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THE CLASSIFICATION OF ORBITS BY A NATURAL ACTION OF CERTAIN REDUCTIVE LINEAR GROUPS

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Abstract. When $n \geq 2m$, the orbit decomposition of $\mathbb{C}^n \otimes \mathbb{C}^m$ by the natural action of $SO_n\mathbb{C} \times GL_m\mathbb{C}$ was studied more than twenty years ago. In this article, the orbit decomposition is given for any $n, m \geq 1$. Then it turns out that there is an orbit which is not appeared in the literature when $n = 2m$.

0. Introduction

In an example [12, Example 9.2], the orbit decomposition of $\mathbb{C}^n \otimes \mathbb{C}^m$ by the tensor product representation of the identity representations $\square$'s of $SO_n\mathbb{C}$ and $GL_m\mathbb{C}$ was studied for $n \geq 2m$ with $m \geq 1$:

$$(SO_nGL_m, M_{n,m}\mathbb{C}) := (SO_n\mathbb{C} \times GL_m\mathbb{C}, \square \otimes \square, \mathbb{C}^n \otimes \mathbb{C}^m).$$

However, by the orbit decomposition of the secant varieties of the adjoint varieties [10] or the study of hyperdeterminants [6, 7], it turns out that the classification of orbits given in an example [12, Example 9.2] should be modified if $(n, m) = (4, 2)$, where the orbit corresponding to a nilpotent orbit of $\mathfrak{so}_8\mathbb{C}$ attached to a very even partition of the integer 8 is not appeared in [12, Example 9.2] (see Example 4.3).

This paper gives the orbit decomposition of $(SO_nGL_m, M_{n,m}\mathbb{C})$ for any $n \geq 1$ and $m \geq 1$, concretely (see Theorems 1.4, 2.3, 3.1, Corollary 2.4). Then it turns out that there is a unique orbit $S_{n/2,0}$ of $O_n\mathbb{C} \times GL_m\mathbb{C}$ that splits into two orbits $S^I_{n/2,0}$, $S^{II}_{n/2,0}$ of $SO_n\mathbb{C} \times GL_m\mathbb{C}$ for even $n \leq 2m$ (see Proposition 2.1, Corollary 3.2 with Figures 1, 2, 3, 4, Remark 3.3). And the orbit $S^{II}_{n/2,0}$ is not appeared

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in [12, Example 9.2] when $n = 2m$. Moreover, $S^{II}_{n/2,0}$ exists apart from $S^{I}_{n/2,0}$ also when n is even and $n < 2m$. It seems that the discussion in [12, Example 9.2] should be considered as the orbit decomposition by the natural action of $O_n\mathbb{C} \times GL_m\mathbb{C}$, but not the one by the natural action of $SO_n\mathbb{C} \times GL_m\mathbb{C}$. And the difference between them shows up when n is even (see Theorem 1.4 (ii), Proposition 1.5).

The method of this paper except §4 is as same as the literature [12, Example 9.2]. The contents are organized as follows: In §1, a rough classification of orbits is given for all $n \geq 1$ and $m \geq 1$ after [12, Example 9.2], which gives a complete classification when n is odd. In §2, a complete classification is given when n is even (Theorem 2.3). In §3, the closure relation in the set of all orbits is studied (Theorem 3.1). When $n \geq 2m$ or $m = 1, 2$, the Hasse diagrams are also given (Corollary 3.2 with Figures 1, 2, 3, 4). In §4 (Appendix), when $m = 2$, the present orbit decomposition is compared with other classifications of orbits. In particular, it turns out that the present orbit decomposition is finer than the classification of nilpotent orbits (Examples 4.5, 4.6).

1. Basic results on the orbit decomposition

Let n and m be positive integers, and $M_{n,m}\mathbb{C}$ the set of all $n \times m$ complex matrices. Put $M_n\mathbb{C} := M_{n,n}\mathbb{C}$ with the identity matrix I_n of degree n, and $\mathbb{C}^n := M_{n,1}\mathbb{C}$. For $x, y \in \mathbb{C}^n$, put $(x|y) := {}^t xy$. And put $\mathfrak{gl}_n\mathbb{C} := M_n\mathbb{C}$ with the commutator as a Lie algebra, $\mathfrak{sl}_n\mathbb{C} := \{X \in \mathfrak{gl}_n\mathbb{C} | \text{ trace } X = 0\}$, $\mathfrak{so}_n\mathbb{C} := \{X \in \mathfrak{gl}_n\mathbb{C} | \ {}^tX = -X\}$; $GL_n\mathbb{C} := \{g \in M_n\mathbb{C} | \det(g) \neq 0\}$, $SL_n\mathbb{C} := \{g \in M_n\mathbb{C} | \det(g) = 1\}$, $O_n\mathbb{C} := \{g \in M_n\mathbb{C} | \ {}^tgg = I_n\}$;

$$SO_n\mathbb{C} := O_n\mathbb{C} \cap SL_n\mathbb{C}.$$

Denote by G_n the subgroup $O_n\mathbb{C}$ or $SO_n\mathbb{C}$ of $GL_n\mathbb{C}$ with the identity representations: $\square : G_n \to GL_n\mathbb{C}; g_1 \mapsto g_1$ and $\square : GL_m\mathbb{C} \to GL_m\mathbb{C}; g_2 \mapsto g_2$. Then the tensor product representation $\rho := \square \otimes \square$ can be identified with the natural action of the product group $G_n \times GL_m\mathbb{C}$ on $M_{n,m}\mathbb{C}$ as follows:

$$\rho(g_1, g_2)X = g_1 X \ {}^t g_2, \ (g_1 \in G_n, \ g_2 \in GL_m\mathbb{C}, \ X \in M_{n,m}\mathbb{C}).$$

In general, put $G_nGL_m := \rho(G_n \times GL_m\mathbb{C})$, that is,

$$O_nGL_m = \rho(O_n\mathbb{C} \times GL_m\mathbb{C}), \quad SO_nGL_m = \rho(SO_n\mathbb{C} \times GL_m\mathbb{C}).$$

For $X \in M_{n,m}\mathbb{C}$, clearly $\text{rank}(X)$ and $\text{rank}({}^tXX)$ are absolute invariants of G_nGL_m. Let $\mathcal{O}(G_nGL_m, M_{n,m}\mathbb{C})$ be the set of all G_nGL_m-orbits in $M_{n,m}\mathbb{C}$. For any integers ν, μ such that $n, m \geq \nu \geq \mu \geq 0$, put

$$S_{\nu,\mu} := \{X \in M_{n,m}\mathbb{C} | \ \text{rank}(X) = \nu, \ \text{rank}({}^tXX) = \mu\}.$$

Then $M_{n,m}\mathbb{C} = \bigsqcup_{n,m\geq\nu\geq\mu\geq 0} S_{\nu,\mu}$, where each $S_{\mu,\nu}$ is an empty set or consists of some orbits of G_nGL_m. Put

$$T_{\nu,\mu} := \{X \in S_{\nu,\mu} \mid X = [X'|O] \text{ for some } X' \in M_{n,\nu}\mathbb{C}\},$$

where $O := O_{n,m-\nu}$ is an $n \times (m-\nu)$ zero matrix (or a blank when $m - \nu = 0$). When $\nu = 0$, X' denotes a blank and put $M_{n,0}\mathbb{C} := \emptyset$, so that $T_{0,0} := \{O_{n,m}\}$.

PROPOSITION 1.1. *Let n, m be positive integers, and ν, μ be integers such that $n, m \geq \nu \geq \mu \geq 0$. Then:*

(1) *If $X \in S_{\nu,\mu}$, then there exists $g_2 \in GL_m\mathbb{C}$ such that $\rho(I_n, g_2)X \in T_{\nu,\mu}$. In particular, each orbit of G_nGL_m on $S_{\nu,\mu}$ contains an element of $T_{\nu,\mu}$.*

(2) *For $Y_1, Y_2 \in T_{\nu,\mu}$ with $\nu < m$, assume that $g_1Y_1{}^tg_2 = Y_2$ for some $(g_1, g_2) \in G_n \times GL_m\mathbb{C}$. Then there exists $g_2' \in GL_\nu\mathbb{C}$ such that*

$$g_2'' := \mathrm{diag}(g_2', I_{m-\nu}) \in GL_m\mathbb{C}$$

satisfies that $g_1Y_1{}^tg_2'' = Y_2$.

(3) *If $n \leq m$, then the following orbit correspondence is bijective:*

$$\begin{aligned} &\mathcal{O}(G_nGL_n, M_n\mathbb{C}) \to \mathcal{O}(G_nGL_m, M_{n,m}\mathbb{C});\\ &\quad (G_nGL_n)\cdot X \mapsto (G_nGL_m)\cdot[X|O_{n,m-n}]. \end{aligned}$$

Proof. (1) If $\nu = 0$, then $X = O_{n,m}$, so that $g_2 := I_m \in GL_m\mathbb{C}$ satisfies the assertion. Assume that $\nu \neq 0$, and put $X = [x_1, \ldots, x_m] \in S_{\nu,\mu}$. Then there exist $i_i, \ldots, i_\nu \in \{1, \ldots, m\}$ such that each x_i is a linear combination of $x_{i_1}, \ldots, x_{i_\nu}$, where the coefficients of the linear combinations are given by a series of elementary column operations. Since the elementary operations are realized as the right matrix multiplication of the corresponding invertible matrixes, there exists $g_2 \in GL_m\mathbb{C}$ such that

$$X' := \rho(I_n, g_2)X = X^tg_2 = [x_{i_1}, \ldots, x_{i_\nu}|O],$$

which is contained in $T_{\nu,\mu}$, as required.

(2) For $i = 1, 2$, take $Y_1', Y_2' \in M_{n,\nu}\mathbb{C}$ such that $Y_i = [Y_i'|O]$. For $(g_1, g_2) \in G_n \times GL_m\mathbb{C}$ such as $g_1Y_1{}^tg_2 = Y_2$, take $g_{11} \in M_\nu\mathbb{C}, g_{12} \in M_{r,n-\nu}\mathbb{C}, g_{21} \in M_{n-\nu,\nu}\mathbb{C}, g_{22} \in M_{n-\nu,n-\nu}\mathbb{C}$ such that

$$g_2 = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}.$$

Then $g_1 Y_1'^t g_{11} = Y_2'$, where $\mathrm{rank}(Y_1') = \mathrm{rank}(Y_2') = \nu$, so that $g_{11} \in GL_\nu\mathbb{C}$. Hence, the assertion holds for $g_2' := g_{11}$.

(3) The correspondence is surjective by (1), and injective by (2). □

For all integers ν, μ such as $m \geq \nu \geq \mu \geq 0$ and $n \geq 2\nu - \mu$, one has that $2\nu - \mu \geq 0$ and $n \geq \nu$ by $\nu \geq \mu \geq 0$ (and $n - \nu \geq n - 2\nu + \mu \geq 0$), and put

$$X'_{\nu,\mu} := \left[\begin{array}{c|c} I_\mu & O_{\mu,\ \nu-\mu} \\ \hline O_{\nu-\mu,\ \mu} & I_{\nu-\mu} \\ \hline O_{\nu-\mu,\ \mu} & \sqrt{-1}I_{\nu-\mu} \\ \hline O & O \end{array}\right] \in M_{n,\nu}\mathbb{C},$$

$$X''_{\nu,\mu} := \left[\begin{array}{c|c} O_{\mu,\ \nu-\mu} & O \\ \hline \frac{1}{2}I_{\nu-\mu} & O \\ \hline -\frac{\sqrt{-1}}{2}I_{\nu-\mu} & O \\ \hline O & I_{n-2\nu+\mu} \end{array}\right] \in M_{n,n-\nu}\mathbb{C},$$

$X_{\nu,\mu} := [X'_{\nu,\mu}|O_{n,\ m-\nu}] \in M_{n,m}\mathbb{C}$, $\tilde{X}_{\nu,\mu} := [X'_{\nu,\mu}|X''_{\nu,\mu}] \in M_n\mathbb{C}$ and

$$\tilde{Z}_{\nu,\mu} := \left[\begin{array}{c|c|c|c} I_\mu & O_{\mu,\ \nu-\mu} & O_{\mu,\ \nu-\mu} & O \\ \hline O_{\nu-\mu,\ \mu} & O_{\nu-\mu,\ \nu-\mu} & I_{\nu-\mu} & O \\ \hline O_{\nu-\mu,\ \mu} & I_{\nu-\mu} & O_{\nu-\mu,\ \nu-\mu} & O \\ \hline O & O & O & I_{n-2\nu+\mu} \end{array}\right] \in M_n\mathbb{C},$$

where $O_{i,j}$ (or O) is the $i \times j$ zero matrix (or a blank when $i = 0$ or $j = 0$), and I_k is the $k \times k$ identity matrix (or a blank when $k = 0$).

Then $X'_{\nu,\mu} \in M_{n,\nu}\mathbb{C}$, $X''_{\nu,\mu} \in M_{n,n-\nu}\mathbb{C}$, $X_{\nu,\mu} \in T_{\nu,\mu}$ and $\tilde{X}_{\nu,\mu}, \tilde{Z}_{\nu,\mu} \in GL_n\mathbb{C}$ such that ${}^t\tilde{X}_{\nu,\mu}\tilde{X}_{\nu,\mu} = \tilde{Z}_{\nu,\mu}$. If $\nu = 0$, then $X_{\nu,\mu} = X_{0,0} = O_{n,m} \in M_{n,m}\mathbb{C}$, since $X'_{\nu,\mu} = X'_{0,0}$ is a blank. Note that $\tilde{Z}_{0,0} = I_n$. And that $X_{\nu,\mu} \in T_{\nu,\mu}$.

PROPOSITION 1.2. *Let n, m be positive integers, and ν, μ be integers such that $m \geq \nu \geq \mu \geq 0$. Then:*

(1) If $n \geq 2\nu - \mu$, then $T_{\nu,\mu} \neq \emptyset$.

(2) If $T_{\nu,\mu} \neq \emptyset$, then $n \geq 2\nu - \mu$.

(3) If $n \geq 2\nu - \mu$ and $\nu \neq 0$, for $Y = [Y_1|O_{n,m-\nu}] \in T_{\nu,\mu}$ with $Y_1 \in M_{n,\nu}\mathbb{C}$, there exist $g \in GL_\nu\mathbb{C}$ and $\tilde{Y} \in GL_n\mathbb{C}$ such as $\tilde{Y} = [Y_1 g|*]$ and ${}^t\tilde{Y}\tilde{Y} = \tilde{Z}_{\nu,\mu}$. If moreover $\mu = 0$, then $g \in GL_\nu\mathbb{C}$ can be taken as $g = I_\nu$.

(4) If $\nu = 0$, then $T_{\nu,\mu} = \{O\}$ and ${}^t\tilde{Y}\tilde{Y} = \tilde{Z}_{\nu,\mu}$ for $\tilde{Y} = I_n \in GL_n\mathbb{C}$.

Proof. (1) If $m \geq \nu \geq \mu \geq 0$ and $n \geq 2\nu - \mu$, then there exists $X_{\nu,\mu} \in T_{\nu,\mu}$, which is well-defined in the above construction.

(2, 3, 4) If $\nu = 0$, then $\mu = 0$, so that $T_{\nu,\mu} = T_{0,0} = \{O\} \neq \emptyset$ and $n \geq 0 = 2\nu - \mu$. In this case, Y_1 is a blank, $\tilde{Y} = I_n$ and ${}^t\tilde{Y}\tilde{Y} = \tilde{Z}_{0,0}$, as required.

Assume that $\nu \neq 0$ and $T_{\nu,\mu} \neq \emptyset$. Then there exists $Y = [Y_1|\ O] \in T_{\nu,\mu}$ with $Y_1 \in M_{n,\nu}\mathbb{C} \neq \emptyset$. Because of $\mathrm{rank}({}^tY_1Y_1) = \mu$ and the Sylvester's law, there exists $g \in GL_\nu\mathbb{C}$ such that $Y_2 := Y_1 g$ satisfies

$$ {}^tY_2Y_2 = \mathrm{diag}(I_\mu, O) \in M_\nu\mathbb{C} \neq \emptyset. $$

Take $y_1, \ldots, y_\nu \in \mathbb{C}^n$ such that $[y_1, \ldots, y_\nu] = Y_2$. Then

$$ (y_i|y_j) = \delta_{ij},\ (y_i|y_k) = 0\ (i = 1, \cdots, \nu;\ j = 1, \cdots, \mu;\ k = \mu+1, \cdots, \nu). $$

Let V_μ be the linear subspace of $\mathbb{C}^n$ generated by $y_1, ..., y_\mu$ when $\mu \neq 0$ (When $\mu = 0$, put $V_\mu := \{0\}$). Put $V' := \{x \in \mathbb{C}^n|\ (x|y) = 0\ (y \in V_\mu)\}$. Since $(*|*)$ is non-degenerate on V_μ, one has that $\mathbb{C}^n = V_\mu \oplus V'$, so that

$$ n - \mu = \dim_{\mathbb{C}} V' \geq 0, $$

and that $(*|*)$ is non-degenerate on V'.

If $\nu = \mu$, then $n \geq \mu = 2\nu - \mu$. In this case, by Sylvester's law, there exist $w_i \in V'$ $(1 \leq i \leq n - \mu)$ such that $(w_i|w_j) = \delta_{ij}$. For $Y^* := [w_1, \cdots, w_{n-\mu}] \in M_{n,n-\nu}\mathbb{C}$, put $\tilde{Y} := [Y_2|Y^*] \in M_n\mathbb{C}$. Then ${}^t\tilde{Y}\tilde{Y} = \tilde{Z}_{\nu,\mu} = \tilde{Z}_{\nu,\nu}$, as required.

Assume that $\nu > \mu$. According to $\mathrm{rank}([y_1, \cdots, y_\nu]) = \mathrm{rank}(Y_2) = \nu$, one has that $\mathrm{rank}({}^t[y_{\mu+1}, \ldots, y_\nu]) = \nu - \mu$, so that

$$ f : \mathbb{C}^n(= V_\mu \oplus V') \to \mathbb{C}^{\nu-\mu}; y' \mapsto {}^t[y_{\mu+1}, \cdots, y_\nu]y' $$

is a surjective linear mapping. And $\ker(f) \supseteq V_\mu$ by $\{y_{\mu+1}, \cdots, y_\nu\} \subseteq V'$. Let $e_1, \cdots, e_{\nu-\mu}$ be the standard basis of $\mathbb{C}^{\nu-\mu}$. For $j \in \{\mu+1, \ldots, \nu\}$, there exists $y'_j \in V'$ such that ${}^t[y_{\mu+1}, \ldots, y_\nu]y'_j = e_{j-\mu}$, so that $(y_j|y'_k) = \delta_{jk}$ for all $j, k \in \{\mu+1, \ldots, \nu\}$. For $y^*_j := y'_j + \sum_{k=\mu+1}^{\nu} a_{j,k}y_k \in V'$ with $a_{j,k} \in \mathbb{C}$, one has $(y^*_j|y^*_k) = (y'_j|y'_k) + a_{j,k} + a_{k,j}$. Take $a_{j,k} = a_{k,j} := -\frac{1}{2}(y'_j|y'_k)$. Then

$$ (y^*_j|y^*_k) = 0 = (y_j|y_k) \ \text{ and } \ (y_j|y^*_k) = \delta_{jk} $$

for all $j, k \in \{\mu+1, \ldots, \nu\}$. Hence, $(*|*)$ is non-degenerate on a $2(\nu-\mu)$-dimensional subspace $W := \mathbb{C}y_{\mu+1} \oplus ... \oplus \mathbb{C}y_\nu \oplus \mathbb{C}y^*_{\mu+1} \oplus ... \oplus \mathbb{C}y^*_\nu$ of V', so that $V' = W \oplus W'$ for $W' := \{x \in V'|\ (x|y) = 0\ (y \in W)\}$. In particular,

$$ n - 2\nu + \mu = (n-\mu) - 2(\nu-\mu) = \dim_{\mathbb{C}} V' - \dim_{\mathbb{C}} W = \dim_{\mathbb{C}} W' \geq 0. $$

And $(*|*)$ is non-degenerate on W'. By Sylvester's law, there exists an orthonormal basis of W', that is, $w_i \in W'$ $(1 \leq i \leq n - 2\nu + \mu)$ such that $(w_i|w_j) = \delta_{i,j}$. For $Y^* := [y^*_{\mu+1}, ..., y^*_\nu, w_1, ..., w_{n-2\nu+\mu}] \in M_{n,n-\nu}\mathbb{C}$, take $\tilde{Y} := [Y_2|Y^*] \in M_n\mathbb{C}$. Then ${}^t\tilde{Y}\tilde{Y} = \tilde{Z}_{\nu,\mu}$, as required. □

PROPOSITION 1.3. *Let n, m be positive integers, and ν, μ be integers such that $m \geq \nu \geq \mu \geq 0$. Then:*

(1) $S_{\nu,\mu} \neq \emptyset \Leftrightarrow T_{\nu,\mu} \neq \emptyset \Leftrightarrow n \geq 2\nu - \mu$.

(2) *If $n \geq 2\nu - \mu$, then $S_{\nu,\mu} = O_n GL_m \cdot X_{\nu,\mu}$.*

(3) *If $n > 2\nu - \mu$, then $S_{\nu,\mu} = SO_n GL_m \cdot X_{\nu,\mu}$.*

(4) *If $n \geq 2\nu - \mu$ and $\mu > 0$, then $S_{\nu,\mu} = SO_n GL_m \cdot X_{\nu,\mu}$.*

Proof. (1) The first equivalence comes from Proposition 1.1 (1). And the second equivalence comes from Proposition 1.2.

(2) Assume that $n \geq 2\nu - \mu$. It is claimed that for any $Y \in S_{\nu,\mu}$ there exists $\alpha \in O_n GL_m$ such that $\alpha \cdot X_{\nu,\mu} = Y$. Because of Proposition 1.1 (1), it can be assumed that $Y = [Y_1|O] \in T_{\nu,\mu}$. If $\nu = 0$, then $Y = X_{\nu,\mu} = O$, as required. Assume that $\nu \neq 0$. By Proposition 1.2, there exists $\tilde{Y}$ such that

$$ {}^t\tilde{Y}\tilde{Y} = {}^t\tilde{X}_{\nu,\mu}\tilde{X}_{\nu,\mu}, $$

and that there exists $g \in GL_\nu\mathbb{C}$ such that $\tilde{Y} = [Y_1 g|*]$. Put $g_0 := \tilde{Y}\tilde{X}_{\nu,\mu}^{-1}$. Then $g_0 \in O_n\mathbb{C}$ by the above equation. And $g_0\tilde{X}_{\nu,\mu} = \tilde{Y}$. In particular,

$$ g_0 X'_{\nu,\mu} = Y_1 g \text{ and } g_0 X_{\nu,\mu} = [Y_1 g | O_{n,m-\nu}]. $$

Hence, $g_0 X_{\nu,\mu}\mathrm{diag}(g^{-1},\ I_{m-\nu}) = [Y_1|O_{n,m-\nu}] = Y$, as required.

(3) Assume that $n > 2\nu - \mu$. Then

$$ X_{\nu,\mu} = \left[\frac{*}{O_{n-2\nu+\mu,\ m}}\right], $$

where the two sizes of $O_{n-2\nu+\mu,\ m}$ are positive, so that $O_{n-2\nu+\mu,\ m}$ do exist. If $n = 1$, then $X_{\nu,\mu} = O_{n-2\nu+\mu,\ m} = SO_n GL_m \cdot X_{\nu,\mu}$. Assume that $n \geq 2$. Then $I_{n-1,1} := \mathrm{diag}(I_{n-1}, -1) \in O_n\mathbb{C}\backslash SO_n\mathbb{C}$ satisfies $I_{n-1,1}X_{\nu,\mu} = X_{\nu,\mu}$. For any $h_0 \in O_n GL_m \backslash SO_n GL_m$, put $h_1 := h_0 \circ \rho(I_{n-1,1}, I_m)$. Then $h_1 \in SO_n GL_m$ such that $h_1 X_{\nu,\mu} = h_0\rho(I_{n-1,1}, I_m)X_{\nu,\mu} = h_0 X_{\nu,\mu}$. Hence, one has the result by (2).

(4) For $\mu > 0$, put $I_\mu^1 := \mathrm{diag}(-1, I_{\mu-1})$. Note that $I_{\mu-1}$ is a blank when $\mu = 1$. Put $g_1 := \mathrm{diag}(I_\mu^1, I_{n-\mu}) \in O_n\mathbf{C}\backslash SO_n\mathbb{C}$ and $g_2 := \mathrm{diag}(I_\mu^1, I_{m-\mu}) \in GL_m\mathbb{C}$. Then $\rho(g_1, g_2)X_{\nu,\mu} = X_{\nu,\mu}$ by virtue of

$$ X_{\nu,\mu} = \left[\begin{array}{c|c} I_\mu & O \\ \hline O & * \end{array}\right], $$

where I_μ do exist. For $h_0 \in O_n GL_m \backslash SO_n GL_m$, put $h_1 := h_0 \circ \rho(g_1, g_2)$. Then $h_1 \in SO_n GL_m$ such that $h_1 X_{\nu,\mu} = h_0(\rho(g_1, g_2)X_{\nu,\mu}) = h_0 X_{\nu,\mu}$. Hence, one has the result by (2). $\square$

THEOREM 1.4. *Let n, m be positive integers. Then:*

(1) $\mathcal{O}(O_nGL_m, M_{n,m}\mathbb{C}) = \{S_{\nu,\mu} \mid m \geq \nu \geq \mu \geq 0,\ n \geq 2\nu - \mu\}$.

(2) *If $n > 2m$ or n is odd, then*

$$\mathcal{O}(SO_nGL_m, M_{n,m}\mathbb{C}) = \mathcal{O}(O_nGL_m, M_{n,m}\mathbb{C}).$$

(3) *If $n \geq 2m$, then*

$$\{(\nu,\mu) \mid m \geq \nu \geq \mu \geq 0,\ n \geq 2\nu - \mu\} = \{(\nu,\mu) \mid m \geq \nu \geq \mu \geq 0\},$$

which consists of $(m+1)(m+2)/2$-elements.

Proof. (1) By virtue of Proposition 1.3 (1), one has that

$$M_{n,m}\mathbb{C} = \bigsqcup_{m \geq \nu \geq \mu \geq 0,\ n \geq 2\nu - \mu} S_{\nu,\mu},$$

where each $S_{\nu,\mu}$ is one orbit of O_nGL_m by Proposition 1.3 (2). And the distinct (ν,μ) gives a distinct orbit, because (ν,μ) is an invariant of the action.

(2) Assume that n is odd or $n > 2m$. For any (ν,μ) such that $m \geq \nu \geq \mu \geq 0$ and $n \geq 2\nu - \mu$, one has that $\mu > 0$ or $n > 2\nu - \mu$. In fact, if $\mu = 0$ and $n = 2\nu - \mu$, then $n = 2\nu \leq 2m$, which contradicts with the assumption that n is odd or $n > 2m$. Hence, for $m \geq \nu \geq \mu \geq 0$ and $n \geq 2\nu - \mu$, each $S_{\nu,\mu}$ is non-empty and one orbit of SO_nGL_m by Proposition 1.3 (3) and (4), as required.

(3) Assume that $n \geq 2m$ and $m \geq \nu \geq \mu \geq 0$. Then

$$n - 2\nu + \mu \geq 2m - 2\nu + \mu = 2(m - \nu) + \mu \geq 0,$$

so that $\{(\nu,\mu) \mid m \geq \nu \geq \mu \geq 0,\ n \geq 2\nu - \mu\} = \{(\nu,\mu) \mid m \geq \nu \geq \mu \geq 0\}$. The cardinality of this set is equal to: $\sum_{\nu=0}^{m}(\sum_{\mu=0}^{\nu} 1) = \sum_{\nu=0}^{m}(\nu+1) = (m+1)(m+2)/2$, as required. □

Note that Theorem 1.4 (2) with an odd n also follows from the following proposition with an odd $n \geq 1$ (cf. [12, Example 9.2, ℓ.14]).

PROPOSITION 1.5. *For any $n \geq 1$ and any $m \geq 1$, one has that:*

$$[O_nGL_m : SO_nGL_m] = \begin{cases} 2 & (\text{if } n \text{ is even}) \\ 1 & (\text{otherwise}) \end{cases}$$

Proof. Note that $[O_n\mathbb{C} \times GL_m\mathbb{C} : SO_n\mathbb{C} \times GL_m\mathbb{C}] = 2$. Hence,

$$[O_nGL_m : SO_nGL_m] \leq 2.$$

If n is odd, by virtue of $O_n\mathbb{C} = SO_n\mathbb{C} \cup (-I_n)SO_n\mathbb{C}$ and $\rho(-I_n, I_m) = \rho(I_n, -I_m)$ with $-I_m \in GL_m\mathbb{C}$, one has that $O_nGL_m = SO_nGL_m$.

If n is even, it is claimed that any $(g_1, g_2) \in (O_n\mathbb{C}\backslash SO_n\mathbb{C}) \times GL_m\mathbb{C}$ satisfies that $\rho(g_1, g_2) \notin SO_nGL_m$. In fact, assume that $\rho(g_1, g_2) \in SO_nGL_m$. Then there exist $(g_3, g_4) \in SO_n\mathbb{C} \times GL_m\mathbb{C}$ such that $\rho(g_1, g_2) = \rho(g_3, g_4)$. Hence, $\rho(g_1g_3^{-1}, g_2g_4^{-1}) = \mathrm{id}_{M_{n,m}\mathbb{C}}$, so that $AXB = X$ for all $X \in M_{n,m}\mathbb{C}$ with $A := g_1g_3^{-1}, B := {}^t(g_2g_4^{-1})$.

Take $X = E_{i,j}$, the matrix whose (k, ℓ)-component is equal to $\delta_{ki}\delta_{\ell j}$. Put $A = \sum_{k,\ell} a_{k\ell}E_{k,\ell}$ and $B = \sum_{k',\ell'} b_{k'\ell'}E_{k',\ell'}$. Then

$$AE_{i,j}B = \sum_k a_{ki}b_{j\ell'}E_{k,\ell'},$$

which should be equal to $E_{i,j}$. By $B \in GL_m\mathbb{C}$, $a_{ki} = 0$ if $k \neq i$. By $A \in GL_n\mathbb{C}$, $b_{j\ell'} = 0$ if $\ell' \neq j$. Hence, $A = \mathrm{diag}(a_{11}, \ldots, a_{nn})$ and $B = \mathrm{diag}(b_{11}, \ldots, b_{mm})$. Considering $AE_{i,j}B = E_{i,j}$ again, one has that $(A, B) = (cI_n, c^{-1}I_m)$, where $c = \pm 1$ by $A \in O_n\mathbb{C}$. If $c = 1$, then $(g_1, g_2) = (g_3, g_4) \in SO_n\mathbb{C} \times GL_m\mathbb{C}$, that contradicts with the choice of (g_1, g_2). If $c = -1$, then $(-g_1, -g_2) = (g_3, g_4)$ and $-g_1 = g_3 \in SO_n\mathbb{C}$. Hence, $1 = \det(g_3) = \det(-g_1) = (-1)^n\det(g_1) = (-1)^{n+1}$, by virtue of $g_1 \in O_n\mathbb{C}\backslash SO_n\mathbb{C}$, so that n is odd, that contradicts with the choice of n. □

By virtue of Proposition 1.5, it turns out that the sentence at the lines 12–14 in [12, Example 9.2], "*and hence there exists $g_1 \in O(n)$ satisfying ... By the action of $GL(m)$, we may assume that $g_1 \in SO(n)$,* " should be modified when n is even.

2. Main results on the orbit decomposition

Since SO_nGL_m is a subgroup of O_nGL_m with a finite index $N_1 \leq 2$, each O_nGL_m-orbit is decomposed into a finite number N_2 of SO_nGL_m-orbits satisfying that $1 \leq N_2 \leq N_1 \leq 2$.

To complete the classification, we now decompose an O_nGL_m-orbit $S_{\nu,\mu}$ into SO_nGL_m-orbits, in the case when

$$2m \geq n = 2\nu \geq 2 \quad \text{and} \quad \mu = 0.$$

This is the case not covered by Proposition 1.3 (3), (4) and Theorem 1.4 (2), where the case of $\nu = 0$ was already discussed in Proposition 1.3 (3) since the condition $n > 0 = 2\nu - \mu$ follows from $0 = \nu \geq \mu \geq 0$.

PROPOSITION 2.1. *Assume that $n = 2\nu$ and $m \geq \nu \geq 1$. Then $S_{\nu,0}$ consists of* *two orbits* $S^I_{\nu,0}$, $S^{II}_{\nu,0}$ *of* SO_nGL_m, *whose representatives are given by* $X_{\nu,0} \in S^I_{\nu,0}$ *and* $Y_{\nu,0} := I^1_n X_{\nu,0} \in S^{II}_{\nu,0}$ *for* $I^1_n := \mathrm{diag}(-1, I_{n-1})$.

Proof. (1) $S_{\nu,0} = SO_nGL_m \cdot X_{\nu,0} \cup SO_nGL_m \cdot Y_{\nu,0}$.

In fact, because of Proposition 1.3 (2), one has that

$$\begin{aligned} S_{\nu,0} = O_nGL_m \cdot X_{\nu,0} &= \rho((SO_n\mathbb{C} \cup (SO_n\mathbb{C})I^1_n) \times GL_m\mathbb{C}) \cdot X_{\nu,0} \\ &= SO_nGL_m \cdot X_{\nu,0} \cup SO_nGL_m \cdot I^1_n X_{\nu,0} \\ &= SO_nGL_m \cdot X_{\nu,0} \cup SO_nGL_m \cdot Y_{\nu,0}. \end{aligned}$$

(2) *For any* $[y_1, ..., y_\nu | O_{n,m-\nu}] \in T_{\nu,0}$ *with* $2 \leq n = 2\nu \leq 2m$, *put* $V := \mathbb{C}y_1 \oplus ... \oplus \mathbb{C}y_\nu$ *and* $V' := \{x \in \mathbb{C}^n |\ (x|y) = 0\ (y \in V)\}$. *Then*

$$V' = V.$$

In fact, note that $(y_i|y_j) = 0$ for all $i, j \in \{1, \ldots, \nu\}$, so that $V \subseteq V'$. Conversely, put $Y_1 := [y_1, \cdots, y_\nu]$. By Proposition 1.2 (3), there exist $g \in GL_\nu\mathbb{C}$ and $y^*_1, ..., y^*_\nu \in \mathbb{C}^n$ such that $\tilde{Y} := [Y_1 g | y^*_1, ..., y^*_\nu] \in M_n\mathbb{C}$ satisfies

$${}^t\tilde{Y}\tilde{Y} = Z_{\nu,0} = \begin{bmatrix} O & I_\nu \\ I_\nu & O \end{bmatrix} \in GL_n\mathbb{C}.$$

Put $[y'_1, \cdots, y'_\nu] := Y_1 g$. Then linear combinations of y'_i $(i = 1, \cdots \nu)$ are contained in V. By $\tilde{Y} \in GL_n\mathbb{C}$, for any $x \in \mathbb{C}^n$, there exist $c_i, c^*_i \in \mathbb{C}$ $(i = 1, \cdots, \nu)$ such that $x = \sum_{i=1}^{\nu}(c_i y'_i + c^*_i y^*_i)$. In particular, if $x \in V'$, then $c^*_i = (x|y'_i) = 0$ for all $i = 1, \cdots, \nu$, so that $x \in V$. Hence, $V' \subseteq V$, as required.

(3) $SO_nGL_m \cdot X_{\nu,0} \neq SO_nGL_m \cdot Y_{\nu,0}$.

If not, there exist $(g_1, g_2) \in SO_n \times GL_m\mathbb{C}$ such that $\rho(g_1, g_2) \cdot X_{\nu,0} = Y_{\nu,0}$. By Proposition 1.1 (2), it may be assumed that there exists $g'_2 \in GL_\nu\mathbb{C}$ such that $g_2 = \mathrm{diag}(g'_2, I_{m-\nu}) \in GL_m\mathbb{C}$. Put $[Y'_{\nu,0}|Y''_{\nu,0}] := I^1_n \tilde{X}_{\nu,0} \in GL_n\mathbb{C}$. Then $g_1 X'_{\nu,0}\ {}^t(g'_2) = Y'_{\nu,0}$. Hence,

$$\begin{aligned} {}^tY'_{\nu,0}(g_1 X''_{\nu,0}(g'_2)^{-1}) &= (g'_2\ {}^tX'_{\nu,0}\ {}^tg_1)(g_1 X''_{r,0}(g'_2)^{-1}) \\ = g'_2\ {}^tX'_{\nu,0}X''_{\nu,0}\ (g'_2)^{-1} &= g'_2 I_\nu\ (g'_2)^{-1} = I_\nu = {}^tY'_{\nu,0}Y''_{\nu,0}, \end{aligned}$$

so that ${}^tY'_{\nu,0}(g_1 X''_{\nu,0}(g'_2)^{-1} - Y''_{\nu,0}) = O$. By (2), there exists $g' \in GL_\nu\mathbb{C}$ such that $g_1 X''_{\nu,0}(g'_2)^{-1} - Y''_{\nu,0} = Y'_{\nu,0}g'$, that is: $g_1 X''_{\nu,0}(g'_2)^{-1} = Y''_{\nu,0} + Y'_{\nu,0}g'$. Then

$$g_1[X'_{\nu,0}|X''_{\nu,0}]\mathrm{diag}({}^tg'_2, (g'_2)^{-1}) = [Y'_{\nu,0}|Y''_{\nu,0} + Y'_{\nu,0}g'].$$

Taking the determinant of the both sides, one has that

$$\det(\tilde{X}_{\nu,0}) = \det([X'_{\nu,0}|X''_{\nu,0}]) = \det([Y'_{\nu,0}|Y''_{\nu,0}]) = \det(I^1_n\tilde{X}_{\nu,0}) = -\det(\tilde{X}_{\nu,0}),$$

so that $\det(\tilde{X}_{\nu,0}) = 0$, which contradicts with $\tilde{X}_{\nu,0} \in GL_n\mathbb{C}$. □

NOTATION 2.2. Let n, m be positive integers. Put $S'_{\nu,\mu} := SO_nGL_m \cdot X_{\nu,\mu}$ for any integers ν, μ such that $m \geq \nu \geq \mu \geq 0$ and $n \geq 2\nu - \mu$. If n is even, then put $S^{II}_{n/2,0} := SO_nGL_m \cdot Y_{n/2,0} = SO_nGL_m \cdot I^1_n X_{n/2,0}$, and $S^I_{n/2,0} := S'_{n/2,0}$.

THEOREM 2.3. (1) *If n is an even number such that $2m \geq n$, then*

$$\mathcal{O}(SO_nGL_m, M_{n,m}\mathbb{C}) = \{S'_{\nu,\mu} |\ m \geq \nu \geq \mu \geq 0,\ n \geq 2\nu - \mu\} \sqcup \{S^{II}_{n/2,0}\}.$$

(2) *If $n = 2m$, then*

$$\mathcal{O}(SO_nGL_m, M_{n,m}\mathbb{C}) = \{S'_{\nu,\mu} |\ m \geq \nu \geq \mu \geq 0\} \sqcup \{S^{II}_{n/2,0}\},$$

so that $\#\mathcal{O}(SO_nGL_m, M_{n,m}\mathbb{C}) = (m+1)(m+2)/2 + 1$.

Proof. (1) By Proposition 1.3 (1), $M_{n,m}\mathbb{C} = \bigsqcup_{m \geq \nu \geq \mu \geq 0,\ n \geq 2\nu - \mu} S_{\nu,\mu}$. By Propositions 1.3 (3), (4) and 2.1, if n is even and $2m \geq n$, then

$$S'_{\nu,\mu} = \begin{cases} S_{\nu,\mu} & (\text{if } n > 2\nu - \mu \text{ or } \mu > 0) \\ S^I_{n/2,0} & (\text{if } n = 2\nu \text{ and } \mu = 0), \end{cases}$$

so that

$$S_{\nu,\mu} = \begin{cases} S'_{\nu,\mu} & (\text{if } n > 2\nu - \mu \text{ or } \mu > 0) \\ S'_{n/2,0} \sqcup S^{II}_{n/2,0} & (\text{if } n = 2\nu \text{ and } \mu = 0). \end{cases}$$

(2) If $n = 2m$, by Theorem 1.4 (3), the assertion follows from (1). □

A classification of orbits of $(SO_nGLm, M_{n,m}\mathbb{C})$ with $n \geq 1$ and $m \geq 1$ is then completed by Theorems 1.4 (2) (if $n > 2m$ or n is odd) and 2.3 (1) (if $n \geq 2m$ and n is even).

In particular, for $m = 1, 2$, one has the following result.

COROLLARY 2.4. (1) $\mathcal{O}(SO_1GL_1, M_{1,1}\mathbb{C}) = \{S_{0,0},\ S_{1,1}\}$.

(2) $\mathcal{O}(SO_2GL_1, M_{2,1}\mathbb{C}) = \{S_{0,0},\ S^I_{1,0},\ S^{II}_{1,0},\ S_{1,1}\}$.

(3) *If $n \geq 3$, then* $\mathcal{O}(SO_nGL_1, M_{n,1}\mathbb{C}) = \{S_{0,0},\ S_{1,0},\ S_{1,1}\}$.

(4) $\mathcal{O}(SO_1GL_2, M_{1,2}\mathbb{C}) = \{S_{0,0},\ S_{1,1}\}$.

(5) $\mathcal{O}(SO_2GL_2, M_{2,2}\mathbb{C}) = \{S_{0,0},\ S^I_{1,0},\ S^{II}_{1,0},\ S_{1,1},\ S_{2,2}\}$.

(6) $\mathcal{O}(SO_3GL_2, M_{3,2}\mathbb{C}) = \{S_{0,0},\ S_{1,0},\ S_{1,1},\ S_{2,1},\ S_{2,2}\}$.

(7) $\mathcal{O}(SO_4GL_2, M_{4,2}\mathbb{C}) = \{S_{0,0},\ S_{1,0},\ S_{1,1},\ S^I_{2,0},\ S^{II}_{2,0},\ S_{2,1},\ S_{2,2}\}$.

(8) *If $n \geq 5$, then*

$$\mathcal{O}(SO_nGL_2, M_{n,2}\mathbb{C}) = \{S_{0,0},\ S_{1,0},\ S_{1,1},\ S_{2,0},\ S_{2,1},\ S_{2,2}\}.$$

Proof. By Theorems 2.3, 1.4 (2), $X_{0,0} = O_{n,m}$ and solving the condition:

$$m \geq \nu \geq \mu \geq 0 \text{ and } n \geq 2\nu - \mu; \text{ or } 2m \geq n = 2\nu \text{ and } \mu = 0,$$

one has the required result. In fact:

(1, 2, 3) Put $m = 1$. By $1 \geq \nu \geq \mu \geq 0$, one has that

$$(\nu, \mu) = (1,1), (1,0), \text{ or } (0,0).$$

Then $2\nu - \mu = 1, 2$, or 0. By $2 \geq n = 2\nu$, $(n, \nu) = (2,1)$.

(4, 5, 6, 7, 8) Put $m = 2$. By $2 \geq \nu \geq \mu \geq 0$, one has that

$$(\nu, \mu) = (2,2), (2,1), (2,0), (1,1), (1,0), \text{ or } (0,0).$$

Then $2\nu - \mu = 2, 3, 4, 1, 2$, or 0. By $4 \geq n = 2\nu$, $(n, \nu) = (2,1)$ or $(4,2)$. □

3. Hasse diagrams

For a subset V of $M_{n,m}\mathbb{C}$, let $\bar{V}$ be the Zariski closure of V in $M_{n,m}\mathbb{C}$. For two orbits $S, S' \in \mathcal{O}(SO_nGL_m, M_{n,m}\mathbb{C})$, $S \leq S'$ means that $\bar{S} \subseteq \bar{S}'$. Note that $\bar{S} = \bar{S}'$ if $S \leq S'$ and $S' \leq S$. Note that *every orbit is Zariski open in its closure* [14, 2.3.3.Lemma], so that $S = S'$ in this case. Hence, $\leq$ defines a partial order on $\mathcal{O}(SO_nGL_m, M_{n,m}\mathbb{C})$ (cf. [5, p.55]). If $S \leq S'$ and $S \neq S'$, then write $S < S'$.

On the other hand, a partial order $\leq$ in $\mathbb{Z}^2$ is defined as follows: For $(\nu, \mu), (\nu', \mu') \in \mathbb{Z}$, $(\nu, \mu) \leq (\nu', \mu')$ if and only if $\nu \leq \nu'$ and $\mu \leq \mu'$. And $(\nu, \mu) < (\nu', \mu')$ means that $(\nu, \mu) \leq (\nu', \mu')$ and $(\nu, \mu) \neq (\nu', \mu')$.

In general, a lattice representing a partial order on a set $\mathcal{O}$ is called the *Hasse diagram* of the partially ordered set $\mathcal{O}$.

THEOREM 3.1. *Let n, m be fixed positive integers. Then:*

(1) *If $n > 2m$ or n is odd, then*

$$\mathcal{O}(SO_nGL_m, M_{n,m}\mathbb{C}) = \{S'_{\nu,\mu} \mid m \geq \nu \geq \mu \geq 0,\ n \geq 2\nu - \mu\}$$

and that $S'_{\nu,\mu} < S'_{\nu',\mu'}$ if and only if $(\nu, \mu) < (\nu', \mu')$.

(2) *If $n \leq 2m$ and n is even, then*

$$\mathcal{O}(SO_nGL_m, M_{n,m}\mathbb{C}) = \{S'_{\nu,\mu} \mid m \geq \nu \geq \mu \geq 0,\ n \geq 2\nu - \mu\} \sqcup \{S^{II}_{n/2,0}\}$$

and that $S^a_{\nu,\mu} < S^b_{\nu',\mu'}$ if and only if $(\nu, \mu) < (\nu', \mu')$, where a and b take $'$ or II, independently, that is:

$$S'_{\nu,\mu} < S'_{\nu',\mu'} \Leftrightarrow (\nu, \mu) < (\nu', \mu');$$
$$S^{II}_{n/2,0} < S'_{\nu',\mu'} \Leftrightarrow (n/2, 0) < (\nu', \mu');$$
$$S'_{\nu,\mu} < S^{II}_{n/2,0} \Leftrightarrow (\nu, \mu) < (n/2, 0),$$

and that there is no relation between $S'_{n/2,0}$ and $S^{II}_{n/2,0}$.

Proof. The orbit decompositions in (1) and (2) follow from Theorems 1.4 (1), (2) and 2.3 (1). Hence, it is enough to determine the partial ordering among the orbits.

(1a) Note that $\bar{S}'_{\nu,\mu} = \bar{S}_{\nu,\mu}$. In general, since the upper bound ν (resp. μ) of the rank of a matrix X (resp. ${}^tXX$) is defined as a zero set of all minor determinants of size ν (resp. μ), one has that

$$\bar{S}_{\nu,\mu} \subseteq \{X \in M_{n,m}\mathbb{C} |\ \mathrm{rank}(X) \leq \nu,\ \mathrm{rank}({}^tXX) \leq \mu\}.$$

If $S'_{\nu,\mu} \leq S'_{\nu',\mu'}$, then $\nu \leq \nu'$ and $\mu \leq \mu'$, by the above equation and the fact that $\mathrm{rank}(X_{\nu,\mu}) = \nu$, $\mathrm{rank}({}^tX_{\nu,\mu}X_{\nu,\mu}) = \mu$ and $X_{\nu,\mu} \in S'_{\nu,\mu}$.

(1a') If $S'_{\nu,\mu} \neq S'_{\nu',\mu'}$, then $(\nu, \mu) \neq (\nu', \mu')$.

By (1a) and (1a'), "only if"-part of (1) is proved. Conversely, assume that $\nu \leq \nu'$ and $\mu \leq \mu'$ in (1b, c, d, e) in the below:

(1b) If $\nu - \mu > 0$ and $S'_{\nu,\mu} \neq \emptyset$, then there exist non-empty

$$S'_{\nu,\mu+1}, \cdots, S'_{\nu,\nu};\ S'_{\nu-1,\mu}, \cdots, S'_{\mu,\mu}$$

such that $S'_{\mu,\mu} < \cdots < S'_{\nu-1,\mu} < S'_{\nu,\mu} < S'_{\nu,\mu+1} < \cdots < S'_{\nu,\nu}$. In fact, for $k \in \{1, \cdots, \nu - \mu\}$, one has that

$$\mathrm{diag}(I_{2\nu-\mu-k}, \varepsilon I_k, I_{n-2\nu+\mu})\ X_{\nu,\mu} \in \begin{cases} S'_{\nu,\mu+k} & (\text{if } 0 < \varepsilon < 1) \\ S'_{\nu,\mu} & (\text{if } \varepsilon = 1), \end{cases}$$

and that

$$X_{\nu,\mu}\ \mathrm{diag}(I_{\nu-k}, \varepsilon I_k, I_{m-\nu}) \in \begin{cases} S'_{\nu,\mu} & (\text{if } 0 < \varepsilon < 1) \\ S'_{\nu-k,\mu} & (\text{if } \varepsilon = 0). \end{cases}$$

Note that Zariski closed set is also closed by the Euclidean topology, so that Zariski closure contains the Euclidean closure.

(1c) If $\mu > 0$ and $S'_{\nu,\mu} \neq \emptyset$, then there exist non-empty

$$S'_{\nu-1,\mu-1}, \cdots, S'_{\nu-\mu,0}$$

such that $S'_{\nu,\mu} > S'_{\nu-1,\mu-1} > \cdots > S'_{\nu-\mu,0}$. In fact, for $k \in \{1, \cdots, \mu\}$,

$$\mathrm{diag}(\varepsilon I_k, I_{n-k})\ X_{\nu,\mu} \in \begin{cases} S'_{\nu,\mu} & (\text{if } 0 < \varepsilon < 1) \\ S'_{\nu-k,\mu-k} & (\text{if } \varepsilon = 0). \end{cases}$$

(1d) If $\nu \leq \mu'$, then $S'_{\nu,\mu} \leq S'_{\nu,\nu} \leq S'_{\mu',\mu'} \leq S'_{\nu',\mu'}$, by (1b), (1c) and (1b).

(1e) If $\mu' < \nu$, then $\nu - \mu \geq \nu - \mu' > 0$ and $\nu' - \mu' \geq \nu - \mu' > 0$, so that

$$S'_{\nu,\mu} \leq S'_{\nu,\mu'} \leq S'_{\nu',\mu'},$$

by (1b). By virtue of (1d) and (1a'), one obtains the "if"-part of (1).

(2a) $S^{II}_{n/2,0} \not\leq S^{I}_{n/2,0}$ and $S^{I}_{n/2,0} \not\leq S^{II}_{n/2,0}$.

In fact, assume that $S^{II}_{n/2,0} \leq S^{I}_{n/2,0}$. By $S^{II}_{n/2,0} = I^1_n S^{I}_{n/2,0}$, one has that

$$I^1_n \bar{S}^{I}_{n/2,0} \subseteq \bar{S}^{I}_{n/2,0}.$$

Since any orbit is open in its closure [14, 2.3.3.Lemma], $S^{I}_{n/2,0}$ is open dense in $\bar{S}^{I}_{n/2,0}$. Since $X \mapsto I^1 X$ is a linear bijection, it is homeomorphic with respect to Zariski topology. Hence, $I^1 S^{I}_{n/2,0}$ is also open dense in $\bar{S}^{I}_{n/2,0}$, so that $I^1 S^{I}_{n/2,0} \cap S^{I}_{n/2,0} \neq \emptyset$, which contradicts with Proposition 2.1. Hence, $S^{II}_{n/2,0} \not\leq S^{I}_{n/2,0}$. Similarly, $S^{I}_{n/2,0} \not\leq S^{II}_{n/2,0}$.

(2a') Assume that $S'_{\nu,\mu} < S'_{\nu',\mu'}$ (resp. $S^a_{\nu,\mu} < S^b_{\nu',\mu'}$). Then $\nu \leq \nu'$ and $\mu \leq \mu'$, as well as (1a). If $(\nu,\mu) = (\nu',\mu')$, then $S'_{\nu,\mu} = S'_{\nu',\mu'}$ (resp. $a = b$ and $S^a_{\nu,\mu} = S^b_{\nu',\mu'}$, by (2a)). Hence, $(\nu,\mu) \neq (\nu',\mu')$, so that the "only if"-part of (2) follows. Conversely, assume that $\nu \leq \nu', \mu \leq \mu'$ and $(\nu,\mu) \neq (\nu',\mu')$:

(2b) For any n, (1b) holds also in the case of (2). If n is even, then $S^{II}_{n/2,0} \neq \emptyset$ and there exist non-empty $S'_{n/2,1}, \cdots, S'_{n/2,n/2}$; $S'_{n/2-1,0}, \cdots, S'_{0,0}$ such that $S'_{0,0} < \cdots < S'_{n/2-1,0} < S^{II}_{n/2,0} < S'_{n/2,1} < \cdots < S'_{n/2,n/2}$.

In fact, the first claim follows as well as (1b). The second claim is also proved as well as (1b) by replacing $X_{\nu,\mu}$ to $I^1_n X_{\nu,\mu}$, because of $I^1_n S'_{\nu,\mu} = S'_{\nu,\mu}$ for $(\nu,\mu) \neq (n/2,0)$.

(2c) For any n, (1d) and (1e) hold also in the case of (2), by the first claims of (2b) and (2c), as well as (1d) and (1e). By virtue of the second claim of (2b), one has the "if"-part of (2). □

COROLLARY 3.2. *If $m = 1, 2$, or $n \geq 2m \geq 2$, then the Hasse diagram of $\mathcal{O}(SO_n GL_m, M_{n,m}\mathbb{C})$ is given as the following Figures 1, 2, 3 or 4.*

Proof. It follows from Theorem 3.1 combined with Theorems 1.4 (2), (3), 2.3 (2) and Corollary 2.4. □

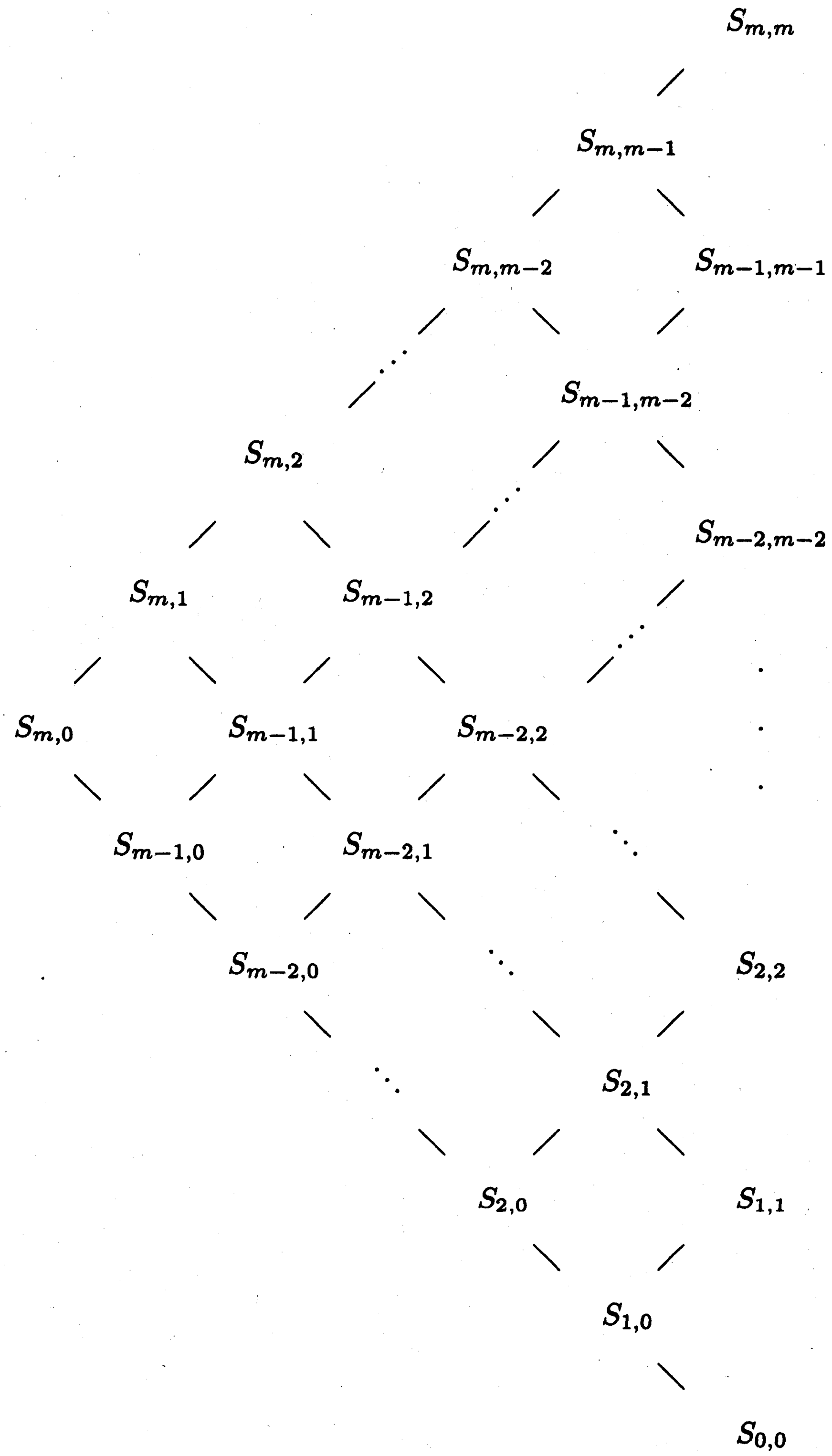

Figure 1 Hasse diagram of $\mathcal{O}(SO_nGL_m, M_{n,m}\mathbb{C})$ with $n > 2m \geq 2$.

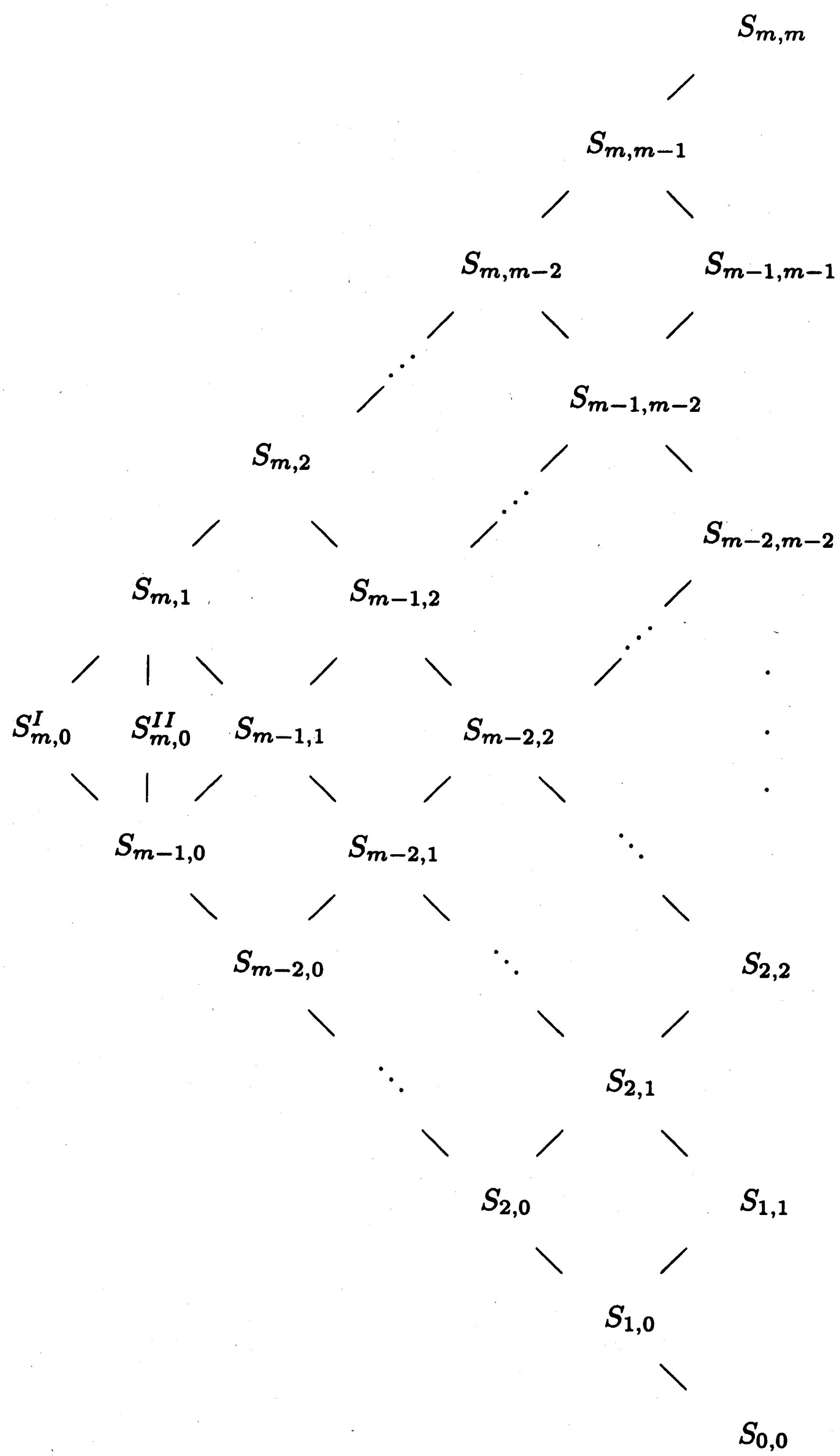

Figure 2 Hasse diagram of $\mathcal{O}(SO_nGL_m, M_{n,m}\mathbb{C})$ with $n = 2m \geq 4$.

$$\begin{tikzcd} S_{1,1} \arrow[d, dash] \\ S_{0,0} \end{tikzcd}$$

$(n=1)$

$$\begin{tikzcd} & S_{2,2} \arrow[d, dash] & \\ & S_{1,1} \arrow[dl, dash] \arrow[dr, dash] & \\ S^{I}_{1,0} \arrow[dr, dash] & & S^{II}_{1,0} \arrow[dl, dash] \\ & S_{0,0} & \end{tikzcd}$$

$(n=2)$

$$\begin{tikzcd} S_{2,2} \arrow[d, dash] \\ S_{2,1} \arrow[d, dash] \\ S_{1,1} \arrow[d, dash] \\ S_{1,0} \arrow[d, dash] \\ S_{0,0} \end{tikzcd}$$

$(n=3)$

$$\begin{tikzcd} & S_{2,2} \arrow[d, dash] & \\ & S_{2,1} \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\ S^{I}_{2,0} \arrow[dr, dash] & S^{II}_{2,0} \arrow[d, dash] & S_{1,1} \arrow[dl, dash] \\ & S_{1,0} \arrow[d, dash] & \\ & S_{0,0} & \end{tikzcd}$$

$(n=4)$

$$\begin{tikzcd} & S_{2,2} \arrow[d, dash] & \\ & S_{2,1} \arrow[dl, dash] \arrow[dr, dash] & \\ S_{2,0} \arrow[dr, dash] & & S_{1,1} \arrow[dl, dash] \\ & S_{1,0} \arrow[d, dash] & \\ & S_{0,0} & \end{tikzcd}$$

$(n \geq 5)$

Figure 3 Hasse diagrams of $\mathcal{O}(SO_nGL_2, M_{n,2}\mathbb{C})$ with $n \geq 1$.

$$
\begin{array}{c}
S_{1,1} \\
| \\
S_{0,0} \\
(n=1)
\end{array}
\qquad
\begin{array}{ccc}
 & S_{1,1} & \\
/ & & \backslash \\
S_{1,0}^{I} & & S_{1,0}^{II} \\
\backslash & & / \\
 & S_{0,0} & \\
 & (n=2) &
\end{array}
\qquad
\begin{array}{c}
S_{1,1} \\
| \\
S_{1,0} \\
| \\
S_{0,0} \\
(n \geq 3)
\end{array}
$$

Figure 4 Hasse diagrams of $\mathcal{O}(SO_nGL_1, M_{n,1}\mathbb{C})$ with $n \geq 1$.

REMARK 3.3. The holonomy diagram of $(SO_nGL_m, M_{n,m}\mathbb{C})$ with $n \geq 2m$ given in [12, p.171, Fig.9.3] is isomorphic to the Hasse diagram in Figure 1 with $n > 2m$, which is not isomorphic to the one in Figure 2 with $n = 2m \geq 4$, Figure 3 with $n = 4$, nor Figure 4 with $n = 2$.

COROLLARY 3.4. *If $m \geq \nu \geq \mu \geq 0$ and $n \geq 2\nu - \mu$, then*

$$\bar{S}_{\nu,\mu} = \{X \in M_{n,m}\mathbb{C} | \ \mathrm{rank}(X) \leq \nu, \ \mathrm{rank}({}^tXX) \leq \mu\}.$$

Proof. If $m \geq \nu \geq \mu \geq 0$ and $n \geq 2\nu - \mu$, then $S_{nu,\mu} \neq \emptyset$, by $X_{\nu,\mu} \in S_{\nu,\mu}$. By the "if"-part of Theorem 3.1, one has that

$$
\begin{aligned}
\bar{S}_{\nu,\mu} &\supseteq \bigcup \{S_{\nu',\mu'} | \ m \geq \nu' \geq \mu' \geq 0, \ n \geq 2\nu' - \mu'; \ \nu' \leq \nu, \ \mu' \leq \mu\} \\
&= \{X \in M_{n,m}\mathbb{C} | \ \mathrm{rank}(X) \leq \nu, \ \mathrm{rank}({}^tXX) \leq \mu\}.
\end{aligned}
$$

By virtue of the reverse equation in (1a), one has the result. □

4. Appendix: Notes on the orbit decomposition for $m = 2$

In this section, when $m = 2$, the classification of orbits given in this paper is compared with other classifications of orbits given in the study of the secant variety of the adjoint variety, the hyperdeterminant, or a classification of nilpotent orbits in a complex simple Lie algebra, respectively.

For a complex vector space V, $\mathbb{P}_*(V)$ denotes the complex projective space of V with the canonical projection $\pi_V : V \backslash \{0\} \to \mathbb{P}_*(V)$. For a linear subspace W of V, put $\mathbb{P}_*(W) := \pi_V(W \backslash \{0\})$. For a projective variety X in $\mathbb{P}_*(V)$, *the secant variety* $S(X)$ is defined as the Zariski closure of the union of complex projective lines $x * y$ in $\mathbb{P}_*(V)$ through two distinct points $x \neq y$ in X:

$$S(X) := \overline{\bigcup_{x,y\in X, x\neq y} x * y} \subseteq \mathbb{P}_*(V).$$

Let $\mathfrak{g}$ be a complex simple Lie algebra, B the Killing form, $\mathfrak{h}$ a Cartan subalgebra, $\triangle$ the set of all non-zero roots w.r.t. $\mathfrak{h}$. For $\alpha \in \triangle$, put

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} |\, [H, X] = \alpha(H)X, H \in \mathfrak{h}\}$$

and $T_\alpha \in \mathfrak{h}$ such that $B(T_\alpha, H) = \alpha(H)$ for all $H \in \mathfrak{h}$. Let $\triangle^+$ (resp. λ) be the set of all positive roots (resp. the highest root) w.r.t. a lexicographic ordering in the real form $\mathfrak{h}_{\mathbb{R}}$. After H. Asano [1, pp.22-23], [2, p.48], take $X_{\pm\lambda} \in \mathfrak{g}_{\pm\lambda}$ in a Chevalley basis, that is, $B(X_\lambda, X_{-\lambda}) = 2/B(T_\alpha, T_\alpha)$, and put $H_\lambda = [X_\lambda, X_{-\lambda}]$ and $\mathfrak{g}_i := \{X \in \mathfrak{g} |\ [H_\lambda, X] = iX\}$ $(i \in \mathbb{Z})$. Then $\mathfrak{g} = \oplus_{i=-2}^{2}\mathfrak{g}_i$ such that $\mathfrak{g}_0 = \mathfrak{h} \oplus \bigoplus_{\alpha\in\triangle_0}(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$, $\mathfrak{g}_1 = \oplus_{\alpha\in\triangle_1}\mathfrak{g}_\alpha$, $\mathfrak{g}_{\pm 2} = \mathfrak{g}_{\pm\lambda}$ with $\dim_{\mathbb{C}} \mathfrak{g}_{\pm 2} = 1$, where $\triangle_1 = \{\alpha \in \triangle^+ |\ \lambda - \alpha \in \triangle\}$ and $\triangle_0 = \{\alpha \in \triangle^+ |\ B(T_\lambda, T_\alpha) = 0\}$ (see J.A. Wolf [18, 4.2.Theorem]). For the inner automorphism group $G := \mathrm{Int}(\mathfrak{g})$ of $\mathfrak{g}$, put $X(\mathfrak{g}) := G \cdot \mathbb{P}_*(\mathfrak{g}_2) \subseteq \mathbb{P}_*(\mathfrak{g})$, which is called *the adjoint variety* of $\mathfrak{g}$ (see [8]).

PROPOSITION 4.1. (1) $\mathfrak{g}_1 = \{0\}$ *if and only if the type of* $\mathfrak{g}$ *is* A_1.

(2) $X(\mathfrak{g}) \cap \mathbb{P}_*(\mathfrak{g}_1) = \emptyset$ *if and only if the type of* $\mathfrak{g}$ *is* A_1 *or* C_ℓ *with* $\ell \geq 2$.

Proof. (1) Put $\mathfrak{s}_\lambda := \mathbb{C}H_\lambda \oplus \mathbb{C}X_\lambda \oplus \mathbb{C}X_{-\lambda}$, which is a three dimensional simple A_1-type subalgebra of $\mathfrak{g}$. If the type of $\mathfrak{g}$ is A_1, then $\dim_{\mathbb{C}} \mathfrak{g} = 3$, so that $\dim_{\mathbb{C}} \mathfrak{g}_1 \leq 3 - \dim_{\mathbb{C}} \mathfrak{s}_\lambda = 0$, that is, $\mathfrak{g}_1 = \{0\}$. If $\mathfrak{g}_1 = \{0\}$, then $\triangle_1 = \emptyset$ and $\mathfrak{g}_{-1} = \{0\}$, so that $\mathfrak{s}_\lambda$ is a non-trivial ideal of a simple $\mathfrak{g}$. In this case, $\mathfrak{g} = \mathfrak{s}_\lambda$, as required.

(2) According to Asano [1, 2], for $P, Q \in \mathfrak{g}_1$, put $P\times Q \in \mathfrak{g}_0$ and $< P,\ Q > \in \mathbb{C}$ such that $P \times Q = -([Q, [P, X_{-\lambda}]] + [P, [Q, X_{-\lambda}]])/2$ and $[P, Q] = 2 < P,\ Q > X_\lambda$. Put $[PQR] := [P \times Q, R] \in \mathfrak{g}_1$ for $P, Q, R \in \mathfrak{g}_1$. Put $M := \{P \in \mathfrak{g}_1 | P \neq 0, P \times P = 0\}$. By [9, Theorem B], one has that

$$X(\mathfrak{g}) \cap \mathbb{P}_*(\mathfrak{g}_1) = \pi_{\mathfrak{g}}(M).$$

Put $\ell := \mathrm{rank}(\mathfrak{g})$ and $q(P) :=< [PPP], P >$ for $P \in \mathfrak{g}_1$. If $\ell = 1$, then the type of $\mathfrak{g}$ is A_1, so that $M = \emptyset$, by (1). In this case, $X(\mathfrak{g}) \cap \mathbb{P}_*(\mathfrak{g}_1) = \emptyset$.

Assume that $\ell \geq 2$. By (1), $\mathfrak{g}_1 \neq \{0\}$. And the following equations are obtained for $P, Q, R \in \mathfrak{g}_1$ [2, Theorem 5](cf. [11, Lemmas 1, 2, 3]):

(S0) $< [PQR], S > + < R, [PQS] >= 0$;

(S1) $[PQR] = [QPR]$;

(S2) $[PQR] - [PRQ] = 2 < Q, R > P - < R, P > Q - < P, Q > R$;
(S3) $[P \times Q, R \times S] = [PQR] \times S + R \times [PQS]$.

Hence, $(\mathfrak{g}_1, [PQR], < P, Q >)$ is *a symplectic triple system* in the sense of K. Yamaguti and H. Asano [16]. Note that $\lambda - \alpha \in \Delta_1$ for all $\alpha \in \Delta_1$. Hence, $[\mathfrak{g}_1, \mathfrak{g}_1] \neq \{0\}$. Then $< P, Q > \not\equiv 0$. By Asano [2, Theorems 1, 4, 5] (cf. [1, Theorems 2.10, 2.11, Corollary]), $< P, Q >$ is *non-degenerate* since $\mathfrak{g}$ is simple.

If the type of $\mathfrak{g}$ is C_ℓ, by a direct calculation of the matrix Lie algebra of type C_ℓ (e.g. [13, pp.14–16]), there is a complex linear isomorphism $f : \mathfrak{g}_1 \to \mathbb{C}^{2(\ell-1)}$ such that $< P, Q > = < f(P), f(Q) >'$ and $[PQR] = [f(P)f(Q)f(R)]'$ for $P, Q, R \in \mathfrak{g}_1$, where

(C1) $< x, y >' = \sum_{i=1}^{\ell-1}(x_i y_{i+\ell-1} - x_{i+\ell-1} y_i)$;
(C2) $[xyz]' = < x, z >' y + < y, z >' x$

for $x, y, z \in \mathbb{C}^{2(\ell-1)}$ (cf. [17, 2.9.Theorem]). In this case, $[xxz]' = < x, z >' x$, so that $M = \emptyset$. Conversely, assume that $M = \emptyset$. By [11, Corollary A1], $q(P) \equiv 0$. In this case, $[PQR] = < P, R > Q + < Q, R > P$ for $P, Q, R \in \mathfrak{g}_1$, by Asano [17, 1.6.Theorem]. Then $\mathfrak{g}$ is isomorphic to a complex simple Lie algebra of type C_ℓ by [17, 2.9.Theorem]. □

Put $G_0 := \mathrm{Int}_{\mathfrak{g}}(\mathfrak{g}_0)$, as the connected Lie subgroup of G corresponding to $\mathfrak{g}_0$. Let $\mathfrak{g}_{nil}$ be the set of all nilpotent elements in $\mathfrak{g}$, and put

$$S(X(\mathfrak{g}))_{nil} := S(X(\mathfrak{g})) \cap \pi(\mathfrak{g}_{nil} \setminus \{0\}).$$

According to $\mathfrak{g}_1 \subseteq \{X \in \mathfrak{g} |\ (\mathrm{ad}X)^5 = 0\}$, one has that $\mathfrak{g}_{nil} \supseteq \mathfrak{g}_1$, so that $S(X(\mathfrak{g}))_{nil} \supseteq \mathbb{P}_*(\mathfrak{g}_1)$. Hence, a natural map between the spaces of orbits is well-defined [10, p.33]:

$$\Psi : \mathcal{O}(G_0, \mathbb{P}_*(\mathfrak{g}_1)) \to \mathcal{O}(G, S(X(\mathfrak{g}))_{nil}).$$

PROPOSITION 4.2. *The mapping Ψ is surjective if and only if the type of $\mathfrak{g}$ is not A_1 nor C_ℓ $(\ell \geq 2)$.*

Proof. By [10, Proposition 1], $\mathcal{O}(G, S(X(\mathfrak{g}))_{nil}) \setminus \{X(\mathfrak{g})\} \subseteq \mathrm{Image}(\Psi)$. By Proposition 4.1 (2), one has then the required result. □

EXAMPLE 4.3. Note that $(SO_n GL_2, M_{n,2}\mathbb{C})$ is *equivalent*([13, §2, Definition 4]) to $(G_0, \mathfrak{g}_1)$ for $\mathfrak{g} = \mathfrak{so}_{n+4}\mathbb{C}$ of type $B_{(n+3)/2}$, $D_{(n+4)/2}$ with $n \geq 3$ or A_3 with $n = 2$. When $n \geq 3$, (that is, $\mathfrak{g} = \mathfrak{so}_7\mathbb{C}$, $\mathfrak{so}_8\mathbb{C}$ or $\mathfrak{so}_{\ell\geq 9}\mathbb{C}$), the Hasse diagram of $(G, S(X(\mathfrak{g}))_{nil})$ given in [10, p.29] is isomorphic to the Hasse diagram without $S_{0,0}$ in Figure 3 with $n = 3$, 4, or $n \geq 5$, respectively. In this case, Ψ is an isomorphism (cf. [10, Remark.(c)]). When $n = 4$, the overlooked orbit of

$(SO_4GL_2, M_{4,2}\mathbb{C})$ in [12] corresponds to a nilpotent orbit of $\mathfrak{so}_8\mathbb{C}$ attached to *a very even partition* of $8 = n+4$ in T. Springer and R. Steinberg's theory on the nilpotent orbits in $\mathfrak{so}_{n+4}\mathbb{C}$ with an even $n \geq 4$ [15, 3, 4, 5]. Since a triplet $(SO_4\mathbb{C}, \square, \mathbb{C}^4)$ is equivalent to a triplet $(SL_2\mathbb{C} \times SL_2\mathbb{C}, \square \otimes \square, \mathbb{C}^2 \otimes \mathbb{C}^2)$, this orbit decomposition is equivalent to the following Example 4.4:

EXAMPLE 4.4. ([6, Example 5.5], [7, Theorem]). Put $V := \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, $E_{ijk} := e_i \otimes e_j \otimes e_k$ $(i,j,k \in \{1,2\})$ with $e_1 = {}^t[1,0], e_2 = {}^t[0,1]$. Then a triplet $(GL_2\mathbb{C} \times GL_2\mathbb{C} \times GL_2\mathbb{C}, \square \otimes \square \otimes \square, V)$ has seven orbits represented by

$$E_{111} + E_{222}, E_{211} + E_{121} + E_{112}, E_{111} + E_{122}, E_{111} + E_{212}, E_{111} + E_{221}, E_{111}, O.$$

Let K be an algebraically closed field of characteristic $p \geqq 2$. In [7, Theorem], it is proved that this orbit decomposition holds also when the basic field $\mathbb{C}$ is replaced by K, and that the orbits are completely characterized by the values of indices $(r_1, r_2, r_3; D)$, as follows:

$$(2,2,2;\neq 0), (2,2,2;0), (2,1,1;0), (1,2,1;0), (1,1,2;0), (1,1,1;0), (0,0,0;0),$$

where $D := (az - by - cx + dw)^2 - 4(ad - bc)(wz - xy)$ is the hyperdeterminant of $Y = aE_{111} + bE_{121} + cE_{211} + dE_{221} + wE_{112} + xE_{122} + yE_{212} + zE_{222} \in V$, and the r_i $(i = 1,2,3)$ are the ranks of a hypermatrix Y defined as follows:

$$\begin{aligned} r_1 &:= \max\{\mathrm{rk}Y(f,*,*) \mid f \in (K^2)^*\}, \\ r_2 &:= \max\{\mathrm{rk}Y(*,f,*) \mid f \in (K^2)^*\}, \\ r_3 &:= \max\{\mathrm{rk}Y(*,*,f) \mid f \in (K^2)^*\} \end{aligned}$$

for $Y \in V$ as an element of the dual space V^{**} of the dual space V^* of V.

EXAMPLE 4.5. When $n = 2$ in Example 4.3, $\mathfrak{g} = \mathfrak{so}_6\mathbb{C}(\cong \mathfrak{sl}_4\mathbb{C})$ and the Hasse diagram of $(G, S(X(\mathfrak{g}))_{nil})$ given in [10, p.28] is looser than the Hasse diagram without $S_{0,0}$ in Figure 3 with $n = 2$ (cf. Corollary 2.4 (5)). In this case, Ψ is not injective (cf. [10, Remark.(c)]), which implies that *the orbit decomposition of* $(SO_2GL_2, M_{2,2}\mathbb{C})$ *is finer than the classification of nilpotent orbits of* $\mathfrak{g} = \mathfrak{so}_6\mathbb{C}$ *intersecting with* $\mathfrak{g}_1$ *by the action of the inner automorphism group* $\mathrm{Int}(\mathfrak{g})$ *of* $\mathfrak{g}$. More directly, this fact is also realized by the following Example 4.6.

EXAMPLE 4.6. For $A, B, C, D \in M_k\mathbb{C}$, put $X(A,B,C,D) := \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, $Y(A;B,C) := X(A,B,C,-{}^tA)$. Put $T_{2k} := \frac{1}{\sqrt{2}}X(I_k, I_k, \sqrt{-1}I_k, -\sqrt{-1}I_k)$,

$S_{2k} := {}^tT_{2k}T_{2k} = Y(O; I_k, I_k)$ and $\mathfrak{os}_{2k}\mathbb{C} := \{X \in \mathfrak{gl}_{2k}\mathbb{C} |\ {}^tXS_{2k} = -S_{2k}X\}$ with a Cartan subalgebra $\mathfrak{h} := \{H(\lambda_1, \cdots, \lambda_k) |\ \lambda_i \in \mathbb{C}\ (i = 1, \cdots, k)\}$;

$$H(\lambda_1, \cdots, \lambda_k) := Y(\mathrm{diag}(\lambda_1, \cdots, \lambda_k); O, O) \in \mathfrak{os}_{2k}\mathbb{C},$$

and the Killing form $B(X, Y) = (2k-2)\ \mathrm{trace}(XY)$. Note that

$$\mathfrak{g} := \mathfrak{os}_{2k}\mathbb{C} = \{Y(A; B, C) |\ A \in M_k\mathbb{C}; B, C \in \mathfrak{so}_k\mathbb{C}\},$$

which is isomorphic to $\mathfrak{so}_{2k}\mathbb{C}$ by $X \mapsto T_{2k}XT_{2k}.^{-1}$. When $k = 3$, the root system of $\mathfrak{g}$ w.r.t. $\mathfrak{h}$ is given by $\Delta := \{\pm(\lambda_j \pm \lambda_k) |\ 1 \leq j < k \leq 3\}$ with root vectors $x_{\lambda_j - \lambda_k} := Y(E_{jk}; O, O)$, $x_{\lambda_j+\lambda_k} := Y(O; E_{jk} - E_{kj}, O)$, $x_{-\lambda_j-\lambda_k} := Y(O; O, -E_{jk} + E_{kj})$ for the standard matrix basis E_{jk} of $M_3\mathbb{C}$. Put $\alpha_1 := \lambda_2 + \lambda_3$, $\alpha_2 := \lambda_1 - \lambda_2$, $\alpha_3 := \lambda_2 - \lambda_3$. Then $\Pi := \{\alpha_1, \alpha_2, \alpha_3\}$ is a fundamental root system with the following Dynkin diagram of type A_3: $\overset{\alpha_1}{\circ} - \overset{\alpha_2}{\circ} - \overset{\alpha_3}{\circ}$ (cf. [13, Example 27]). The highest root is given by $\lambda := \lambda_1 + \lambda_2$. Let $t_\lambda \in \mathfrak{h}$ be such that $B(t_\lambda, H') = \lambda(H')$ for all $H' \in \mathfrak{h}$. Put $h_\lambda := 2t_\lambda / B(t_\lambda, t_\lambda)$. Then $h_\lambda = H(1, 1, 0) = [x_\lambda, x_{-\lambda}]$. For $i \in \mathbb{Z}$, put

$$\mathfrak{g}_i := \{X \in \mathfrak{g} |\ [h_\lambda, X] = iX\}.$$

Then $\mathfrak{g}_0 = \{Y(\mathrm{diag}(A_1, a_2); O, O) |\ A_1 \in M_2\mathbb{C},\ a_2 \in \mathbb{C}\}$,

$$\mathfrak{g}_1 = \left\{ Y(\begin{bmatrix} O & A_{12} \\ O & 0 \end{bmatrix}; \begin{bmatrix} O & B_{12} \\ -{}^tB_{12} & 0 \end{bmatrix}, O) |\ A_{12}, B_{12} \in M_{2,1}\mathbb{C} \right\},$$

$\mathfrak{g}_{-1} = \{X \in \mathfrak{g} |\ {}^tX \in \mathfrak{g}_1\}$ and $\mathfrak{g}_{\pm 2} = \mathbb{C}x_{\pm\lambda}$, so that $\mathfrak{g} = \oplus_{i=-2}^{i} \mathfrak{g}_i$ is a $\mathbb{Z}$-gradation of complex contact type. Note that $\mathfrak{g}_0$ is identified with $\mathfrak{so}_2\mathbb{C} \oplus \mathfrak{gl}_2\mathbb{C}$ by $Y(\mathrm{diag}(A_1, a_2); O, O) \mapsto T_2\ \mathrm{diag}(a_2, -a_2)\ T_2^{-1} \oplus A_1$, so that the action of $\mathrm{Int}_{\mathfrak{g}}(\mathfrak{g}_0)$ on $\mathfrak{g}_1$ is equivariant to $(SO_2GL_2, M_{2,2}\mathbb{C})$ by

$$F : \mathfrak{g}_1 \to M_{2,2}\mathbb{C}; Y(\begin{bmatrix} O & A_{12} \\ O & 0 \end{bmatrix}; \begin{bmatrix} O & B_{12} \\ -{}^tB_{12} & 0 \end{bmatrix}, O) \mapsto T_2 \begin{bmatrix} -{}^tB_{12} \\ -{}^tA_{12} \end{bmatrix}.$$

By Corollary 2.4 (5), the representatives of $\mathcal{O}(SO_2GL_2, M_{2,2}\mathbb{C})$ are given by O, $X_{1,0}$, $Y_{1,0}$, $X_{1,1}$, $X_{2,2}$. Up to $GL_2\mathbb{C}$-action, their images by F^{-1} equal O, $X_{1,0}^+ := Y(O; -E_{13} + E_{31}, O)$, $Y_{1,0}^+ := Y(-E_{13}; O, O)$, $X_{1,1}^+ := Y(-E_{13}; -E_{13} + E_{31}, O)$, $X_{2,2}^+ := Y(-E_{13} - E_{23}; -E_{13} + E_{23} + E_{31} - E_{32}, O)$, respectively. For $(j, k) = (0, 0), (1, 0), (1, 1), (2, 2)$, put $X_{j,k}^- := {}^tX_{j,k}$ and $H_{j,k} := [X_{j,k}^+, X_{j,k}^-]$. And put $Y_{1,0}^- := {}^tY_{1,0}^+$, $H_{1,0}^{II} := [Y_{1,0}^+, Y_{1,0}^-]$. Then $\{H_{j,k}, X_{j,k}^+, X_{j,k}^-\}$ and $\{H_{1,0}^{II}, Y_{1,0}^+, Y_{1,0}^-\}$ are $\mathfrak{sl}_2\mathbb{C}$-triplets. And the weighted Dynkin diagram A_3 of $H_{j,k}^*(= H_{j,k}$ or $H_{1,0}^{II})$ is translated by the Weyl group of $\mathfrak{g} = \mathfrak{os}_6\mathbb{C} \cong \mathfrak{sl}_4\mathbb{C}$ to the following canonical forms (see [10, p.40, Recipe]):

$H_{2,2}$: $\overset{2}{\circ}\!-\!\overset{0}{\circ}\!-\!\overset{2}{\circ}$

$H_{1,1}$: $\overset{0}{\circ}\!-\!\overset{2}{\circ}\!-\!\overset{0}{\circ}$

$H_{1,0}$: $\overset{1}{\circ}\!-\!\overset{1}{\circ}\!-\!\overset{-1}{\circ} \quad \overset{\sigma_{\alpha_3}}{\mapsto} \quad \overset{1}{\circ}\!-\!\overset{0}{\circ}\!-\!\overset{1}{\circ}$

$H_{1,0}^{II}$: $\overset{-1}{\circ}\!-\!\overset{1}{\circ}\!-\!\overset{1}{\circ} \quad \overset{\sigma_{\alpha_1}}{\mapsto} \quad \overset{1}{\circ}\!-\!\overset{0}{\circ}\!-\!\overset{1}{\circ}$

$O_{2,2}$: $\overset{0}{\circ}\!-\!\overset{0}{\circ}\!-\!\overset{0}{\circ}$

Hence, $X_{1,0}^{+}$ and $Y_{1,0}^{+}$ stand on the same orbit of $(\mathrm{Int}(\mathfrak{g}), \mathfrak{g})$. However, they does not stand on the same orbit of $(\mathrm{Int}_{\mathfrak{g}}(\mathfrak{g}_0), \mathfrak{g}_1) \cong (SO_2GL_2, M_{2,2}\mathbb{C})$.

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