2, ϕ_3 generate all permutations of three coordinates, and $G_3 \cong S_3$. □

Suppose $n \geq 4$, and put

$$[x_{12}, x_{13}, x_{14}, \ldots, x_{23}, x_{24}, \ldots, x_{34}, \ldots] = [u, v, w, \ldots, x, y, \ldots, z, \ldots],$$

see Figure 2. Then ϕ_1, ϕ_2 change the coordinates as follows:

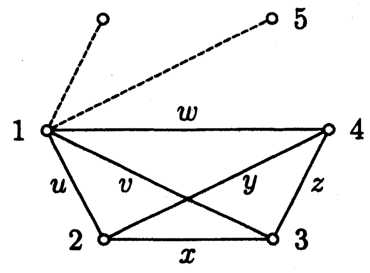

Figure 2 Distances u, v, w, x, y, z

$$\phi_1[u, v, w, \ldots, x, y, \ldots, z, \ldots] = \left[\frac{1}{u}, \frac{1}{v}, \frac{1}{w}, \ldots, \frac{x}{uv}, \frac{y}{uw}, \ldots, \frac{z}{vw}, \ldots\right] \quad (2)$$

$$\phi_2[u, v, w, \ldots, x, y, \ldots, z, \ldots] = \left[\frac{1}{u}, \frac{v}{ux}, \frac{w}{uy}, \ldots, \frac{1}{x}, \frac{1}{y}, \ldots, \frac{z}{xy}, \ldots\right] \quad (3)$$

4. Group G_n and its subgroup H

In this section, we assume $n \geq 4$.

Lemma 2. *For $1 \leq i \neq j \leq n$, the composite transformation $\phi_i\phi_j\phi_i$ of Σ_n works as the exchange of the order between the ith point and the jth point. In other words, $\phi_i\phi_j\phi_i$ is a permutation of homogeneous coordinates $[x_{12}, x_{13}, \ldots]$ induced by the transposition of the suffixes i and j. For example,*

$$\begin{aligned} &\phi_1\phi_2\phi_1[x_{12}, x_{13}, x_{14}, \ldots, x_{23}, x_{24}, \ldots, x_{34}, \ldots] \\ &= [x_{21}, x_{23}, x_{24}, \ldots, x_{13}, x_{14}, \ldots, x_{34}, \ldots] \end{aligned} \quad (4)$$

where $x_{21} \equiv x_{12}$. Therefore $\phi_i\phi_j\phi_i = \phi_j\phi_i\phi_j$.

Proof. It will be enough to show (4). By (2)(3), we have

$$\begin{aligned}
&\phi_1\phi_2\phi_1[u, v, w, \ldots, x, y, \ldots, z, \ldots] \\
&= \phi_1\phi_2\left[\frac{1}{u}, \frac{1}{v}, \frac{1}{w}, \ldots, \frac{x}{uv}, \frac{y}{uw}, \ldots, \frac{z}{vw} \cdots\right] \\
&= \phi_1\left[u, \frac{u^2}{x}, \frac{u^2}{y}, \ldots, \frac{uv}{x}, \frac{uw}{y}, \ldots, \frac{u^2 z}{xy}, \ldots\right] \\
&= \left[\frac{1}{u}, \frac{x}{u^2}, \frac{y}{u^2}, \ldots, \frac{v}{u^2}, \frac{w}{u^2}, \ldots, \frac{z}{u^2}, \ldots\right] \\
&= [u, x, y, \ldots, v, w, \ldots, z, \ldots].
\end{aligned}$$

Hence we have (4). □

COROLLARY 2. *The subgroup H of G_n generated by $\phi_i\phi_j\phi_i$, $1 \le i < j \le n$, is isomorphic to the symmetric group S_n of all permutations of $\{1, 2, 3, \ldots, n\}$.*

Proof. Since $\phi_i\phi_j\phi_i$ corresponds to the transposition of the ith point and the jth point in a sequence of n points, every element of H corresponds to a permutation of n points. Since the $\binom{n}{2}$ transpositions of n points generate all permutations of n points, the subgroup H generated by $\phi_i\phi_j\phi_i$ $(1 \le i < j \le n)$ is isomorphic to the symmetric group S_n. □

COROLLARY 3. *For any ordered n-point-set τ and for $1 \le i < j \le n$, the point-set $\phi_i\phi_j\phi_i(\tau)$ is similar to τ. Hence, for any element h of the subgroup H of G_n generated by $\phi_i\phi_j\phi_i$, $1 \le i < j \le n$, two n-point-sets τ and $h(\tau)$ are similar to each other.* □

LEMMA 3. *The index of H in G_n is $n+1$, that is, G_n is the disjoint union of the $n+1$ cosets $H, \phi_1 H, \phi_2 H, \phi_3 H, \ldots, \phi_n H$.*

Proof. Let τ be an ordered n-point-set *lying on a circle C*. Since every inversion with center on C transforms C into a line, the convex hull of $\phi_i(\tau)$ becomes a triangle for every $1 \le i \le n$, and hence $\phi_i(\tau)$ is not similar to τ. Hence $\phi_i \notin H$ and hence H and $\phi_i H$ are disjoint. Now, for $1 \le i < j \le n$, since $\phi_i\phi_j\phi_i \in H$ and $\phi_i^2 = 1$, we have

$$\phi_j\phi_i H = \phi_j\phi_i(\phi_i\phi_j\phi_i)H = \phi_i H. \tag{5}$$

Therefore, $\phi_j\phi_i \notin H$, and hence $\phi_i H, \phi_j H$ are disjoint. Thus, the $n+1$ cosets in Lemma 3 are mutually disjoint. Since G_n is generated by $\phi_1, \phi_2, \phi_3, \ldots, \phi_n$, it follows from (5) that every left cosets of H is equal to one of the $n+1$ left cosets in the lemma. This proves the lemma. □

Proof of Theorem 2. The former part of the theorem follows from Corollary 3 and Lemma 3. To see the latter part, consider an n-gon with n consecutive edge-lengths $1, 2, 3, \ldots, n$, and inscribed in a circle. (*Remark*: Every polygon can be deformed with keeping the edge-lengths into a polygon that is inscribed in a circle.) Let τ be the ordered set of n vertices of this n-gon. Then, for any $1 \le i \le n$, the convex hull of $\varphi_i(\tau)$ is a triangle, one of whose edges contains $n-3$ points in its interior. Let us call the other two edges, the *empty* edges of $\varphi_i(\tau)$. If $i \neq j$ then the ratio of the lengths of two empty edges of $\varphi_i(\tau)$ and that of $\varphi_j(\tau)$ are not equal. Hence $\varphi_i(\tau)$ and $\varphi_j(\tau)$ are not similar. Thus, no two of $\tau, \varphi_1(\tau), \varphi_2(\tau), \ldots, \varphi_n(\tau)$, are similar to each other. □

Proof of Theorem 3. First observe that by Corollary 2, H has $n!$ elements, and hence, by Lemma 3, G_n has $(n+1)!$ elements. Now, G_n naturally acts on $\{H, \phi_1 H, \phi_2 H, \ldots, \phi_n H\} =: \{\bar{0}, \bar{1}, \bar{2}, \ldots, \bar{n}\}$ from the left as a transformation group.

Now, for $j \neq i$, (5) implies $\phi_j(\bar{i}) = \bar{i}$, and since $\phi_j(\bar{0}) = \bar{j}$, $\phi_j(\bar{j}) = \bar{0}$, the action of ϕ_j corresponds to the transposition $(\bar{0}, \bar{j})$. Since the transpositions $(\bar{0}, \bar{j})$, $j = 1, 2, \ldots, n$, generate all permutations of $\{\bar{0}, \bar{1}, \bar{2}, \ldots, \bar{n}\}$, and since G_n is generated by $\phi_1, \phi_2, \ldots, \phi_n$, it follows that G_n is isomorphic to the symmetric group S_{n+1}. □

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