YOKOHAMA MATHEMATICAL
JOURNAL VOL. 53, 2007

GLOBAL ATTRACTIVITY AND PERIODIC CHARACTER OF A FRACTIONAL DIFFERENCE EQUATION OF ORDER THREE

By

E.M. ELABBASY, H. EL-METWALLY, AND E.M. ELSAYED

(Received September 27, 2005; Revised March 7, 2006)

Abstract. In this paper we investigate the global convergence result, boundedness, and periodicity of solutions of the recursive sequence

$$x_{n+1}=\frac{\alpha x_n+\beta x_{n-1}+\gamma x_{n-2}}{Ax_n+Bx_{n-1}+Cx_{n-2}},\quad n=0,1,\ldots$$

where the parameters A, B, C, α, β and γ are positive real numbers and the initial conditions x_{-2}, x_{-1} and x_0 are arbitrary positive numbers.

1. Introduction

Our goal in this paper is to investigate the global stability character and the periodicity of solutions of the recursive sequence

$$x_{n+1}=\frac{\alpha x_n+\beta x_{n-1}+\gamma x_{n-2}}{Ax_n+Bx_{n-1}+Cx_{n-2}} \tag{1}$$

where A, B, C, α, β and $\gamma \in (0,\infty)$ with the initial conditions x_0, x_{-1} and $x_{-2} \in (0,\infty)$.

This paper is motivated by the open problem 6.10.8 in Kulenovic and Ladas [9, pp.126], and we also regard theorems as an answer in the affirmative for this open problem.

The special case of Eq.(1) where $C=\gamma=0$ was investigated in [8]. Also many other special cases of Eq.(1) are studied in [2,5-7].

DEFINITION 1. The difference equation

$$x_{n+1}=F(x_n,x_{n-1},\ldots,x_{n-k}),\ \ n=0,1,\ldots \tag{2}$$

is said to be persistence if there exist numbers m and M with $0<m\leq M<\infty$ such that for any initial conditions $x_{-k}, x_{-k+1},\ldots, x_{-1}, x_0 \in (0,\infty)$ there exists

2000 Mathematics Subject Classification: 39A10
Key words and phrases: stability, periodic solutions, difference equations

a positive integer N which depends on the initial conditions such that

$$m \leq x_n \leq M \qquad \text{for all} \quad n \geq N.$$

Definition 2. (Stability)

(i) The equilibrium point $\overline{x}$ of Eq.(2) is locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, \dots, x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + \cdots + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \epsilon \qquad \text{for all} \quad n \geq -k.$$

(ii) The equilibrium point $\overline{x}$ of Eq.(2) is locally asymptotically stable if $\overline{x}$ is locally stable solution of Eq.(2) and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, \dots, x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + \cdots + |x_0 - \overline{x}| < \gamma,$$

we have

$$\lim_{n \to \infty} x_n = \overline{x}.$$

(iii) The equilibrium point $\overline{x}$ of Eq.(2) is global attractor if for all $x_{-k}, x_{-k+1}, \dots, x_{-1}, x_0 \in I$, we have

$$\lim_{n \to \infty} x_n = \overline{x}.$$

(iv) The equilibrium point $\overline{x}$ of Eq.(2) is globally asymptotically stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of Eq.(2).

(v) The equilibrium point $\overline{x}$ of Eq.(2) is unstable if $\overline{x}$ is not locally stable.

The linearized equation of Eq.(2) about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial F(\overline{x}, \overline{x}, \dots, \overline{x})}{\partial x_{n-i}} y_{n-i} \tag{3}$$

Theorem A. *Assume that $p, q \in R$ and $k \in \{0, 1, 2, \dots\}$. Then*

$$|p| + |q| < 1$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+1} + px_n + qx_{n-k} = 0, \quad n = 0, 1, \dots .$$

REMARK 1. Theorem A can be easily extended to a general linear equations of the form

$$x_{n+k} + p_1 x_{n+k-1} + \ldots + p_k x_n = 0, \quad n = 0, 1, \ldots \tag{4}$$

where $p_1, p_2, \ldots, p_k \in R$ and $k \in \{1, 2, \ldots\}$. Then Eq.(4) is asymptotically stable provided that

$$\sum_{i=1}^{k} |p_i| < 1.$$

The theory of the Full Limiting Sequences was indicated in [3] and [4]. The following theorem was given in [1].

THEOREM B. *Let $F \in C[I^{k+1}, I]$ for some interval I of real numbers and for some non-negative integer k, and consider the difference equation*

$$x_{n+1} = F(x_n, x_{n-1}, \ldots, x_{n-k}). \tag{5}$$

Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of Eq.(5), and suppose that there exist constants $A \in I$ and $B \in I$ such that

$$A \leq x_n \leq B \quad \text{for all} \quad n \geq -k.$$

Let ℓ_0 be a limit point of the sequence $\{x_n\}_{n=-k}^{\infty}$. Then the following statements are true.

(i) There exists a solution $\{L_n\}_{n=-\infty}^{\infty}$ of Eq.(5), called a full limiting sequence of $\{x_n\}_{n=-k}^{\infty}$, such that $L_0 = \ell_0$, and such that for every $N \in \{\ldots,-1,0,1,\ldots\}$ L_N is a limit point of $\{x_n\}_{n=-k}^{\infty}$.

(ii) For every $i_0 \leq -k$, there exists a subsequence $\{x_{r_i}\}_{i=0}^{\infty}$ of $\{x_n\}_{n=-k}^{\infty}$ such that

$$L_N = \lim_{i\to\infty} x_{r_i+N} \quad \text{for every} \quad N \geq i_0.$$

2. Local Stability of the Equilibrium Point

In this section we study the local stability character of the solutions of Eq.(1). Eq.(1) has a unique positive equilibrium point and is given by

$$\overline{x} = \frac{\alpha + \beta + \gamma}{A + B + C}.$$

Let $f:(0,\infty)^3 \longrightarrow (0,\infty)$ be a function defined by

$$f(u,v,w)=\frac{\alpha u+\beta v+\gamma w}{Au+Bv+Cw} \tag{6}$$

Then the linearized equation of Eq.(1) about $\overline{x}$ is

$$y_{n+1}+a_2y_n+a_1y_{n-1}+a_0y_{n-2}=0 \tag{7}$$

where $a_2=-f_u(\overline{x},\overline{x},\overline{x})$, $a_1=-f_v(\overline{x},\overline{x},\overline{x})$, and $a_0=-f_w(\overline{x},\overline{x},\overline{x})$.

Whose characteristic equation is

$$\lambda^3+a_2\lambda^2+a_1\lambda+a_0=0 \tag{8}$$

THEOREM 2.1. *Assume that*

$$\begin{aligned}(\alpha+\beta+\gamma)(A+B+C)> \quad & \max\{\,|2\gamma(A+B)-2C(\alpha+\beta)|\,,\\ & |2\beta(A+C)-2B(\alpha+\gamma)|\,,\\ & |2A(\beta+\gamma)-2\alpha(B+C)|\}. \end{aligned}\tag{9}$$

Then the positive equilibrium point of Eq.(1) is locally asymptotically stable.

Proof. It is follows by Theorem A that, Eq.(7) is asymptotically stable if all roots of Eq.(8) lie in the open disc $|\lambda|<1$ that is if

$$|a_2|+|a_1|+|a_0|<1. \tag{10}$$

We consider the following cases

(1) $a_2>0$, $a_1>0$ and $a_0>0$. In this case we see from (10) that

$$\begin{aligned}&A(\beta+\gamma)-\alpha(B+C)+B(\alpha+\gamma)-\beta(A+C)+C(\alpha+\beta)-\gamma(A+B)\\ &<(A+B+C)(\alpha+\beta+\gamma)\end{aligned}$$

if and only if

$$(A+B+C)(\alpha+\beta+\gamma)>0,$$

which is always true.

(2) $a_2>0$, $a_1>0$ and $a_0<0$. It follows from (10) that

$$\begin{aligned}&A(\beta+\gamma)-\alpha(B+C)+B(\alpha+\gamma)-\beta(A+C)-C(\alpha+\beta)+\gamma(A+B)\\ &<(A+B+C)(\alpha+\beta+\gamma)\end{aligned}$$

if and only if

$$2\gamma(A+B) - 2C(\alpha+\beta) < (A+B+C)(\alpha+\beta+\gamma),$$

which is satisfied by Condition (9).

(3) $a_2 > 0$, $a_1 < 0$ and $a_0 > 0$. From (10) we see that

$$\begin{aligned}&A(\beta+\gamma) - \alpha(B+C) - B(\alpha+\gamma) + \beta(A+C) + C(\alpha+\beta) - \gamma(A+B)\\ &< (A+B+C)(\alpha+\beta+\gamma)\end{aligned}$$

if and only if

$$2\beta(A+C) - 2B(\alpha+\gamma) < (A+B+C)(\alpha+\beta+\gamma),$$

which is satisfied by (9).

(4) $a_2 > 0$, $a_1 < 0$ and $a_0 < 0$. From (10) we see that

$$\begin{aligned}&A(\beta+\gamma) - \alpha(B+C) - B(\alpha+\gamma) + \beta(A+C) - C(\alpha+\beta) + \gamma(A+B)\\ &< (A+B+C)(\alpha+\beta+\gamma)\end{aligned}$$

if and only if

$$2A(\beta+\gamma) - 2\alpha(B+C) < (A+B+C)(\alpha+\beta+\gamma),$$

which is satisfied by (9).

(5) $a_2 < 0$, $a_1 > 0$ and $a_0 > 0$. From (10) we see that

$$\begin{aligned}&-A(\beta+\gamma) + \alpha(B+C) + B(\alpha+\gamma) - \beta(A+C) + C(\alpha+\beta) - \gamma(A+B)\\ &< (A+B+C)(\alpha+\beta+\gamma)\end{aligned}$$

if and only if

$$2\alpha(B+C) - 2A(\beta+\gamma) < (A+B+C)(\alpha+\beta+\gamma),$$

which is satisfied by (9).

(6) $a_2 < 0$, $a_1 > 0$ and $a_0 < 0$. It follows from (10) that

$$\begin{aligned}&-A(\beta+\gamma) + \alpha(B+C) + B(\alpha+\gamma) - \beta(A+C) - C(\alpha+\beta) + \gamma(A+B)\\ &< (A+B+C)(\alpha+\beta+\gamma)\end{aligned}$$

if and only if

$$2B(\alpha+\gamma)-2\beta(A+C)<(A+B+C)(\alpha+\beta+\gamma),$$

which is satisfied by (9).

(7) $a_2<0$, $a_1<0$ and $a_0>0$. From (10) we see that

$$\begin{aligned}&-A(\beta+\gamma)+\alpha(B+C)-B(\alpha+\gamma)+\beta(A+C)+C(\alpha+\beta)-\gamma(A+B)\\&<(A+B+C)(\alpha+\beta+\gamma)\end{aligned}$$

if and only if

$$2C(\alpha+\beta)-2\gamma(A+B)<(A+B+C)(\alpha+\beta+\gamma),$$

which is satisfied by (9).

(8) $a_2<0$, $a_1<0$, and $a_0<0$. From (10) we see that

$$\begin{aligned}&-A(\beta+\gamma)+\alpha(B+C)-B(\alpha+\gamma)+\beta(A+C)-C(\alpha+\beta)+\gamma(A+B)\\&<(A+B+C)(\alpha+\beta+\gamma)\end{aligned}$$

if and only if

$$(A+B+C)(\alpha+\beta+\gamma)>0,$$

which is always true. The proof is complete.

3. Boundedness of solutions

Here we study the permanence of Eq.(1).

THEOREM 3.1. *Every solution of Eq.*(1) *is bounded and persists.*

Proof. Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of Eq.(1). It follows from Eq.(1) that

$$\begin{aligned}x_{n+1}&=\frac{\alpha x_n+\beta x_{n-1}+\gamma x_{n-2}}{Ax_n+Bx_{n-1}+Cx_{n-2}}\\&=\frac{\alpha x_n}{Ax_n+Bx_{n-1}+Cx_{n-2}}+\frac{\beta x_{n-1}}{Ax_n+Bx_{n-1}+Cx_{n-2}}\\&\quad+\frac{\gamma x_{n-2}}{Ax_n+Bx_{n-1}+Cx_{n-2}}\end{aligned}$$

Then

$$x_n \le \frac{\alpha}{A} + \frac{\beta}{B} + \frac{\gamma}{C} = M \quad \text{for all} \quad n \ge 1. \tag{11}$$

Now we wish to show that there exists $m > 0$ such that

$$x_n \ge m \quad \text{for all} \quad n \ge 1.$$

The change of variables

$$x_n = \frac{1}{y_n},$$

gives Eq.(1) in the form

$$\frac{1}{y_{n+1}} = \frac{\frac{\alpha}{y_n} + \frac{\beta}{y_{n-1}} + \frac{\gamma}{y_{n-2}}}{\frac{A}{y_n} + \frac{B}{y_{n-1}} + \frac{C}{y_{n-2}}},$$

or in the equivalent form

$$\begin{aligned} y_{n+1} &= \frac{Ay_{n-1}y_{n-2} + By_ny_{n-2} + Cy_ny_{n-1}}{\alpha y_{n-1}y_{n-2} + \beta y_ny_{n-2} + \gamma y_ny_{n-1}} \\ &= \frac{Ay_{n-1}y_{n-2}}{\alpha y_{n-1}y_{n-2} + \beta y_ny_{n-2} + \gamma y_ny_{n-1}} + \frac{By_ny_{n-2}}{\alpha y_{n-1}y_{n-2} + \beta y_ny_{n-2} + \gamma y_ny_{n-1}} \\ &\quad + \frac{Cy_ny_{n-1}}{\alpha y_{n-1}y_{n-2} + \beta y_ny_{n-2} + \gamma y_ny_{n-1}}. \end{aligned}$$

It follows that

$$y_{n+1} \le \frac{A}{\alpha} + \frac{B}{\beta} + \frac{C}{\gamma} = \frac{A\beta\gamma + B\alpha\gamma + C\alpha\beta}{\alpha\beta\gamma} = H \quad \text{for all} \quad n \ge 1.$$

Thus we obtain

$$x_n = \frac{1}{y_n} \ge \frac{1}{H} = \frac{\alpha\beta\gamma}{A\beta\gamma + B\alpha\gamma + C\alpha\beta} = m \quad \text{for all} \quad n \ge 1. \tag{12}$$

From (11) and (12) we see that

$$m \le x_n \le M \quad \text{for all} \quad n \ge 1.$$

Therefore every solution of Eq.(1) is bounded and persists.

4. Periodicity of solutions

In this section we study the existence of prime period two solutions of Eq.(1).

THEOREM 4.1. *Eq.*(1) *has positive prime period two solutions if and only if*

$$(i)\ 4B(\alpha+\gamma) < (A+C-B)(\beta-\alpha-\gamma).$$

Proof. First suppose that there exists a prime period two solution

$$\ldots, p, q, p, q, \ldots$$

of Eq.(1). We will prove that Condition (i) holds.
From Eq.(1), we see that

$$p = \frac{\alpha q + \beta p + \gamma q}{Aq + Bp + Cq},$$

and

$$q = \frac{\alpha p + \beta q + \gamma p}{Ap + Bq + Cp}.$$

Then it is easy to see that p and q are the two positive distinct roots of the quadratic equation

$$t^2 - \frac{(\beta-\alpha-\gamma)}{B}t + \frac{(\beta-\alpha-\gamma)(\alpha+\gamma)}{B(A+C-B)} = 0 \qquad (13)$$

and so

$$\left[\frac{\beta-\alpha-\gamma}{B}\right]^2 - \frac{4(\beta-\alpha-\gamma)(\alpha+\gamma)}{B(A+C-B)} > 0.$$

Since $A+C-B$ and $\beta-\alpha-\gamma$ have the same sign, (i) follows.

Second suppose that Condition (i) is true. We will show that Eq.(1) has a prime period two solution. Assume that p and q are the two roots of Eq.(13). It follows from Condition (i) that p and q are distinct positive real numbers.
Set

$$x_{-2} = p, \ x_{-1} = q \quad \text{and} \quad x_0 = p.$$

It is easy to prove that

$$x_1 = x_{-1} \qquad \text{and} \qquad x_2 = x_0,$$

and it follows by induction that

$$x_{2n} = p \ \text{ and } \ x_{2n+1} = q \quad \text{for all} \quad n \geq -1.$$

Thus Eq.(1) has the positive prime period two solution

$$\ldots, p, q, p, q, \ldots$$

where p and q are the distinct roots of the quadratic equation (13) and the proof is complete.

5. Global Stability of Eq.(1)

In this section we investigate the global asymptotic stability of Eq.(1).

Lemma 1. *For any values of the quotient $\frac{\alpha}{A}$, $\frac{\beta}{B}$, and $\frac{\gamma}{C}$, the function $f(u, v, w)$ defined by Eq.(6) has the monotonicity behavior in at least two of its arguments.*

Proof. The proof follows by some computations and it will be omitted.

Theorem 5.1. *The equilibrium point $\overline{x}$ is a global attractor of Eq.(1) if one of the following statements holds*

$$(1)\ \alpha B \geq \beta A\,,\ \beta C \geq \gamma B \ \text{ and } \ \gamma(2A + B + C) \geq (\alpha + \beta)(B + C). \tag{14}$$

$$(2)\ \alpha C \geq \gamma A\,,\ \beta C \leq \gamma B \ \text{ and } \ \beta(2A + B + C) \geq (\alpha + \gamma)(B + C). \tag{15}$$

$$(3)\ \alpha B \geq \beta A\,,\ \alpha C \leq \gamma A \ \text{ and } \ \beta(A + B + 2C) \geq (\alpha + \gamma)(A + B). \tag{16}$$

$$(4)\ \alpha B \leq \beta A\,,\ \alpha C \geq \gamma A \ \text{ and } \ \gamma(A + 2B + C) \geq (\alpha + \beta)(A + C). \tag{17}$$

$$(5)\ \alpha C \leq \gamma A\,,\ \beta C \geq \gamma B \ \text{ and } \ \alpha(A + 2B + C) \geq (\beta + \gamma)(A + C). \tag{18}$$

$$(6)\ \alpha B \leq \beta A\,,\ \beta C \leq \gamma B \ \text{ and } \ \alpha(A + B + 2C) \geq (\beta + \gamma)(A + B). \tag{19}$$

Proof. Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of Eq.(1) and again let f be a function defined by Eq.(6)

We will prove the theorem when case (1) is true and the proof of the other cases are similar and so we will be omitted.

Assume that (14) is true, then it is easy to see that the function $f(u,v,w)$ is non-decreasing in its first argument and non-increasing in its third argument. Thus from Eq.(1) we see that

$$x_{n+1} = \frac{\alpha x_n + \beta x_{n-1} + \gamma x_{n-2}}{Ax_n + Bx_{n-1} + Cx_{n-2}} \leq \frac{\alpha x_n + \beta x_{n-1} + \gamma(0)}{Ax_n + Bx_{n-1} + C(0)} = \frac{\alpha x_n + \beta x_{n-1}}{Ax_n + Bx_{n-1}}$$

$$= \frac{\alpha x_n}{Ax_n + Bx_{n-1}} + \frac{\beta x_{n-1}}{Ax_n + Bx_{n-1}} \leq \frac{\alpha}{A} + \frac{\beta}{B}.$$

Then

$$x_n \leq \frac{\alpha}{A} + \frac{\beta}{B} = M \quad \text{for all} \quad n \geq 1. \tag{20}$$

$$x_{n+1} = \frac{\alpha x_n + \beta x_{n-1} + \gamma x_{n-2}}{Ax_n + Bx_{n-1} + Cx_{n-2}} \geq \frac{\alpha(0) + \beta x_{n-1} + \gamma(M)}{A(0) + B(M) + C(M)}$$

$$\geq \frac{\beta(0) + \gamma(M)}{B(M) + C(M)} \geq \frac{\gamma}{(B+C)} = m \quad \text{for all} \quad n \geq 1. \tag{21}$$

Then from Eq.(20) and (21) we see that

$$m = \frac{\gamma}{(B+C)} \leq x_n \leq \frac{\alpha}{A} + \frac{\beta}{B} = \frac{\alpha B + \beta A}{AB} \quad \text{for all } n \geq 1.$$

It follows by the Method of Full Limiting Sequences that there exist solutions $\{I_n\}_{n=-\infty}^{\infty}$ and $\{S_n\}_{n=-\infty}^{\infty}$ of Eq.(1) with

$$I = I_0 = \lim_{n\to\infty} \inf x_n \leq \lim_{n\to\infty} \sup x_n = S_0 = S,$$

where

$$I_n, S_n \in [I, S], \quad n = 0, -1, \ldots .$$

It suffices to show that I = S.

Now it follows from Eq.(1) that

$$I = \frac{\alpha I_{-1} + \beta I_{-2} + \gamma I_{-3}}{AI_{-1} + BI_{-2} + CI_{-3}} \geq f(I, I_{-2}, S)$$

$$= \frac{\alpha I + \beta I_{-2} + \gamma S}{AI + BI_{-2} + CS} \geq \frac{(\alpha+\beta)I + \gamma S}{AI + (B+C)S},$$

and so

$$(\alpha+\beta)I+\gamma S-AI^2 \leq (B+C)SI. \tag{22}$$

Similarly, we see from Eq.(1) that

$$S=\frac{\alpha S_{-1}+\beta S_{-2}+\gamma S_{-3}}{AS_{-1}+BS_{-2}+CS_{-3}} \leq f(S,S_{-2},I)$$

$$=\frac{\alpha S+\beta S_{-2}+\gamma I}{AS+BS_{-2}+CI} \leq \frac{(\alpha+\beta)S+\gamma I}{AS+(B+C)I},$$

and so

$$(\alpha+\beta)S+\gamma I-AS^2 \geq (B+C)SI. \tag{23}$$

Therefore it follows from (22) and (23) that

$$(\alpha+\beta)I+\gamma S-AI^2 \leq (\alpha+\beta)S+\gamma I-AS^2$$

if and only if

$$(\alpha+\beta)(S-I)+\gamma(I-S)+A(I+S)(I-S) \geq 0.$$

Thus

$$(I-S)[A(I+S)+\gamma-(\alpha+\beta)] \geq 0,$$

and so

$$I \geq S \quad \text{if} \quad A(I+S)+\gamma-(\alpha+\beta) \geq 0.$$

Now, we know by (14) that

$$(\alpha+\beta)(B+C) \leq \gamma(2A+B+C).$$

Then

$$(\alpha+\beta)(B+C) \leq \gamma(B+C)(\frac{2A}{B+C}+1).$$

Thus

$$(\alpha+\beta) \leq \gamma(\frac{2A}{B+C}+1)=A\frac{2\gamma}{B+C}+\gamma \leq A(I+S)+\gamma,$$

and so it follows that

$$I \geq S.$$

Therefore

$$I=S.$$

This completes the proof.

References

[1] H. El-Metwally, E. A. Grove, G. Ladas, and H.D.Voulov, On the global attractivity and the periodic character of some difference equations, *J. Diff. Equa. Appl.*, **7** (2001), 837–850.

[2] H. El-Metwally, E. A. Grove, G. Ladas, and McGrath, On the difference equation $y_{n+1} = \frac{y_{n-(2k+1)}+p}{y_{n-(2k+1)}+qy_{n-2l}}$, Proceedings of the 6th ICDE, Taylor and Francis, London, 2004.

[3] G. Karakostas, Asymptotic 2-periodic difference equations with diagonally self-invertible responses, *J. Diff. Equa. Appl.* (to appear).

[4] G. Karakostas, Convergence of a difference equation via the full limiting sequences method, Differential Equations and Dynamical System 1 (1993), 289–294.

[5] V.L. Kocic and G. Ladas, Global Behavior of Nonlinear Difference Equations of Higher Order with Applications, Kluwer Academic Publishers, Dordrecht, 1993.

[6] M.R.S. Kulenovic, G. Ladas and N.R. Prokup, On a rational Difference Equation, *Comput. Math. Appl.*, **41** (2001), 671–678.

[7] M.R.S. Kulenovic, G. Ladas and N.R. Prokup, On the Recursive Sequence $x_{n+1} = \frac{\alpha x_n+\beta x_{n-1}}{1+x_n}$, *J. Diff. Equa. Appl.*, **6** (5) (2001), 563–576.

[8] M.R.S. Kulenovic, G. Ladas and W.S. Sizer, On the Recursive Sequence $x_{n+1} = \frac{\alpha x_n+\beta x_{n-1}}{\gamma x_n+\delta x_{n-1}}$, *Math. Sci. Res. Hot-Line* **2** (5) (1998), 1–16.

[9] M. R. S. Kulenovic and G. Ladas, Dynamics of Second Order Rational Difference Equations, Chapman & Hall /CRC, 2002.

Department of Mathematics,
Faculty of Science,
Mansoura University,
Mansoura 35516,
Egypt

E.M. Elabbasy
E-mail: `emelabbasy@mans.edu.eg`

H. El-Metwally
E-mail: `helmetwally@mans.edu.eg`

E.M. Elsayed
E-mail: `emelsayed@mans.edu.eg`