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MATRIX ELEMENTS OF GENERALIZED COHERENT OPERATORS

By

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Abstract. Explicit forms are given of matrix elements of generalized coherent operators based on Lie algebras su(1,1) and su(2). We also give a kind of factorization formula of the associated Laguerre polynomials.

1. Introduction

Coherent states or generalized coherent states play an important role in quantum physics, in particular, quantum optics, see [1] and references therein, or the books [2], [3]. They also play an important one in mathematical physics. See the textbook [4]. For example, they are very useful in performing stationary phase approximations to path integral, [5], [6], [7].

Coherent operators which produce coherent states are very useful because they are unitary and easy to handle. Why are they so handy ? The basic reason is probably that they are subject to the elementary Baker-Campbell-Hausdorff formula. Many basic properties of them have been investigated, see [2], [3], [4] or [8], [9]. We are particularly interested in the following three ones: the matrix elements, the trace formula and the Glauber's formula.

Generalized coherent operators which produce generalized coherent states are also useful. But the corresponding properties have not been investigated as far as the author knows, see for example [10]. One of the reasons is that they are not easy to handle. Of course we have disentangling formulas corresponding to the elementary Baker-Campbell-Hausdorff formula, but they are still not handy.

In this paper we focus our attention only on matrix elements of generalized coherent operators based on Lie algebras $su(1,1)$ and $su(2)$, and give explicit forms to all of them.

We make a short comment on some applications. Matrix elements of coherent operator can be applied to Rabi oscillations in Quantum Optics, see [11] and [12]. One of our aims is also to apply the calculations in this paper to the same subject.

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This will be published in [13].

2. Coherent and Generalized Coherent Operators

We review the general theory of both the coherent operator and generalized coherent ones based on Lie algebras $su(1,1)$ and $su(2)$.

2.1 Coherent Operator

Let $a(a^\dagger)$ be the annihilation (creation) operator of the harmonic oscillator. If we set $N \equiv a^\dagger a$ (: number operator), then

$$[N, a^\dagger] = a^\dagger\ ,\ [N, a] = -a\ ,\ [a^\dagger, a] = -\mathbf{1}\ . \tag{1}$$

Let $\mathcal{H}$ be a Fock space generated by a and $a^\dagger$, and $\{|n\rangle|\ n \in \mathbf{N} \cup \{0\}\}$ be its basis. The actions of a and $a^\dagger$ on $\mathcal{H}$ are given by

$$a|n\rangle = \sqrt{n}|n-1\rangle\ ,\ a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle\ , N|n\rangle = n|n\rangle \tag{2}$$

where $|0\rangle$ is the normalized vacuum ($a|0\rangle = 0$ and $\langle 0|0\rangle = 1$). From (2) the state $|n\rangle$ for $n \geq 1$ is given by

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle\ . \tag{3}$$

These states satisfy the orthonormality and completeness conditions

$$\langle m|n\rangle = \delta_{mn}\ ,\quad \sum_{n=0}^{\infty} |n\rangle\langle n| = \mathbf{1}\ . \tag{4}$$

Definition. We call a state

$$|z\rangle = \mathrm{e}^{za^\dagger - \bar{z}a}|0\rangle \equiv U(z)|0\rangle \quad \text{for} \quad z\ \in\ \mathbf{C} \tag{5}$$

the coherent state.

Since the operator $U(z) = \mathrm{e}^{za^\dagger - \bar{z}a}$ is unitary, we call this a (unitary) coherent operator. For these operators the following property is crucial:

$$U(z+w) = \mathrm{e}^{-\frac{1}{2}(z\bar{w} - \bar{z}w)}\ U(z)U(w) \quad \text{for} \quad z,\ w \in \mathbf{C}. \tag{6}$$

From this we have a well–known commutation relation $U(z)U(w) = \mathrm{e}^{z\bar{w} - \bar{z}w}$ $U(w)U(z)$. Here let us recall the disentangling formula of $U(z)$ for later convenience:

$$\mathrm{e}^{za^\dagger - \bar{z}a} = \mathrm{e}^{-\frac{1}{2}|z|^2}\mathrm{e}^{za^\dagger}\mathrm{e}^{-\bar{z}a} = \mathrm{e}^{\frac{1}{2}|z|^2}\mathrm{e}^{-\bar{z}a}\mathrm{e}^{za^\dagger}. \tag{7}$$

This is obtained by the famous (elementary) Baker–Campbell–Hausdorff formula:

$$e^{A+B} = e^{-\frac{1}{2}[A,B]}e^{A}e^{B} \tag{8}$$

whenever $[A,[A,B]] = [B,[A,B]] = 0$.

2.2 Generalized Coherent Operator Based on $su(1,1)$

Let us introduce generalized coherent operators and the states based on $su(1,1)$. Let $\{k_+, k_-, k_3\}$ be a Weyl basis of Lie algebra $su(1,1) \subset sl(2,\mathbf{C})$,

$$k_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad k_- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad k_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{9}$$

Then we have

$$[k_3, k_+] = k_+, \quad [k_3, k_-] = -k_-, \quad [k_+, k_-] = -2k_3. \tag{10}$$

We note that $(k_+)^\dagger = -k_-$.

Next we consider a spin K (> 0) representation of $su(1,1) \subset sl(2,\mathbf{C})$ and set its generators $\{K_+, K_-, K_3\}$ $((K_+)^\dagger = K_-)$,

$$[K_3, K_+] = K_+, \quad [K_3, K_-] = -K_-, \quad [K_+, K_-] = -2K_3. \tag{11}$$

We note that this (unitary) representation is necessarily infinite dimensional. The Fock space on which $\{K_+, K_-, K_3\}$ act is $\mathcal{H}_K \equiv \{|K,n\rangle | n \in \mathbf{N} \cup \{0\}\}$ and whose actions are

$$\begin{aligned} K_+|K,n\rangle &= \sqrt{(n+1)(2K+n)}|K,n+1\rangle, \\ K_-|K,n\rangle &= \sqrt{n(2K+n-1)}|K,n-1\rangle, \\ K_3|K,n\rangle &= (K+n)|K,n\rangle, \end{aligned} \tag{12}$$

where $|K,0\rangle$ is a normalized vacuum ($K_-|K,0\rangle = 0$ and $\langle K,0|K,0\rangle = 1$). We have written $|K,0\rangle$ instead of $|0\rangle$ to emphasize the spin K representation, see [5]. From (12), states $|K,n\rangle$ are given by

$$|K,n\rangle = \frac{(K_+)^n}{\sqrt{n!(2K)_n}}|K,0\rangle, \tag{13}$$

where $(a)_n$ is the Pochhammer's notation $(a)_n \equiv a(a+1)\cdots(a+n-1)$. These states satisfy the orthonormality and completeness conditions

$$\langle K,m|K,n\rangle = \delta_{mn}, \quad \sum_{n=0}^{\infty} |K,n\rangle\langle K,n| = \mathbf{1}_K. \tag{14}$$

Now let us consider a generalized version of coherent states:

DEFINITION. We call a state

$$|z\rangle = \mathrm{e}^{zK_+ - \bar{z}K_-}|K,0\rangle \equiv V(z)|K,0\rangle \quad \text{for} \quad z \in \mathbf{C}. \tag{15}$$

the generalized coherent state (or the coherent state of Perelomov's type based on $su(1,1)$ in our terminology).

Here is the disentangling formula:

$$\begin{aligned}\mathrm{e}^{zK_+ - \bar{z}K_-} &= \mathrm{e}^{\zeta K_+}\mathrm{e}^{\log(1-|\zeta|^2)K_3}\mathrm{e}^{-\bar{\zeta}K_-} \\ &= \mathrm{e}^{-\bar{\zeta}K_-}\mathrm{e}^{-\log(1-|\zeta|^2)K_3}\mathrm{e}^{\zeta K_+}, \quad \zeta \equiv \frac{\tanh(|z|)}{|z|}z.\end{aligned} \tag{16}$$

This is also the key formula for generalized coherent operators. See [4] or [14].

Here let us construct an important example of this representation. If we set

$$K_+ \equiv \frac{1}{2}\left(a^\dagger\right)^2 \ , \ K_- \equiv \frac{1}{2}a^2 \ , \ K_3 \equiv \frac{1}{2}\left(a^\dagger a + \frac{1}{2}\right) \ , \tag{17}$$

then it is easy to check (11). That is, the set $\{K_+, K_-, K_3\}$ gives unitary representations of $su(1,1)$with spin $K = 1/4$ and $3/4$, [4]. Now we in particular call an operator

$$S(z) = \mathrm{e}^{\frac{1}{2}\{z(a^\dagger)^2 - \bar{z}a^2\}} \quad \text{for} \quad z \in \mathbf{C} \tag{18}$$

the squeezed operator. This operator plays a very important role.

2.3 Generalized Coherent Operator Based on $su(2)$

Let us introduce generalized coherent operators and the states based on $su(2)$. Let $\{j_+, j_-, j_3\}$ be a Weyl basis of Lie algebra $su(2) \subset sl(2,\mathbf{C})$,

$$j_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad j_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad j_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{19}$$

Then we have

$$[j_3, j_+] = j_+, \quad [j_3, j_-] = -j_-, \quad [j_+, j_-] = 2j_3. \tag{20}$$

We note that $(j_+)^\dagger = j_-$.

Next we consider a spin J (> 0) representation of $su(2) \subset sl(2,\mathbf{C})$ and set its generators $\{J_+, J_-, J_3\}$ $((J_+)^\dagger = J_-)$,

$$[J_3, J_+] = J_+, \quad [J_3, J_-] = -J_-, \quad [J_+, J_-] = 2J_3. \tag{21}$$

We note that this (unitary) representation is necessarily finite dimensional. The Fock space on which $\{J_+, J_-, J_3\}$ act is $\mathcal{H}_J \equiv \{|J,n\rangle | 0 \le n \le 2J\}$ and whose actions are

$$J_+|J,n\rangle = \sqrt{(n+1)(2J-n)}|J,n+1\rangle, \ J_-|J,n\rangle = \sqrt{n(2J-n+1)}|J,n-1\rangle, \\ J_3|J,n\rangle = (-J+n)|J,n\rangle, \tag{22}$$

where $|J,0\rangle$ is a normalized vacuum ($J_-|J,0\rangle = 0$ and $\langle J,0|J,0\rangle = 1$). We have written $|J,0\rangle$ instead of $|0\rangle$ to emphasize the spin J representation, see [5]. From (22), states $|J,n\rangle$ are given by

$$|J,n\rangle = \frac{(J_+)^n}{\sqrt{n!_{2J}P_n}}|J,0\rangle \tag{23}$$

where ${}_{2J}P_n \equiv (2J)(2J-1)\cdots(2J-n+1)$.
These states satisfy the orthonormality and completeness conditions

$$\langle J,m|J,n\rangle = \delta_{mn}, \quad \sum_{n=0}^{2J} |J,n\rangle\langle J,n| \ = \mathbf{1}_J. \tag{24}$$

Now let us consider a generalized version of coherent states:

Definition. We call a state

$$|z\rangle = e^{zJ_+ - \bar{z}J_-}|J,0\rangle \equiv W(z)|J,0\rangle \quad \text{for} \quad z \in \mathbf{C}. \tag{25}$$

the generalized coherent state (or the coherent state of Perelomov's type based on $su(2)$ in our terminology).

We recall the disentangling formula:

$$e^{zJ_+ - \bar{z}J_-} = e^{\eta J_+}e^{\log(1+|\eta|^2)J_3}e^{-\bar{\eta}J_-} = e^{-\bar{\eta}J_-}e^{-\log(1+|\eta|^2)J_3}e^{\eta J_+}, \quad \eta \equiv \frac{\tan(|z|)}{|z|}z. \tag{26}$$

This is also the key formula for generalized coherent operators.

A comment is in order. We can construct the spin K and J representations by making use of Schwinger's boson method. But we don't repeat it here, see for example [10].

3. Matrix Elements of Generalized Coherent Operators $\cdots$ Results

In this section we first present matrix elements of coherent operators and next study matrix elements of generalized coherent operators.
Let us endeavor to make this section self–contained as far as we can because the proofs in the following are very important to understand mathematical structure of coherent or generalized coherent operators.

3.1 Matrix Elements of Coherent Operator

We show explicit formulas of matrix elements of coherent operators in (5). This result is well–known, see for example [4].

Matrix Elements The matrix elements of $U(z)$ are:

$$\text{(i)} \quad n \le m \quad \langle n|U(z)|m\rangle = \mathrm{e}^{-\frac{1}{2}|z|^2}\sqrt{\frac{n!}{m!}}(-\bar{z})^{m-n}{L_n}^{(m-n)}(|z|^2), \tag{27}$$

$$\text{(ii)} \quad n \ge m \quad \langle n|U(z)|m\rangle = \mathrm{e}^{-\frac{1}{2}|z|^2}\sqrt{\frac{m!}{n!}}z^{n-m}{L_m}^{(n-m)}(|z|^2), \tag{28}$$

where ${L_n}^{(\alpha)}$ is the associated Laguerre polynomial defined by

$${L_n}^{(\alpha)}(x) = \sum_{j=0}^{n}(-1)^j\binom{n+\alpha}{n-j}\frac{x^j}{j!}. \tag{29}$$

In particular ${L_n}^{(0)} \equiv L_n$ is the usual Laguerre polynomial and these are related to diagonal elements of $U(z)$.

The proof is easy and as follows.

For the case $n \ge m$

$$\begin{aligned}\langle n|U(z)|m\rangle &= \frac{1}{\sqrt{n!}}\frac{1}{\sqrt{m!}}\langle 0|a^n\mathrm{e}^{za^\dagger - \bar{z}a}(a^\dagger)^m|0\rangle && \text{by (3)}\\ &= \frac{1}{\sqrt{n!m!}}\mathrm{e}^{\frac{1}{2}|z|^2}\langle 0|a^n\mathrm{e}^{-\bar{z}a}\mathrm{e}^{za^\dagger}(a^\dagger)^m|0\rangle && \text{by (7)}\\ &= \frac{1}{\sqrt{n!m!}}\mathrm{e}^{\frac{1}{2}|z|^2}\frac{\partial^n}{\partial(-\bar{z})^n}\frac{\partial^m}{\partial z^m}\langle 0|\mathrm{e}^{-\bar{z}a}\mathrm{e}^{za^\dagger}|0\rangle.\end{aligned}$$

Noting

$$\mathrm{e}^{-\bar{z}a}\mathrm{e}^{za^\dagger} = \mathrm{e}^{-|z|^2}\mathrm{e}^{za^\dagger}\mathrm{e}^{-\bar{z}a}$$

by (7) and $a|0\rangle = 0$, $\langle 0|a^\dagger = 0$

$$\begin{aligned}&\langle n|U(z)|m\rangle\\ &= \frac{1}{\sqrt{n!m!}}\mathrm{e}^{\frac{1}{2}|z|^2}\frac{\partial^m}{\partial z^m}\frac{\partial^n}{\partial(-\bar{z})^n}\mathrm{e}^{-|z|^2}\\ &= \frac{1}{\sqrt{n!m!}}\mathrm{e}^{\frac{1}{2}|z|^2}\frac{\partial^m}{\partial z^m}\left(z^n\mathrm{e}^{-|z|^2}\right)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{n!m!}} e^{\frac{1}{2}|z|^2} \sum_{k=0}^{m} \binom{m}{m-k} (z^n)^{(m-k)} \left(e^{-|z|^2}\right)^{(k)} \\
&= \frac{1}{\sqrt{n!m!}} e^{\frac{1}{2}|z|^2} \sum_{k=0}^{m} \binom{m}{m-k} n(n-1)\cdots(n-m+k+1) z^{n-m+k} (-\bar{z})^k e^{-|z|^2} \\
&= \frac{1}{\sqrt{n!m!}} e^{-\frac{1}{2}|z|^2} z^{n-m} \sum_{k=0}^{m} (-1)^k \frac{m!}{k!(m-k)!} \frac{n!}{(n-m+k)!} (|z|^2)^k \\
&= \sqrt{\frac{m!}{n!}} e^{-\frac{1}{2}|z|^2} z^{n-m} \sum_{k=0}^{m} (-1)^k \frac{n!}{(m-k)!(n-m+k)!} \frac{(|z|^2)^k}{k!} \\
&= \sqrt{\frac{m!}{n!}} e^{-\frac{1}{2}|z|^2} z^{n-m} \sum_{k=0}^{m} (-1)^k \binom{m+n-m}{m-k} \frac{(|z|^2)^k}{k!} \\
&= \sqrt{\frac{m!}{n!}} e^{-\frac{1}{2}|z|^2} z^{n-m} L_m{}^{(n-m)}(|z|^2).
\end{aligned}
$$

For the case $n \le m$ we have only to take a complex conjugate of (28) (note that $U(z)^\dagger = U(-z)$).

Here we state an interesting application of matrix elements to the theory of special functions. From (6) $\quad U(z+w) = e^{-\frac{1}{2}(z\bar{w}-\bar{z}w)}\, U(z)U(w) \quad$ let us take a matix element

$$
\begin{aligned}
\langle n|U(z+w)|m\rangle &= e^{-\frac{1}{2}(z\bar{w}-\bar{z}w)} \langle n|U(z)U(w)|m\rangle \\
&= e^{-\frac{1}{2}(z\bar{w}-\bar{z}w)} \sum_{k=0}^{\infty} \langle n|U(z)|k\rangle\langle k|U(w)|m\rangle.
\end{aligned}
$$

Then by substituting (27) and (28) into the above equation and making some calculations we obtain

(i) $N \ge 1$

$$
\begin{aligned}
&L_m^{(N)}(|z+w|^2) \\
&= e^{\bar{z}w} \left\{ \left(\frac{z}{z+w}\right)^N \sum_{k=0}^{m} \frac{k!}{m!} (-z\bar{w})^{m-k} L_k^{(m+N-k)}(|z|^2) L_k^{(m-k)}(|w|^2) \right. \\
&\quad + \left(\frac{1}{z+w}\right)^N \sum_{k=m+1}^{m+N} z^{m+N-k} w^{k-m} L_k^{(m+N-k)}(|z|^2) L_m^{(k-m)}(|w|^2) \\
&\quad \left. + \left(\frac{w}{z+w}\right)^N \sum_{k=m+N+1}^{\infty} \frac{(m+N)!}{k!} (-\bar{z}w)^{k-m-N} L_{m+N}^{(k-m-N)}(|z|^2) L_m^{(k-m)}(|w|^2) \right\},
\end{aligned} \tag{30}
$$

(ii) $N=0$

$$L_m(|z+w|^2)=\mathrm{e}^{\bar{z}w}\left\{\sum_{k=0}^{m}\frac{k!}{m!}(-z\bar{w})^{m-k}L_k^{(m-k)}(|z|^2)L_k^{(m-k)}(|w|^2)\right.$$
$$\left.+\sum_{k=m+1}^{\infty}\frac{m!}{k!}(-\bar{z}w)^{k-m}L_m^{(k-m)}(|z|^2)L_m^{(k-m)}(|w|^2)\right\}. \quad (31)$$

To obtain equations of these types is not our purpose of this paper, so we omit the proof (we leave it to the readers).

Since the equation (6) is based on the elementary Baker-Campbell-Hausdorff formula (8), our equations (30) and (31) are highly non–trivial. What is the mathematical meaning of these formulas ? We think they are a kind of factorization formula.

3.2 Matrix Elements of Coherent Operator Based on $su(1,1)$

We in this section study matrix elements of the generalized coherent operator based on Lie algebra $su(1,1)$ (15).

Matrix Elements The matrix elements of $V(z)$ are:

(i) $n\le m$ $$\langle K,n|V(z)|K,m\rangle=\sqrt{\frac{n!m!}{(2K)_n(2K)_m}}\frac{(-\bar{\kappa})^{m-n}}{(1+|\kappa|^2)^{K+\frac{n+m}{2}}}\times$$
$$\sum_{j=0}^{n}(-1)^{n-j}\frac{\Gamma(2K+m+n-j)}{\Gamma(2K)(m-j)!(n-j)!j!}(1+|\kappa|^2)^j(|\kappa|^2)^{n-j}, \quad (32)$$

(ii) $n\ge m$ $$\langle K,n|V(z)|K,m\rangle=\sqrt{\frac{n!m!}{(2K)_n(2K)_m}}\frac{\kappa^{n-m}}{(1+|\kappa|^2)^{K+\frac{n+m}{2}}}\times$$
$$\sum_{j=0}^{m}(-1)^{m-j}\frac{\Gamma(2K+m+n-j)}{\Gamma(2K)(m-j)!(n-j)!j!}(1+|\kappa|^2)^j(|\kappa|^2)^{m-j}, \quad (33)$$

where

$$\kappa\equiv\frac{\sinh(|z|)}{|z|}z=\cosh(|z|)\zeta. \quad (34)$$

A comment is in order. The author doesn't know whether or not the right hand sides of (32) and (33) could be expressed by some known special functions such as associated Laguerre functions in (29).

This result has been reported in [10] under some assumption ($2K\in\mathbf{N}$). In this paper we remove this extra condition and give a complete proof to this result.

Since (32) is given by taking a complex conjugate of (33), we have only to prove (33). The proof is as follows.

Step 1 For the case $n \geq m$

$$
\begin{aligned}
&\langle K,n|V(z)|K,m\rangle \\
&= \frac{1}{\sqrt{n!(2K)_n}}\frac{1}{\sqrt{m!(2K)_m}}\langle K,0|K_-{}^n \mathrm{e}^{zK_+-\bar{z}K_-}K_+{}^m|K,0\rangle \qquad \text{by (13)} \\
&= \frac{1}{\sqrt{n!m!(2K)_n(2K)_m}}\langle K,0|K_-{}^n \mathrm{e}^{-\bar{\zeta}K_-}\mathrm{e}^{-\log(1-|\zeta|^2)K_3}\mathrm{e}^{\zeta K_+}K_+{}^m|K,0\rangle \quad \text{by (16)} \\
&= \frac{1}{\sqrt{n!m!(2K)_n(2K)_m}} \times \\
&\qquad \frac{\partial^n}{\partial\alpha^n}\frac{\partial^m}{\partial\beta^m}\langle K,0|\mathrm{e}^{(\alpha-\bar{\zeta})K_-}\mathrm{e}^{-\log(1-|\zeta|^2)K_3}\mathrm{e}^{(\beta+\zeta)K_+}|K,0\rangle|_{\alpha=\beta=0} \\
&= \frac{1}{\sqrt{n!m!(2K)_n(2K)_m}}\frac{\partial^n}{\partial\alpha^n}\frac{\partial^m}{\partial\beta^m}\left\{(1-|\zeta|^2)^{1/2}-\frac{(\alpha-\bar{\zeta})(\beta+\zeta)}{(1-|\zeta|^2)^{1/2}}\right\}^{-2K}|_{\alpha=\beta=0},
\end{aligned}
\tag{35}
$$

where we have used the exchange relation like (7):

Formula

$$
\mathrm{e}^{aK_-}\mathrm{e}^{2bK_3}\mathrm{e}^{cK_+} = \mathrm{e}^{xK_+}\mathrm{e}^{2yK_3}\mathrm{e}^{zK_-} \tag{36}
$$

where

$$
x = \frac{c\mathrm{e}^b}{\mathrm{e}^{-b}-ac\mathrm{e}^b}, \quad y = -\log(\mathrm{e}^{-b}-ac\mathrm{e}^b), \quad z = \frac{a\mathrm{e}^b}{\mathrm{e}^{-b}-ac\mathrm{e}^b}. \tag{37}
$$

For the proof see Appendix.

Step 2 Let us calculate the differential

$$
\begin{aligned}
&\frac{\partial^n}{\partial\alpha^n}\frac{\partial^m}{\partial\beta^m}\left\{(1-|\zeta|^2)^{1/2}-\frac{(\alpha-\bar{\zeta})(\beta+\zeta)}{(1-|\zeta|^2)^{1/2}}\right\}^{-2K}|_{\alpha=\beta=0} \\
&= \left\{\frac{1}{(1-|\zeta|^2)^{1/2}}\right\}^{-2K}\frac{\partial^m}{\partial\beta^m}\frac{\partial^n}{\partial\alpha^n}\left\{1-|\zeta|^2-(\alpha-\bar{\zeta})(\beta+\zeta)\right\}^{-2K}|_{\alpha=\beta=0} \\
&= \left\{\frac{1}{(1-|\zeta|^2)^{1/2}}\right\}^{-2K}\frac{\partial^m}{\partial\beta^m}(2K)_n(\beta+\zeta)^n\left\{1-|\zeta|^2+\bar{\zeta}(\beta+\zeta)\right\}^{-2K-n}|_{\beta=0} \\
&= (2K)_n\left\{\frac{1}{(1-|\zeta|^2)^{1/2}}\right\}^{-2K}\frac{\partial^m}{\partial\beta^m}\left\{(\beta+\zeta)^n(1+\beta\bar{\zeta})^{-2K-n}\right\}|_{\beta=0}
\end{aligned}
$$

$$= (2K)_n \left\{\frac{1}{(1-|\zeta|^2)^{1/2}}\right\}^{-2K} \sum_{j=0}^{m} \binom{m}{j} \{(\beta+\zeta)^n\}^{(j)} \{(1+\beta\bar{\zeta})^{-2K-n}\}^{(m-j)} |_{\beta=0}$$

$$= (2K)_n \left\{\frac{1}{(1-|\zeta|^2)^{1/2}}\right\}^{-2K} \sum_{j=0}^{m} \binom{m}{j} \frac{n!}{(n-j)!} \zeta^{n-j} (-1)^{m-j} (2K+n)_{m-j} \bar{\zeta}^{m-j}$$

$$= \left\{\frac{1}{(1-|\zeta|^2)^{1/2}}\right\}^{-2K} \zeta^{n-m} \times$$
$$\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \frac{n!}{(n-j)!} (2K)_n (2K+n)_{m-j} (|\zeta|^2)^{m-j}$$

$$= \left\{\frac{1}{(1-|\zeta|^2)^{1/2}}\right\}^{-2K-n-m} \left(\frac{\zeta}{(1-|\zeta|^2)^{1/2}}\right)^{n-m} \times$$
$$\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \frac{n!}{(n-j)!} (2K)_{n+m-j} \left(\frac{|\zeta|^2}{1-|\zeta|^2}\right)^{m-j} \left(\frac{1}{1-|\zeta|^2}\right)^{j}. \tag{38}$$

Now let us make a change of variables

$$\zeta \longrightarrow \kappa \equiv \frac{\zeta}{\sqrt{1-|\zeta|^2}}$$

then by (16)

$$\kappa = \frac{z\sinh(|z|)}{|z|}, \quad |\kappa|^2 = \sinh^2(|z|), \quad 1+|\kappa|^2 = \cosh^2(|z|) = \frac{1}{1-|\zeta|^2}.$$

Therefore

$$(38) = (1+|\kappa|^2)^{-(K+\frac{n+m}{2})} \kappa^{n-m} \times$$
$$\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \frac{n!}{(n-j)!} (2K)_{n+m-j} (|\kappa|^2)^{m-j} (1+|\kappa|^2)^j$$
$$= n!m!\kappa^{n-m} (1+|\kappa|^2)^{-(K+\frac{n+m}{2})} \times$$
$$\sum_{j=0}^{m} (-1)^{m-j} \frac{(2K)_{n+m-j}}{j!(m-j)!(n-j)!} (|\kappa|^2)^{m-j} (1+|\kappa|^2)^j$$
$$= n!m!\kappa^{n-m} (1+|\kappa|^2)^{-(K+\frac{n+m}{2})} \times$$
$$\sum_{j=0}^{m} (-1)^{m-j} \frac{\Gamma(2K+n+m-j)}{\Gamma(2K)j!(m-j)!(n-j)!} (|\kappa|^2)^{m-j} (1+|\kappa|^2)^j, \tag{39}$$

where we have used the formula

$$(2K)_j \equiv (2K)(2K+1)\cdots(2K+j-1) = \frac{\Gamma(2K+j)}{\Gamma(2K)}.$$

Combining (35) with (39) we finally obtain the matrix elements (33).

3.3 Matrix Elements of Coherent Operator Based on $su(2)$

We in this section study matrix elements of the generalized coherent operator based on Lie algebra $su(2)$ (25). In this case it is always $2J \in \mathbf{N}$.

Matrix Elements The matrix elements of $W(z)$ are:

$$\text{(i)}\quad n \le m \quad \langle J,n|W(z)|J,m\rangle = \sqrt{\frac{n!m!}{{}_{2J}P_{n\,2J}P_m}}(-\bar{\kappa})^{m-n}(1-|\kappa|^2)^{J-\frac{n+m}{2}} \times$$
$$\sum_{j=0}^{n}{}_{*}(-1)^{n-j}\frac{(2J)!}{(2J-m-n+j)!(m-j)!(n-j)!j!}(1-|\kappa|^2)^j(|\kappa|^2)^{n-j}, \quad (40)$$

$$\text{(ii)}\quad n \ge m \quad \langle J,n|W(z)|J,m\rangle = \sqrt{\frac{n!m!}{{}_{2J}P_{n\,2J}P_m}}\kappa^{n-m}(1-|\kappa|^2)^{J-\frac{n+m}{2}} \times$$
$$\sum_{j=0}^{m}{}_{*}(-1)^{m-j}\frac{(2J)!}{(2J-m-n+j)!(m-j)!(n-j)!j!}(1-|\kappa|^2)^j(|\kappa|^2)^{m-j}, \quad (41)$$

where

$$\kappa \equiv \frac{\sin(|z|)}{|z|}z = \cos(|z|)\eta. \quad (42)$$

Here $\sum_*$ means a summation over j satisfying $2J-m-n+j \ge 0$.

A comment is in order. The author doesn't know whether or not the right hand sides of (40) and (41) could be expressed in terms of some known special functions.

The proof is almost the same as the preceding one, so we leave the proof of the first step to the readers. For the case $n \ge m$ we reach

$$\langle J,n|W(z)|J,m\rangle$$
$$= \frac{1}{\sqrt{n!m!{}_{2J}P_{n\,2J}P_m}}\frac{\partial^n}{\partial\alpha^n}\frac{\partial^m}{\partial\beta^m}\left\{(1+|\eta|^2)^{1/2}+\frac{(\alpha-\bar{\eta})(\beta+\eta)}{(1+|\eta|^2)^{1/2}}\right\}^{2J}|_{\alpha=\beta=0}. \quad (43)$$

Let us calculate the differential.

$$\frac{\partial^n}{\partial\alpha^n}\frac{\partial^m}{\partial\beta^m}\left\{(1+|\eta|^2)^{1/2}+\frac{(\alpha-\bar{\eta})(\beta+\eta)}{(1+|\eta|^2)^{1/2}}\right\}^{2J}|_{\alpha=\beta=0}$$

$$
\begin{aligned}
&= \left\{\frac{1}{(1+|\eta|^2)^{1/2}}\right\}^{2J} \frac{\partial^m}{\partial\beta^m}\frac{\partial^n}{\partial\alpha^n}\left\{1+|\eta|^2+(\alpha-\bar{\eta})(\beta+\eta)\right\}^{2J}|_{\alpha=\beta=0} \\
&= \left\{\frac{1}{(1+|\eta|^2)^{1/2}}\right\}^{2J} \frac{\partial^m}{\partial\beta^m}{}_{2J}P_n(\beta+\eta)^n\left\{1+|\eta|^2-\bar{\eta}(\beta+\eta)\right\}^{2J-n}|_{\beta=0} \\
&= {}_{2J}P_n\left\{\frac{1}{(1+|\eta|^2)^{1/2}}\right\}^{2J} \frac{\partial^m}{\partial\beta^m}\left\{(\beta+\eta)^n(1-\beta\bar{\eta})^{2J-n}\right\}|_{\beta=0} \\
&= {}_{2J}P_n\left\{\frac{1}{(1+|\eta|^2)^{1/2}}\right\}^{2J} \sum_{j=0}^{m}{}_*\binom{m}{j}\left\{(\beta+\eta)^n\right\}^{(j)}\left\{(1-\beta\bar{\eta})^{2J-n}\right\}^{(m-j)}|_{\beta=0} \\
&= {}_{2J}P_n\left\{\frac{1}{(1+|\eta|^2)^{1/2}}\right\}^{2J} \sum_{j=0}^{m}{}_*\binom{m}{j}\frac{n!}{(n-j)!}\eta^{n-j}{}_{2J-n}P_{m-j}(-\bar{\eta})^{m-j} \\
&= \left\{\frac{1}{(1+|\eta|^2)^{1/2}}\right\}^{2J}\eta^{n-m}\times \\
&\qquad \sum_{j=0}^{m}{}_*(-1)^{m-j}\binom{m}{j}\frac{n!}{(n-j)!}{}_{2J}P_n\ {}_{2J-n}P_{m-j}(|\eta|^2)^{m-j} \\
&= \left\{\frac{1}{(1+|\eta|^2)^{1/2}}\right\}^{2J-n-m}\left(\frac{\eta}{(1+|\eta|^2)^{1/2}}\right)^{n-m}\times \\
&\qquad \sum_{j=0}^{m}{}_*(-1)^{m-j}\binom{m}{j}\frac{n!}{(n-j)!}{}_{2J}P_{n+m-j}\left(\frac{|\eta|^2}{1+|\eta|^2}\right)^{m-j}\left(\frac{1}{1+|\eta|^2}\right)^{j} \\
&= (1-|\kappa|^2)^{J-\frac{n+m}{2}}\kappa^{n-m}\times \\
&\qquad \sum_{j=0}^{m}{}_*(-1)^{m-j}\binom{m}{j}\frac{n!}{(n-j)!}{}_{2J}P_{n+m-j}(|\kappa|^2)^{m-j}(1-|\kappa|^2)^{j} \\
&= n!m!\kappa^{n-m}(1-|\kappa|^2)^{J-\frac{n+m}{2}}\times \\
&\qquad \sum_{j=0}^{m}{}_*(-1)^{m-j}\frac{{}_{2J}P_{n+m-j}}{j!(m-j)!(n-j)!}(|\kappa|^2)^{m-j}(1-|\kappa|^2)^{j} \\
&= n!m!\kappa^{n-m}(1-|\kappa|^2)^{J-\frac{n+m}{2}}\times \\
&\qquad \sum_{j=0}^{m}{}_*(-1)^{m-j}\frac{(2J)!}{(2J-n-m+j)!j!(m-j)!(n-j)!}(|\kappa|^2)^{m-j}(1-|\kappa|^2)^{j},
\end{aligned}
\tag{44}
$$

where we have used a change of variables

$$
\eta \longrightarrow \kappa \equiv \frac{\eta}{\sqrt{1+|\eta|^2}} \quad\Longrightarrow\quad |\kappa|^2=\frac{|\eta|^2}{1+|\eta|^2},\quad 1-|\kappa|^2=\frac{1}{1+|\eta|^2}
$$

and the formula

$$_{2J}P_j = (2J)(2J-1)\cdots(2J-j+1) = \frac{(2J)!}{(2J-j)!}\,.$$

Combining (43) with (44) we finally obtain the matrix elements (41).

4. A Problem

In this section let us present a problem for the readers.

In the definition ((25), (15)) of generalized coherent operators based on Lie algebras $su(2)$ and $su(1,1)$ operators J_3 and K_3 are not used. To use these ones we would like to extend generalized coherent operators as follows (see [15]).

Definition We set

$$su(2):\quad W(z,t) = \mathrm{e}^{zJ_+ - \bar{z}J_- + 2itJ_3} \quad \text{for } z \in \mathbf{C},\ t \in \mathbf{R} \tag{45}$$

$$su(1,1):\quad V(z,t) = \mathrm{e}^{zK_+ - \bar{z}K_- + 2itK_3} \quad \text{for } z \in \mathbf{C},\ t \in \mathbf{R}. \tag{46}$$

Both $W(z,t)$ and $V(z,t)$ are unitary and $W(z,0) = W(z)$, $V(z,0) = V(z)$. On the other hand we have used operators of these types in the process of proof, so it is very natural to consider the above. Then

Problem Determine the matrix elements of $W(z,t)$ and $V(z,t)$:

$$su(2):\quad \langle J,n|W(z,t)|J,m\rangle \quad \text{for} \quad 0 \le n,\ m \le 2J, \tag{47}$$

$$su(1,1):\quad \langle K,n|V(z,t)|K,m\rangle \quad \text{for} \quad 0 \le n,\ m. \tag{48}$$

See for example [8].

5. Discussion

In this paper we have determined the matrix elements of the generalized coherent operators based on $su(1,1)$ and $su(2)$ in a perfect manner.

By the way we have an interesting application of these matrix elements. In [12] Frasca has used matrix elements of coherent operator in section 3.1 to explain the recent experimental results on Rabi oscillations in a Josephson junction [16]. See also [17].

Since we have calculated matrix elements of generalized coherent operators we should generalize his method. We will report the results in a forthcoming paper [13].

We believe strongly that our calculations will play an essential role in understanding the general (mathematical) structure of Rabi oscillations in the strong coupling regime.

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Appendix: An Exchange Relation of Operators

Let us prove the exchange relation (49).

Formula

$$e^{aK_-}e^{2bK_3}e^{cK_+} = e^{xK_+}e^{2yK_3}e^{zK_-} \tag{49}$$

where

$$x = \frac{ce^b}{e^{-b} - ace^b}, \quad y = -\log(e^{-b} - ace^b), \quad z = \frac{ae^b}{e^{-b} - ace^b}. \tag{50}$$

We divide the proof into two parts.

First we assume that $\{K_+, K_-, K_3\}$ is a differential representation of Lie group $SU(1,1)$, namely we have a unitary representation

$$\rho : SU(1,1) \subset SL(2,\mathbf{C}) \longrightarrow U(\mathcal{H} \otimes \mathcal{H})$$

such that

$$K_- = d\rho(k_-), \quad K_3 = d\rho(k_3), \quad K_+ = d\rho(k_+).$$

In this case we must take $2K \in \mathbf{N}$ (an integrability condition), see [10]. Then since ρ is a homomorphism of the **group**

$$\begin{aligned}
e^{aK_-}e^{2bK_3}e^{cK_+} &= e^{ad\rho(k_-)}e^{2bd\rho(k_3)}e^{cd\rho(k_+)} = \rho(e^{ak_-})\rho(e^{2bk_3})\rho(e^{ck_+}) \\
&= \rho(e^{ak_-}e^{2bk_3}e^{ck_+}) \\
&= \rho\left(\exp\begin{pmatrix} & 0 \\ -a & \end{pmatrix}\exp\begin{pmatrix} b & \\ & -b \end{pmatrix}\exp\begin{pmatrix} & c \\ 0 & \end{pmatrix}\right) \\
&= \rho\left(\begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix}\begin{pmatrix} e^b & \\ & e^{-b} \end{pmatrix}\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}\right) \\
&= \rho\left(\begin{pmatrix} e^b & ce^b \\ -ae^b & e^{-b} - ace^b \end{pmatrix}\right) \equiv \rho(A).
\end{aligned}$$

Next let us make another Gauss decomposition of A. It is easy to see

$$A = \begin{pmatrix} 1 & \frac{ce^b}{e^{-b}-ace^b} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{e^{-b}-ace^b} & \\ & e^{-b}-ace^b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{ae^b}{e^{-b}-ace^b} & 1 \end{pmatrix}.$$

Here we set $f = e^{-b} - ace^b$ for simplicity. Therefore

$$\begin{aligned}
&\rho(A) \\
&= \rho\left(\begin{pmatrix} 1 & \frac{ce^b}{f} \\ 0 & 1 \end{pmatrix}\right) \rho\left(\begin{pmatrix} \frac{1}{f} & \\ & f \end{pmatrix}\right) \rho\left(\begin{pmatrix} 1 & 0 \\ -\frac{ae^b}{f} & 1 \end{pmatrix}\right) \\
&= \rho\left(\exp\begin{pmatrix} 0 & \frac{ce^b}{f} \\ 0 & 0 \end{pmatrix}\right) \rho\left(\exp\begin{pmatrix} -\log(f) & \\ & \log(f) \end{pmatrix}\right) \rho\left(\exp\begin{pmatrix} 0 & 0 \\ -\frac{ae^b}{f} & 0 \end{pmatrix}\right) \\
&= \rho\left(e^{\frac{ce^b}{f}k_+}\right) \rho\left(e^{-2\log(f)k_3}\right) \rho\left(e^{\frac{ae^b}{f}k_-}\right) \\
&= e^{\frac{ce^b}{f}d\rho(k_+)} e^{-2\log(f)d\rho(k_3)} e^{\frac{ae^b}{f}d\rho(k_-)} \\
&= e^{\frac{ce^b}{f}K_+} e^{-2\log(f)K_3} e^{\frac{ae^b}{f}K_-}.
\end{aligned} \tag{51}$$

That is, we obtained the formula under the condition $2K \in \mathbf{N}$. Next to remove this condition we use a method of differential equations. We set

$$F(a) = e^{aK_-}e^{2bK_3}e^{cK_+}, \tag{52}$$

$$G(a) = e^{\frac{ce^b}{f(a)}K_+}e^{-2\log f(a)K_3}e^{\frac{ae^b}{f(a)}K_-}, \tag{53}$$

where $f(a) = e^{-b} - ace^b$.
First

$$F(0) = e^{2bK_3}e^{cK_+}, \quad F'(a) = K_-F(a). \tag{54}$$

Next by (53)

$$G(0) = e^{ce^{2b}K_+}e^{2bK_3} \tag{55}$$

so

$$\begin{aligned}
G(0) &= e^{ce^{2b}K_+}e^{2bK_3} = e^{2bK_3}e^{-2bK_3}e^{ce^{2b}K_+}e^{2bK_3} = e^{2bK_3}e^{ce^{2b}e^{-2bK_3}K_+e^{2bK_3}} \\
&= e^{2bK_3}e^{ce^{2b}e^{-2b}K_+} = e^{2bK_3}e^{cK_+} = F(0).
\end{aligned} \tag{56}$$

Here we have used the formula

$$e^{\alpha K_3}K_+e^{-\alpha K_3} = e^{\alpha}K_+.$$

Moreover

$$
\begin{aligned}
&G'(a) \\
&= \frac{(ce^b)^2}{f(a)^2}K_+G(a) + \frac{2ce^b}{f(a)}e^{\frac{ce^b}{f(a)}K_+}K_3e^{-2\log f(a)K_3}e^{\frac{ce^b}{f(a)}K_-} + \frac{1}{f(a)^2}G(a)K_- \\
&= \frac{(ce^b)^2}{f(a)^2}K_+G(a) \\
&\qquad + \frac{2ce^b}{f(a)}\left\{e^{\frac{ce^b}{f(a)}K_+}K_3e^{-\frac{ce^b}{f(a)}K_+}\right\}G(a) + \frac{1}{f(a)^2}\left\{G(a)K_-G(a)^{-1}\right\}G(a) \\
&= \frac{(ce^b)^2}{f(a)^2}K_+G(a) \\
&\qquad + \frac{2ce^b}{f(a)}\left(K_3 - \frac{ce^b}{f(a)}K_+\right)G(a) + \frac{1}{f(a)^2}\left\{G(a)K_-G(a)^{-1}\right\}G(a) \\
&= \frac{1}{f(a)^2}\left\{(ce^b)^2K_+ + 2ce^bf(a)\left(K_3 - \frac{ce^b}{f(a)}K_+\right) + G(a)K_-G(a)^{-1}\right\}G(a).
\end{aligned} \tag{57}
$$

Here we have used the formula

$$e^{\alpha K_+}K_3e^{-\alpha K_+} = K_3 - \alpha K_+.$$

Now let us calculate the term $G(a)K_-G(a)^{-1}$:

$$
\begin{aligned}
G(a)K_-G(a)^{-1} &= e^{\frac{ce^b}{f(a)}K_+}\left\{e^{-2\log f(a)K_3}K_-e^{2\log f(a)K_3}\right\}e^{-\frac{ce^b}{f(a)}K_+} \\
&= e^{\frac{ce^b}{f(a)}K_+}\left(e^{2\log f(a)}K_-\right)e^{-\frac{ce^b}{f(a)}K_+} = f(a)^2e^{\frac{ce^b}{f(a)}K_+}K_-e^{-\frac{ce^b}{f(a)}K_+} \\
&= f(a)^2\left(K_- - 2\frac{ce^b}{f(a)}K_3 + \frac{(ce^b)^2}{f(a)^2}K_+\right) \\
&= f(a)^2K_- - 2ce^bf(a)K_3 + (ce^b)^2K_+.
\end{aligned} \tag{58}
$$

Here we have used the formula

$$e^{\alpha K_3}K_-e^{-\alpha K_3} = e^{-\alpha}K_-, \quad e^{\alpha K_+}K_-e^{-\alpha K_+} = K_- - 2\alpha K_3 + \alpha^2K_+.$$

Therefore from (57) and (58) we reach

$$G'(a) = K_-G(a). \tag{59}$$

From the uniqueness of differential equations with the same initial condition we obtain $F(a) = G(a)$.

Similarly we have a formula for some operators based on $su(2)$.

Formula

$$e^{aJ_-}e^{2bJ_3}e^{cJ_+} = e^{xJ_+}e^{2yJ_3}e^{zJ_-} \tag{60}$$

where

$$x = \frac{ce^b}{e^{-b}+ace^b}, \quad y = -\log(e^{-b}+ace^b), \quad z = \frac{ae^b}{e^{-b}+ace^b}. \tag{61}$$

We leave the proof to the readers.

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