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ON THE SWITCHBACK VERTION OF JOSEPHUS PROBLEM

By

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Abstract. In this paper we will study an alternative row version of Josephus problem. Suppose that $n$numbers $1, 2, \cdots, n$ are arranged in a line from left to right in this order. Starting with number 1, and counting each number from left to right, every second number is eliminated. Subsequently, starting with the right most number of the remains and counting each number in turn in the contrary direction, *i.e.* from right to left, every second number is eliminated. Repeat such a process by alternate changing the order of cunting and eliminating until only one number is left. Denote by $f_t(n)$ the number of the $(n-t+1)$-th element which is removed by the process described above. If n's binary expansion is $\sum_{k=0}^{\infty} 2^k n_k$ $(n_k = 1, 0)$, let us denote $f(n) = \sum_{k=0}^{\infty} 2^{2k+1} n_{2k+1}$. Let $g_t(n)$ be either 0 for $t = 1$, or $(-2)^r \{f(\sigma^r(2n-1)) + f(\sigma^r(n-1)) - 2t + 3\}$ for $t \geq 2$, where $r = \left\lfloor \log_2 \frac{n-1}{t-1} \right\rfloor$ and $\sigma(n) = \lfloor \frac{n}{2} \rfloor$, i.e. $\sigma(n)$ is one-bit shift right of n's binary expansion. In this paper we prove that

$$f_t(n) = f(n-1) + 1 + g_t(n).$$

1. Introduction

In the most general form of the Josephus Problem can be described as follows [3, 5]: Suppose that $n$numbers $1, 2, \cdots, n$ are arranged in a circle, and are numbered from 1 to n. The counting may be clockwise but carried out in the same direction. Starting with number 1, and counting each number in turn around the circle, every $k-$th number is eliminateted. Let $J(n, k, i)$ be the $i-$th element which is removed by the process.

Numerous aspects of the Josephus Problem have treated in the literatures [1, 3, 4, 6, 7, 8, 9]. In [3] explicit non-recursive formula to compute $J(n, 2, i)$ and $J(n, 3, i)$ are given, and explicit upper and lower bounds for $J(n, k, i)$ (where $k \geq 4$) which differ by $2k-2$ are derived. In paticular they obtained the following formula (see [1], [2]):

$$J(n, 2, n-i+1) = 2(n - (2i-1) \cdot 2^{\lfloor log_2 n - log_2 (2i-1) \rfloor}) + 1.$$

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$$\begin{array}{ccccccccc}
1 & 2 & \cancel{3} & 4 & 5 & \cancel{6} & 7 & 8 & \cancel{9} \\
\cancel{1} & 2 & & 4 & \cancel{5} & & 7 & 8 & \\
 & 2 & & 4 & & & \cancel{7} & 8 & \\
 & \cancel{2} & & 4 & & & & 8 & \\
 & & & \cancel{4} & & & & 8 &
\end{array}$$

Figure 1 $f_1^3(9) = 8$.

Now, we would like to propose an alternative row version of Josephus problem as follows.

Problem

Suppose that n numbers $1, 2, \cdots, n$ are arranged in a line from left to right in this order. Starting with number 1, and counting each number in turn from left to right, every $k-$th number is eliminated. Subsequently, starting with the right most number of the remains and counting each number in the contrary direction, *i.e.* from right to left, every $k-$th number is eliminated. Repeat such a process by alternate changing the order of cunting and eliminating until only one number is left. Let $f_t^k(n)$ be the $(n-t+1)-$th element which is determined by the process (see Fig.1.). Determine $f_t^k(n)$.

In this paper we give a non-recursive explicit formula compute to the $f_t^k(n)$ which corresponds to the above formula for $J(n, 2, n-i+1)$ obtained by Halbeisen and Hungerbuhler. By using it, we can fast and easily to calculate $f_t^k(n)$. From now on we denote simply $f_t^2(n)$ by $f_t(n)$.

For a positive integer n, let $n = (b_m b_{m-1} \cdots b_1 b_0)_2$ be n's binary expansion, that is,

$$n = b_m 2^m + b_{m-1} 2^{m-1} + \cdots + b_1 2 + b_0,$$

where each b_i is either 0 or 1 and the leading bit b_m is 1. Denote $\sigma(n) = \lfloor \frac{n}{2} \rfloor$, i.e. $\sigma(n)$ is one-bit shift right of n's binary expansion. For a non-negative integer r, denote $\sigma^r(n) = \sigma(\sigma \cdots \sigma(n))$(the image of n under r-times composition of σ), i.e. $\sigma^r(n)$ is r-bit shift right of n's binary expansion. Moreover, for $n = \sum_{k=0}^{\infty} 2^k n_k \quad (n_k = 1, 0)$, denote

$$f(n) = \sum_{k=0}^{\infty} 2^{2k+1} n_{2k+1}.$$

The following is our result. The proof is given in the later section.

Theorem 1. *Let $g_t(n)$ be either 0 for $t = 1$, or $(-2)^r\{f(\sigma^r(2n-1)) + f(\sigma^r(n-1)) - 2t + 3\}$ for $t \geq 2$, where $r = \left\lfloor \log_2 \frac{n-1}{t-1} \right\rfloor$ and $\sigma(n) = \lfloor \frac{n}{2} \rfloor$, i.e. $\sigma(n)$ is one-bit* *shift right of n's binary expansion. Then*

$$f_t(n) = f(n-1) + 1 + g_t(n).$$

2. Lemmas

We prepare some lemmas to prove Theorem 1. The first lemma is related to $f_t(n)$.

Lemma 2. *The following recursion holds.*

(i) *If* $n < 2t - 1$, *then* $f_t(n) = 2(n-t) + 2$.
(ii) *If* n *is even and* $2t \le n$, *then* $f_t(n) = f_t(n-1)$.
(iii) *If* n *is odd and* $2t - 1 \le n$, *then* $f_t(n) = n + 2 - 2f_t(\sigma(n) + 1)$

Proof. (i) In the first $\lceil \frac{n}{2} \rceil$ steps, even numbers removed in order. Hence if $n < 2t - 1$, then $f_t(n) = 2(n-t) + 2$.
(ii) In the case of $n-1$ elements, all elements with even numbers are eliminated after the first $\frac{n-1}{2}$ steps. The elements left are the same as in the case of n elements. Hence if $2t \le n$, then $f_t(n) = f_t(n-1)$.
(iii) Denote $n = 2k+1$ with $k \ge 0$. From $n \ge 2t - 1$, we have $t \le k + 1$. Since $\sigma(n) = k$, it suffices to show

$$f_t(2k+1) = 2k + 3 - 2f_t(k+1).$$

By eliminating every second element from left to right in the sequence $1, 2, \ldots, n$, $k+1$ elements (*i.e.* $1, 3, \cdots, 2k+1$) are left. Starting with right side element of the remains, and counting every element of them, the number of ith element is $2k + 3 - 2i$. Hence the last equation holds. ∎

Next we note a simply property of $f(n)$.

Lemma 3. *Let* m *be a positive integer. Then*

(i) $2f(m) + f(2m) = 2m$.
(ii) $2f(m) + f(2m+1) = 2m$.

Proof. (i) It is clear from the definition of f.
(ii) It follows from (i) and $f(2m) = f(2m+1)$. ∎

3. Proof of Theorem 1

We divide the proof into two cases.
Case 1. $t = 1$ We prove by induction on n that $f_1(n) = f(n-1) + 1$.
If $n = 1$, then $f_1(1) = 1 = f(0) + 1$, and we are done.
Suppose that $n > 1$. If n is odd, by Lemma 3,

$$\begin{aligned} f_1(n) &= n + 2 - 2f_1(\sigma(n) + 1) \\ &= n + 2 - 2(f(\sigma(n)) + 1) && \text{by induction hypothesis} \\ &= n - 2f(\sigma(n)) \\ &= n - (n - 1 - f(n)) && \text{by Lemma 2 (ii) and } m = 2m + 1 \\ &= f(n) + 1 \\ &= f(n-1) + 1 && \text{since } n \text{ is odd.} \end{aligned}$$

If n is even, then we can use Lemma 1(ii) since $t \le \frac{n}{2}$. Then

$$\begin{aligned} f_1(n) &= f_1(n-1) \\ &= f(n-2) + 1 && \text{by induction hypothesis} \\ &= f(n-1) + 1. \end{aligned}$$

Case 2. $t \ge 1$ We proceed by induction on n.
Case 2.1. $r = 0$. In this case, we have $\frac{n-1}{t-1} < 2$ since $r = 0$, and hence $n < 2t - 1$.
By Lemma 1, $f_t(n) = 2(n-t) + 2$. On the other hand,

$$\begin{aligned} &f(n-1) + 1 + g_t(n) \\ &\quad = f(n-1) + 1 + (f(2n-1) + f(n-1) - 2t + 3) \\ &\quad = 1 + 2(n-1) - 2t + 3 && \text{by Lemma 2 (ii) with } m = n - 1 \\ &\quad = 2(n-t) + 2. \end{aligned}$$

Case 2.2. $r \ge 1$. In this case, we have $\frac{n-1}{t-1} \ge 2$. If n is odd, from Lemma 3 and induction hypothesis, then we have

$$\begin{aligned} f_t(n) &= n + 2 - 2f_t(\sigma(n) + 1) \\ &= n + 2 - 2\{f(\sigma(n)) + 1 \\ &\quad + (-2)^{r-1}\{f(\sigma^{r-1}(2\sigma(n) + 1)) + f(\sigma^{r-1}(\sigma(n))) - 2t + 3\}\}, \end{aligned}$$

by using the following fact:

$$r = \left\lfloor \log_2 \frac{\sigma(n)}{t-1} \right\rfloor = \left\lfloor \log_2 \frac{n-1}{t-1} \right\rfloor - 1 = r - 1.$$

On the other hand, $f(\sigma^{r-1}(2\sigma(n)+1)) = f(\sigma^{r-1}(n)) = f(\sigma^r(2n-1))$, since n is odd.
By using that $\sigma^r(n) = \sigma^r(n-1)$ and $n - 2f(\sigma(n)) = f(n)+1 = f(n-1)+1$, we have $f_t(n) = f(n-1)+1+g_t(n)$.
Next we assume that n is even. Since $n \geq 2t-1$, $f_t(n) = f_t(n-1)$ by Lemma 1. Thus by induction hypothesis we have

$$\begin{aligned} f_t(n) &= f_t(n-1) \\ &= f(n-2)+1+(-2)^s\{f(\sigma^s(2n-3)) + f(\sigma^s(n-2)) - 2t+3\}, \end{aligned}$$

where $s = \left\lfloor \log_2 \frac{n-2}{t-1} \right\rfloor$.
By putting $r = \left\lfloor \log_2 \frac{n-1}{t-1} \right\rfloor$, we have $s = r$, since n is even.
On the other hand,

$$\begin{aligned} &f(n-2) = f(n-1), \text{ and} \\ &f(\sigma^r(2n-3)) = f(\sigma^r(2n-1)),\ \sigma^r(n-2) = \sigma^r(n-1), \text{ since } r \geq 1. \end{aligned}$$

Hence we have

$$\begin{aligned} f_t(n) &= f(n-1)+1+(-2)^r\{f(\sigma^r(2n-3)) + f(\sigma^r(n-1)) - 2t+3\} \\ &= f(n-1)+1+g_t(n). \qquad \blacksquare \end{aligned}$$

4. Example

Let us calculate $f_t(n)$ for $n = 321$ and $t = 24$. We see that

$$\begin{aligned} n &= 321 = (101000001)_2 \\ f(n-1) &= f(320) = f((101000000)_2) = 0, \\ r &= \left\lfloor \log_2 \frac{320}{23} \right\rfloor = 3. \end{aligned}$$

On the otherhand,

$$\begin{aligned} &f(\sigma^r(2n-1)) + f(\sigma^r(n-1)) - 2t + 3 \\ &\qquad = f(\sigma^3(641)) + f(\sigma^3(320)) - 48 + 3 \\ &\qquad = f((1010000)_2) + f((101000)_2 - 45 \\ &\qquad = 0 + (2^5 + 2^3) - 45 \\ &\qquad = -5, \end{aligned}$$

and

$$\begin{aligned} g_t(n) &= (-2)^r\{f(\sigma^r(2n-1)) + f(\sigma^r(n-1)) - 2t + 3\}. \\ &= (-2)^3(-5) \\ &= 40. \end{aligned}$$

By Theorem 1, $f_t(n) = f(n-1) + 1 + g_t(n)$, and hence we have

$$f_{24}(321) = 0 + 1 + 40 = 41.$$

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