

# FIXED POINTS SUBGROUPS BY TWO INVOLUTIVE AUTOMORPHISMS $\gamma, \gamma'$ OF COMPACT EXCEPTIONAL LIE GROUPS $G_2, F_4, E_6$ AND $E_7$

By

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(Received April 14, 2005; Revised August 15, 2005)

**Abstract.** For the simply connected compact exceptional Lie groups  $G = G_2, F_4, E_6$  and  $E_7$ , we determine the group structure of the subgroup  $G^{\gamma, \gamma'}$  of  $G$  by considering two consitutions.

## Introduction

For the simply connected compact exceptional Lie groups  $G = G_2, F_4, E_6$  and  $E_7$ , we consider two involutions  $\gamma, \gamma'$  and determine the group structure of the subgroup  $G^{\gamma, \gamma'}$  of  $G$ , which is the intersection  $G^\gamma \cap G^{\gamma'}$  of the fixed points subgroups  $G^\gamma$  and  $G^{\gamma'}$ . The motivation is as follows. In the preceding paper [3], we determined the group structure of  $G^{\sigma, \sigma'} = G^\sigma \cap G^{\sigma'}$ ,  $G = F_4, E_6$  and  $E_7$  for the involutions  $\sigma, \sigma' \in F_4$ . We consider the case replacing  $\gamma, \gamma'$  instead of  $\sigma, \sigma'$ . We shall give two different proofs, needless to say, results are essentially the same.

$$\begin{aligned}(G_2)^{\gamma, \gamma'} &\cong (U(1) \times U(1))/\mathbf{Z}_2 \times \{1, \gamma_1\} \\ &\cong (U(1) \times U(1)) \cdot \mathbf{Z}_2 \\ (F_4)^{\gamma, \gamma'} &\cong (U(1) \times U(1) \times SU(3))/(\mathbf{Z}_2 \times \mathbf{Z}_3) \times \{1, \gamma_1\} \\ &\cong ((U(1) \times U(1) \times SU(3))/\mathbf{Z}_3) \cdot \mathbf{Z}_2 \\ (E_6)^{\gamma, \gamma'} &\cong (U(1) \times U(1) \times SU(3) \times SU(3))/(\mathbf{Z}_2 \times \mathbf{Z}_3) \times \{1, \gamma_1\} \\ &\cong ((U(1) \times U(1) \times SU(3) \times SU(3))/\mathbf{Z}_3) \cdot \mathbf{Z}_2 \\ (E_7)^{\gamma, \gamma'} &\cong (U(1) \times U(1) \times SU(6))/(\mathbf{Z}_2 \times \mathbf{Z}_6) \times \{1, l_1\} \\ &\cong ((U(1) \times U(1) \times SU(6))/\mathbf{Z}_3) \cdot \mathbf{Z}_2\end{aligned}$$

As for the group  $(E_8)^{\gamma, \gamma'}$ , we can not realize so far.

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2000 Mathematics Subject Classification: 22E99  
Key words and phrases: exceptional Lie group

**NOTATION.**

- (1) For a group  $G$  and an element  $s$  of  $G$ , we denote  $\{g \in G \mid sg = gs\}$  by  $G^s$ .
- (2) For a topological group  $G$ , we denote by  $G^0$  the connected component containing the identity of  $G$  and  $G = G^0 \times \{1, a\}$  means that  $G$  has two connected components such that  $G = G^0 \cup aG^0$ .
- (3)  $G \cdot \mathbf{Z}_2$  denotes a semi-direct product of groups  $G$  and  $\mathbf{Z}_2 = \{1, \gamma_1\}$ .
- (4) For an  $\mathbf{R}$ -vector space  $V$ , its complexification  $\{u+iv \mid u, v \in V\}$  is denoted by  $V^C$ . The complex conjugation in  $V^C$  is denoted by  $\tau$ :  $\tau(u+iv) = u-iv$ . In particular, the complexification of  $\mathbf{R}$  is briefly denoted by  $C$ :  $\mathbf{R}^C = C$ .
- (5) The Lie algebra of a Lie group  $G$  is denoted by the corresponding German small letter  $\mathfrak{g}$ . For example,  $\mathfrak{sp}(n)$  denotes the Lie algebra of the group  $Sp(n)$ .

Although we will give all definitions used in the following sections, if in case of insufficiency, refer to [5], [6] or [7].

**1. The first consideration****1.1 Group  $G_2$** 

Let  $\mathfrak{C}$  be the Cayley division algebra with the canonical  $\mathbf{R}$ -basis  $\{e_0 = 1, e_1, \dots, e_7\}$  ([7]).  $\mathfrak{C}$  contains naturally the field  $C$  of complex numbers and the field  $H$  of quaternions as

$$C = \{x_0 + x_1 e_1 \mid x_k \in \mathbf{R}\}, \quad H = \{x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3 \mid x_k \in \mathbf{R}\},$$

respectively. Any element  $x$  of  $\mathfrak{C}$  is uniquely expressed as  $x = m + ae_4$ ,  $m, a \in H$ :  $\mathfrak{C} = H \oplus He_4$ . In  $\mathfrak{C} = H \oplus He_4$ , the multiplication and the conjugation are defined by

$$(m + ae_4)(n + be_4) = (mn - \bar{b}a) + (a\bar{n} + bm)e_4,$$

$$\overline{m + ae_4} = \bar{m} - ae_4.$$

The simply connected compact Lie group  $G_2$  is given by

$$G_2 = \{\alpha \in \text{Iso}_{\mathbf{R}}(\mathfrak{C}) \mid \alpha(xy) = (\alpha x)(\alpha y)\}.$$

We define  $\mathbf{R}$ -linear transformations  $\gamma, \gamma'$  and  $\gamma_1$  of  $H \oplus He_4 = \mathfrak{C}$  by

$$\gamma(m + ae_4) = m - ae_4,$$

$$\gamma'(m + ae_4) = \gamma' m + (\gamma' a)e_4,$$

$$\gamma_1(m + ae_4) = \gamma_1 m + (\gamma_1 a)e_4, \quad m + ae_4 \in H \oplus He_4 = \mathfrak{C},$$

respectively, where  $\gamma', \gamma_1 : H \rightarrow H$  are defined by

$$\begin{aligned} \gamma'(x + ye_2) &= x - ye_2, \\ \gamma_1(x + ye_2) &= \bar{x} + \bar{y}e_2, \quad x + ye_2 \in C \oplus Ce_2 = H. \end{aligned}$$

Then  $\gamma, \gamma', \gamma_1 \in G_2$  and  $\gamma^2 = \gamma'^2 = \gamma_1^2 = 1$ .  $\gamma, \gamma'$  and  $\gamma_1$  are conjugate with each other in  $G_2$  ([5]) and commutative. From  $\gamma\gamma' = \gamma'\gamma$ , we have

$$(G_2)^\gamma \cap (G_2)^{\gamma'} = ((G_2)^\gamma)^{\gamma'} = ((G_2)^{\gamma'})^\gamma,$$

so this group will be briefly denoted by  $(G_2)^{\gamma, \gamma'}$ .

**PROPOSITION 1.1.1.**  $(G_2)^\gamma \cong (Sp(1) \times Sp(1))/Z_2$ ,  $Z_2 = \{(1, 1), (-1, -1)\}$ .

*Proof.* Let  $Sp(1) = \{p \in H \mid p\bar{p} = 1\}$ . The mapping  $\varphi_2 : Sp(1) \times Sp(1) \rightarrow (G_2)^\gamma$ ,

$$\varphi_2(p, q)(m + ae_4) = qm\bar{q} + (pa\bar{q})e_4, \quad m + ae_4 \in H \oplus He_4 = \mathfrak{C}$$

induces the required isomorphism (see [5] or [7] for details).  $\square$

**LEMMA 1.1.2.** The mapping  $\varphi_2 : Sp(1) \times Sp(1) \rightarrow (G_2)^\gamma$  satisfies

$$\gamma'\varphi_2(p, q)\gamma' = \varphi_2(\gamma'p, \gamma'q), \quad \gamma' = \varphi_2(e_1, e_1), \quad \gamma_1 = \varphi_2(e_2, e_2).$$

Now, we will determine the group structure of  $(G_2)^{\gamma, \gamma'}$ .

**THEOREM 1.1.3.**  $(G_2)^{\gamma, \gamma'} \cong (U(1) \times U(1))/Z_2 \times \{1, \gamma_1\}$ ,  $Z_2 = \{(1, 1), (-1, -1)\}$ .

*Proof.* For  $\alpha \in (G_2)^{\gamma, \gamma'} \subset (G_2)^\gamma$ , there exist  $p, q \in Sp(1)$  such that  $\alpha = \varphi_2(p, q)$  (Proposition 1.1.1). From  $\gamma'\alpha\gamma' = \alpha$ , we have  $\varphi_2(\gamma'p, \gamma'q) = \varphi_2(p, q)$  (Lemma 1.1.2). Hence

$$\begin{cases} \gamma'p = p \\ \gamma'q = q \end{cases} \quad \text{or} \quad \begin{cases} \gamma'p = -p \\ \gamma'q = -q. \end{cases}$$

In the former case, we have  $p, q \in U(1) = \{a \in C \mid a\bar{a} = 1\}$ . Hence the group of the former case is  $(U(1) \times U(1))/Z_2$ . In the latter case,  $p = q = e_2$  satisfy the conditions and  $\varphi_2(e_2, e_2) = \gamma_1$  (Lemma 1.1.2). Thus we have the isomorphism  $(G_2)^{\gamma, \gamma'} \cong (U(1) \times U(1))/Z_2 \times \{1, \gamma_1\}$ .  $\square$

## 1.2 Group $F_4$

Let  $\mathfrak{J} = \mathfrak{J}(3, \mathfrak{C}) = \{X \in M(3, \mathfrak{C}) \mid X^* = X\}$  be the exceptional Jordan algebra with the Jordan multiplication  $X \circ Y$ , the inner product  $(X, Y)$  and the Freudenthal multiplication  $X \times Y$  respectively defined by

$$\begin{aligned} X \circ Y &= \frac{1}{2}(XY + YX), \quad (X, Y) = \text{tr}(X \circ Y), \\ X \times Y &= \frac{1}{2}(2X \circ Y - \text{tr}(X)Y - \text{tr}(Y)X + (\text{tr}(X)\text{tr}(Y) - (X, Y))E), \end{aligned}$$

where  $E$  is the  $3 \times 3$  unit matrix.

The simply connected compact Lie group  $F_4$  is given by

$$F_4 = \{\alpha \in \text{Iso}_{\mathbf{R}}(\mathfrak{J}) \mid \alpha(X \times Y) = \alpha X \times \alpha Y\}.$$

We have naturally the inclusion  $G_2 \subset F_4$  ([5],[7]).

Any element  $X = \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix}$  of  $\mathfrak{J}$  is expressed as

$$X = \begin{pmatrix} \xi_1 & m_3 & \bar{m}_2 \\ \bar{m}_3 & \xi_2 & m_1 \\ m_2 & \bar{m}_1 & \xi_3 \end{pmatrix} + \begin{pmatrix} 0 & a_3 e_4 & -a_2 e_4 \\ -a_3 e_4 & 0 & a_1 e_4 \\ a_2 e_4 & -a_1 e_4 & 0 \end{pmatrix},$$

where  $x_k = m_k + a_k e_4 \in \mathbf{H} \oplus \mathbf{H}e_4 = \mathfrak{C}$ . We associate such  $X$  with the element

$$\begin{pmatrix} \xi_1 & m_3 & \bar{m}_2 \\ \bar{m}_3 & \xi_2 & m_1 \\ m_2 & \bar{m}_1 & \xi_3 \end{pmatrix} + (a_1, a_2, a_3)$$

of  $\mathfrak{J}(3, \mathbf{H}) \oplus \mathbf{H}^3$ . In  $\mathfrak{J}(3, \mathbf{H}) \oplus \mathbf{H}^3$ , we define the multiplication  $\times$  by

$$(M + \mathbf{a}) \times (N + \mathbf{b}) = \left( M \times N - \frac{1}{2}(\mathbf{a}^* \mathbf{b} + \mathbf{b}^* \mathbf{a}) \right) - \frac{1}{2}(\mathbf{a}N + \mathbf{b}M).$$

Then  $\mathfrak{J}(3, \mathbf{H}) \oplus \mathbf{H}^3$  is isomorphic to  $\mathfrak{J}$  as Freudenthal algebras.

Using the inclusion  $G_2 \subset F_4$ , the  $\mathbf{R}$ -linear transformations  $\gamma, \gamma', \gamma_1$  of  $\mathbf{H} \oplus \mathbf{H}e_4 = \mathfrak{C}$  are extended to  $\mathbf{R}$ -linear transformations  $\gamma, \gamma', \gamma_1$  of  $\mathfrak{J}(3, \mathbf{H}) \oplus \mathbf{H}^3 = \mathfrak{J}$  by

$$\gamma(M + \mathbf{a}) = M - \mathbf{a}, \quad \gamma'(M + \mathbf{a}) = \gamma' M + \gamma' \mathbf{a}, \quad \gamma_1(M + \mathbf{a}) = \gamma_1 M + \gamma_1 \mathbf{a},$$

respectively.

**PROPOSITION 1.2.1.**  $(F_4)^\gamma \cong (Sp(1) \times Sp(3))/Z_2$ ,  $Z_2 = \{(1, E), (-1, -E)\}$ .

*Proof.* Let  $Sp(1) = \{p \in H \mid p\bar{p} = 1\}$  and  $Sp(3) = \{A \in M(3, H) \mid AA^* = E\}$ . The mapping  $\varphi_4 : Sp(1) \times Sp(3) \rightarrow (F_4)^\gamma$ ,

$$\varphi_4(p, A)(M + a) = AMA^* + paA^*, \quad M + a \in \mathfrak{J}(3, H) \oplus H^3 = \mathfrak{J}$$

induces the required isomorphism (see [5] or [7] for details).  $\square$

**LEMMA 1.2.2.** The mapping  $\varphi_4 : Sp(1) \times Sp(3) \rightarrow (F_4)^\gamma$  satisfies

$$\gamma' \varphi_4(p, A) \gamma' = \varphi_4(\gamma' p, \gamma' A), \quad \gamma' = \varphi_4(e_1, e_1 E), \quad \gamma_1 = \varphi_4(e_2, e_2 E).$$

Hereafter,  $\omega_1$  denotes  $\omega_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}e_1$ . Then  $\omega_1 \in \mathfrak{C}$  and  $\omega_1^3 = 1$ .

Now, we will determine the group structure of  $(F_4)^{\gamma, \gamma'} = ((F_4)^\gamma)^{\gamma'} = ((F_4)^{\gamma'})^\gamma = (F_4)^\gamma \cap (F_4)^{\gamma'}$ .

**THEOREM 1.2.3.**  $(F_4)^{\gamma, \gamma'} \cong (U(1) \times U(1) \times SU(3))/(Z_2 \times Z_3) \times \{1, \gamma_1\}$ ,  $Z_2 = \{(1, 1, E), (-1, -1, E)\}$ ,  $Z_3 = \{(1, 1, E), (1, \omega_1, \omega_1^2 E), (1, \omega_1^2, \omega_1 E)\}$ .

*Proof.* For  $\alpha \in (F_4)^{\gamma, \gamma'} \subset (F_4)^\gamma$ , there exist  $p \in Sp(1)$  and  $A \in Sp(3)$  such that  $\alpha = \varphi_4(p, A)$  (Proposition 1.2.1). From  $\gamma' \alpha \gamma' = \alpha$ , we have  $\varphi_4(\gamma' p, \gamma' A) = \varphi_4(p, A)$  (Lemma 1.2.2). Hence

$$\begin{cases} \gamma' p = p \\ \gamma' A = A \end{cases} \quad \text{or} \quad \begin{cases} \gamma' p = -p \\ \gamma' A = -A. \end{cases}$$

In the former case, we have  $p \in U(1)$  and  $A \in U(3) = \{A \in M(3, C) \mid AA^* = E\}$ . Hence the group of the former case is  $(U(1) \times U(3))/Z_2$ ,  $Z_2 = \{(1, E), (-1, -E)\}$ . In the latter case,  $p = e_2$ ,  $A = e_2 E$  satisfy the conditions and  $\varphi_4(e_2, e_2 E) = \gamma_1$  (Lemma 1.2.2). Hence we have the isomorphism

$$(F_4)^{\gamma, \gamma'} \cong (U(1) \times U(3))/Z_2 \times \{1, \gamma_1\}, \quad Z_2 = \{(1, E), (-1, -E)\}.$$

Since the mapping  $h : U(1) \times SU(3) \rightarrow U(3)$ ,  $h(a, A) = aA$  gives the isomorphism  $U(3) \cong (U(1) \times SU(3))/Z_3$ ,  $Z_3 = \{(1, E), (\omega_1, \omega_1^2 E), (\omega_1^2, \omega_1 E)\}$ , we have the isomorphism  $(F_4)^{\gamma, \gamma'} \cong (U(1) \times U(1) \times SU(3))/(Z_2 \times Z_3) \times \{1, \gamma_1\}$ .  $\square$

### 1.3 Group $E_6$

Let  $\mathfrak{J}^C$  be the complexification of the Jordan algebra  $\mathfrak{J}$ . In  $\mathfrak{J}^C$ , we define the determinant  $\det X$  by  $\frac{1}{3}(X, X \times X)$  and the Hermite inner product  $\langle X, Y \rangle$  by  $(\tau X, Y)$ , respectively.

The simply connected compact Lie group  $E_6$  is given by

$$\begin{aligned} E_6 &= \{\alpha \in \text{Iso}_C(\mathfrak{J}^C) \mid \det \alpha X = \det X, \langle \alpha X, \alpha Y \rangle = \langle X, Y \rangle\} \\ &= \{\alpha \in \text{Iso}_C(\mathfrak{J}^C) \mid \alpha X \times \alpha Y = \tau \alpha \tau(X \times Y), \langle \alpha X, \alpha Y \rangle = \langle X, Y \rangle\}. \end{aligned}$$

We have naturally the inclusion  $G_2 \subset F_4 \subset E_6$  ([5],[7]).

Let  $k : \mathbf{H} = C \oplus C e_2 \rightarrow M(2, C)$  be the  $\mathbf{R}$ -linear mapping defined by

$$k(a + b e_2) = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \quad a, b \in C.$$

This  $k$  is naturally extended to  $\mathbf{R}$ -linear mappings

$$k : M(3, \mathbf{H}) \rightarrow M(6, C), \quad k : \mathbf{H}^3 \rightarrow M(2, 6, C).$$

Furthermore, these are extended to  $C$ - $C$ -linear isomorphisms

$$k : M(3, \mathbf{H})^C \rightarrow M(6, C), \quad k : (\mathbf{H}^3)^C \rightarrow M(2, 6, C),$$

defined by

$$\begin{aligned} k(M_1 + i M_2) &= k(M_1) + e_1 k(M_2), \quad M_1, M_2 \in M(3, \mathbf{H}), \\ k(a_1 + i a_2) &= k(a_1) + e_1 k(a_2), \quad a_1, a_2 \in \mathbf{H}^3. \end{aligned}$$

Finally, we define the  $C$ -vector space  $\mathfrak{S}(6, C)$  by  $\{S \in M(6, C) \mid {}^t S = -S\}$  and the  $C$ - $C$ -linear isomorphism  $k_J : \mathfrak{J}(3, \mathbf{H})^C \rightarrow \mathfrak{S}(6, C)$  by

$$k_J(M_1 + i M_2) = k(M_1)J + e_1 k(M_2)J, \quad M_1, M_2 \in M(3, \mathbf{H}),$$

where  $J = \text{diag}(J, J, J)$ ,  $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ .

Using the inclusion  $F_4 \subset E_6$ , the  $\mathbf{R}$ -linear transformations  $\gamma, \gamma', \gamma_1$  of  $\mathfrak{J}$  are extended to  $C$ -linear transformations  $\gamma, \gamma', \gamma_1$  of  $\mathfrak{J}^C$ .

**PROPOSITION 1.3.1.**  $(E_6)^\gamma \cong (Sp(1) \times SU(6))/Z_2$ ,  $Z_2 = \{(1, E), (-1, -E)\}$ .

*Proof.* Let  $Sp(1) = \{p \in \mathbf{H} \mid p\bar{p} = 1\}$  and  $SU(6) = \{A \in M(6, C) \mid AA^* = E, \det A = 1\}$ . The mapping  $\varphi_6 : Sp(1) \times SU(6) \rightarrow (E_6)^\gamma$ ,

$$\begin{aligned} \varphi_6(p, A)(M + a) &= k_J^{-1}(A(k_J(M))^t A) + p a (k^{-1}(A^*)), \\ M + a &\in \mathfrak{J}(3, \mathbf{H})^C \oplus (\mathbf{H}^3)^C = \mathfrak{J}^C \end{aligned}$$

induces the required isomorphism (see [5] or [7] for details).  $\square$

**LEMMA 1.3.2.** *The mapping  $\varphi_6 : Sp(1) \times SU(6) \rightarrow (E_6)^\gamma$  satisfies*

$$\gamma' \varphi_6(p, A) \gamma' = \varphi_6(\gamma' p, IAI), \quad \gamma' = \varphi_6(e_1, e_1 I), \quad \gamma_1 = \varphi_6(e_2, J),$$

where  $I = \text{diag}(1, -1, 1, -1, 1, -1) \in M(6, \mathbf{R})$ .

Now, we will determine the group structure of  $(E_6)^{\gamma, \gamma'} = ((E_6)^\gamma)^{\gamma'} = ((E_6)^{\gamma'})^\gamma = (E_6)^\gamma \cap (E_6)^{\gamma'}$ .

**THEOREM 1.3.3.**  $(E_6)^{\gamma, \gamma'} \cong (U(1) \times U(1) \times SU(3) \times SU(3)) / (Z_2 \times Z_3) \times \{1, \gamma_1\}$ ,  $Z_2 = \{(1, 1, E, E), (-1, -1, E, E)\}$ ,  $Z_3 = \{(1, 1, E, E), (1, \omega_1, \omega_1^2 E, \omega_1 E), (1, \omega_1^2, \omega_1 E, \omega_1^2 E)\}$ .

*Proof.* For  $\alpha \in (E_6)^{\gamma, \gamma'} \subset (E_6)^\gamma$ , there exist  $p \in Sp(1)$  and  $A \in SU(6)$  such that  $\alpha = \varphi_6(p, A)$  (Proposition 1.3.1). From  $\gamma' \alpha \gamma' = \alpha$ , we have  $\varphi_6(\gamma' p, IAI) = \varphi_6(p, A)$  (Lemma 1.3.2). Hence

$$\begin{cases} \gamma' p = p \\ IAI = A \end{cases} \quad \text{or} \quad \begin{cases} \gamma' p = -p \\ IAI = -A. \end{cases}$$

In the former case, we have  $p \in U(1)$ . Since  $I = \text{diag}(1, -1, 1, -1, 1, -1)$  is conjugate to  $I_3 = \text{diag}(1, 1, 1, -1, -1, -1)$  in  $SU(6)$ , the group  $\{A \in SU(6) \mid IAI = A\}$  is isomorphic to the group  $\{A \in SU(6) \mid I_3 A I_3 = A\} = S(U(3) \times U(3))$ . Hence the group of the former case is  $(U(1) \times S(U(3) \times U(3))) / Z_2$ ,  $Z_2 = \{(1, E), (-1, -E)\}$ . In the latter case,  $p = e_2$ ,  $A = J$  satisfy the conditions and  $\varphi_6(e_2, J) = \gamma_1$  (Lemma 1.3.2). Hence we have the isomorphism

$$(E_6)^{\gamma, \gamma'} \cong (U(1) \times S(U(3) \times U(3))) / Z_2 \times \{1, \gamma_1\}, \quad Z_2 = \{(1, E), (-1, -E)\}.$$

Since the mapping  $h : U(1) \times SU(3) \times SU(3) \rightarrow S(U(3) \times U(3))$ ,  $h(a, A, B) = \begin{pmatrix} aA & 0 \\ 0 & a^{-1}B \end{pmatrix}$  gives the isomorphism  $S(U(3) \times U(3)) \cong (U(1) \times SU(3) \times SU(3)) / Z_3$ ,  $Z_3 = \{(1, E, E), (\omega_1, \omega_1^2 E, \omega_1 E), (\omega_1^2, \omega_1 E, \omega_1^2 E)\}$ , we have the isomorphism  $(E_6)^{\gamma, \gamma'} \cong (U(1) \times U(1) \times SU(3) \times SU(3)) / (Z_2 \times Z_3) \times \{1, \gamma_1\}$ .  $\square$

#### 1.4 Group $E_7$

We define the  $C$ -vector space  $\mathfrak{P}^C$ , called the Freudenthal  $C$ -vector space, by

$$\mathfrak{P}^C = \mathfrak{J}^C \oplus \mathfrak{J}^C \oplus C \oplus C$$

with the Hermite inner product

$$\langle P, Q \rangle = \langle X, Z \rangle + \langle Y, W \rangle + (\tau\xi)\zeta + (\tau\eta)\omega,$$

for  $P = (X, Y, \xi, \eta), Q = (Z, W, \zeta, \omega) \in \mathfrak{P}^C$ . For  $\phi \in \mathfrak{e}_6$ ,  $A, B \in \mathfrak{J}^C$  and  $\nu \in C$ , we define the  $C$ -linear mapping  $\Phi(\phi, A, B, \nu) : \mathfrak{P}^C \rightarrow \mathfrak{P}^C$  by

$$\Phi(\phi, A, B, \nu) \begin{pmatrix} X \\ Y \\ \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \phi X - \frac{1}{3}\nu X + 2B \times Y + \eta A \\ 2A \times X - {}^t\phi Y + \frac{1}{3}\nu Y + \xi B \\ (A, Y) + \nu\xi \\ (B, X) - \nu\eta \end{pmatrix},$$

where  ${}^t\phi \in \mathfrak{e}_6$  is the transpose of  $\phi$  with respect to the inner product  $(X, Y)$ :  $({}^t\phi X, Y) = (X, \phi Y)$ . Next, for  $P = (X, Y, \xi, \eta), Q = (Z, W, \zeta, \omega) \in \mathfrak{P}^C$ , we define the  $C$ -linear mapping  $P \times Q : \mathfrak{P}^C \rightarrow \mathfrak{P}^C$  by

$$P \times Q = \Phi(\phi, A, B, \nu), \quad \begin{cases} \phi = -\frac{1}{2}(X \vee W + Z \vee Y) \\ A = -\frac{1}{4}(2Y \times W - \xi Z - \zeta X) \\ B = \frac{1}{4}(2X \times Z - \eta W - \omega Y) \\ \nu = \frac{1}{8}((X, W) + (Z, Y) - 3(\xi\omega + \zeta\eta)), \end{cases}$$

where  $X \vee W \in \mathfrak{e}_6$  is defined by  $(X \vee W)U = \frac{1}{2}(W, U)X + \frac{1}{6}(X, W)U - 2W \times (X \times U)$  for  $U \in \mathfrak{J}^C$ .

The simply connected compact Lie group  $E_7$  is given by

$$E_7 = \{ \alpha \in \text{Iso}_C(\mathfrak{P}^C) \mid \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q, \langle \alpha P, \alpha Q \rangle = \langle P, Q \rangle \}.$$

We have naturally the inclusion  $G_2 \subset F_4 \subset E_6 \subset E_7$  ([6], [7]).

Using the inclusion  $E_6 \subset E_7$ , the  $C$ -linear transformations  $\gamma, \gamma', \gamma_1$  of  $\mathfrak{J}^C$  are extended to  $C$ -linear transformations  $\gamma, \gamma', \gamma_1$  of  $\mathfrak{P}^C$ .

We define the  $C$ -linear transformation  $\lambda : \mathfrak{P}^C \rightarrow \mathfrak{P}^C$  by

$$\lambda(X, Y, \xi, \eta) = (Y, -X, \eta, -\xi).$$

Then  $\lambda \in E_7$  and  $\lambda^2 = -1$ . Note that  $\alpha \in E_7$  satisfies  $\tau\lambda\alpha = \alpha\tau\lambda$ .

**LEMMA 1.4.1.** *The Lie algebra  $\mathfrak{e}_7$  of the group  $E_7$  is given by*

$$\mathfrak{e}_7 = \{ \Phi(\phi, A, -\tau A, \nu) \mid \phi \in \mathfrak{e}_6, A \in \mathfrak{J}^C, \nu \in i\mathbb{R} \}.$$

*The Lie bracket in  $\mathfrak{e}_7$  is given as follows.*

$$[\Phi(\phi_1, A_1, -\tau A_1, \nu_1), \Phi(\phi_2, A_2, -\tau A_2, \nu_2)] = \Phi(\phi, A, -\tau A, \nu),$$



$$\begin{cases} \phi = [\phi_1, \phi_2] - 2A_1 \vee \tau A_2 + 2A_2 \vee \tau A_1 \\ A = \left(\phi_1 + \frac{2}{3}\nu_1\right)A_2 - \left(\phi_2 + \frac{2}{3}\nu_2\right)A_1 \\ \nu = \langle A_1, A_2 \rangle - \langle A_2, A_1 \rangle. \end{cases}$$

To know the group structure of  $(E_7)^\gamma$ , we first investigate the group  $(E_7)^\sigma$  which is isomorphic to  $(E_7)^\gamma$ . Let  $\sigma, \sigma' : \mathfrak{J}^C \rightarrow \mathfrak{J}^C$  be  $C$ -linear transformations defined by

$$\sigma X = \sigma \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & -x_3 & -\bar{x}_2 \\ -\bar{x}_3 & \xi_2 & x_1 \\ -x_2 & \bar{x}_1 & \xi_3 \end{pmatrix}, \quad \sigma' X = \begin{pmatrix} \xi_1 & x_3 & -\bar{x}_2 \\ \bar{x}_3 & \xi_2 & -x_1 \\ -x_2 & -\bar{x}_1 & \xi_3 \end{pmatrix},$$

and extend to  $C$ -linear transformations  $\sigma, \sigma' : \mathfrak{P}^C \rightarrow \mathfrak{P}^C$  by

$$\sigma P = \sigma(X, Y, \xi, \eta) = (\sigma X, \sigma Y, \xi, \eta), \quad \sigma'(X, Y, \xi, \eta) = (\sigma' X, \sigma' Y, \xi, \eta),$$

respectively. Then  $\sigma, \sigma' \in F_4 \subset E_6 \subset E_7$  and  $\sigma^2 = \sigma'^2 = 1$ .

We define  $C$ -linear mappings  $\kappa, \mu : \mathfrak{P}^C \rightarrow \mathfrak{P}^C$  by

$$\begin{aligned} \kappa(X, Y, \xi, \eta) &= \left( \begin{pmatrix} -\xi_1 & 0 & 0 \\ 0 & \xi_2 & x_1 \\ 0 & \bar{x}_1 & \xi_3 \end{pmatrix}, \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & -\eta_2 & -y_1 \\ 0 & -\bar{y}_1 & -\eta_3 \end{pmatrix}, -\xi, \eta \right), \\ \mu(X, Y, \xi, \eta) &= \left( \begin{pmatrix} \eta & 0 & 0 \\ 0 & \eta_3 & -y_1 \\ 0 & -\bar{y}_1 & \eta_2 \end{pmatrix}, \begin{pmatrix} \xi & 0 & 0 \\ 0 & \xi_3 & -x_1 \\ 0 & -\bar{x}_1 & \xi_2 \end{pmatrix}, \eta_1, \xi_1 \right), \end{aligned}$$

respectively. We define the subgroup  $(E_7)^{\kappa, \mu}$  of  $E_7$  by

$$(E_7)^{\kappa, \mu} = \{\alpha \in E_7 \mid \kappa\alpha = \alpha\kappa, \mu\alpha = \alpha\mu\}.$$

Then we have the following lemma.

**LEMMA 1.4.2.**  $(E_7)^{\kappa, \mu} \cong Spin(12)$ .

*Proof.* We define a 12-dimensional  $\mathbf{R}$ -vector space  $V^{12}$  by

$$\begin{aligned} V^{12} &= \{P \in \mathfrak{P}^C \mid \kappa P = P, \mu \tau \lambda P = P\} \\ &= \left\{ \left( \begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi & x \\ 0 & \bar{x} & -\tau\xi \end{pmatrix}, \begin{pmatrix} \eta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \tau\eta \right) \mid x \in \mathfrak{C}, \xi, \eta \in C \right\} \end{aligned}$$

with the norm

$$(P, P)_\mu = \frac{1}{2}(\mu P, \lambda P) = \bar{x}x + (\tau\xi)\xi + (\tau\eta)\eta.$$

Let  $SO(12) = SO(V^{12})$ . Then  $(E_7)^{\kappa, \mu}$  is connected and we have  $(E_7)^{\kappa, \mu}/Z_2 \cong SO(12)$ ,  $Z_2 = \{1, \sigma\}$ . Therefore  $(E_7)^{\kappa, \mu}$  is isomorphic to  $Spin(12)$  as a double covering group of  $SO(12)$  (see [6] or [7] for details).  $\square$

**LEMMA 1.4.3.** *The Lie algebra  $\mathfrak{spin}(12) = (\mathfrak{e}_7)^{\kappa, \mu}$  of the group  $Spin(12) = (E_7)^{\kappa, \mu}$  is given by*

$$\begin{aligned} (\mathfrak{e}_7)^{\kappa, \mu} &= \{\Phi \in \mathfrak{e}_7 \mid \kappa\Phi = \Phi\kappa, \mu\Phi = \Phi\mu\} \\ &= \left\{ \Phi(\phi, A, -\tau A, \nu) \in \mathfrak{e}_7 \mid \begin{array}{l} \phi \in \mathfrak{e}_6, \sigma\phi = \phi\sigma, A \in \mathfrak{J}^C, \sigma A = A, \\ (E_1, A) = 0, \nu = -\frac{3}{2}(\phi E_1, E_1) \end{array} \right\}. \end{aligned}$$

In more detail,  $\phi$  and  $A$  are of the forms:

$$\phi = d + \tilde{A}_1(a) + i\tilde{T}, \quad A = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & a_1 \\ 0 & \bar{a}_1 & \alpha_3 \end{pmatrix},$$

$$\text{where } d \in \mathfrak{so}(8) = \mathfrak{so}(\mathfrak{C}), A_1(a) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & -\bar{a} & 0 \end{pmatrix}, T = \begin{pmatrix} \tau_1 & 0 & 0 \\ 0 & \tau_2 & t_1 \\ 0 & \bar{t}_1 & \tau_3 \end{pmatrix}, a, t_1 \in$$

$\mathfrak{C}, \tau_k \in \mathbf{R}, \tau_1 + \tau_2 + \tau_3 = 0, \alpha_k \in C, a_1 \in \mathfrak{C}^C$  and the notation  $\tilde{S}$  ( $S = A_1(a)$  or  $T$ ) is the  $\mathbf{R}$ -linear mapping of  $\mathfrak{J}$  defined by

$$\tilde{S}X = \frac{1}{2}(SX + XS^*), \quad X \in \mathfrak{J}^C.$$

Let  $SU(2) = \{A \in M(2, C) \mid A^t(\tau A) = E, \det A = 1\}$  and we define the mapping  $\varphi: SU(2) \rightarrow (E_7)^\sigma$  by

$$\varphi(A)(X, Y, \xi, \eta) = (X', Y', \xi', \eta'),$$

$$\begin{pmatrix} \xi_1' \\ \eta' \end{pmatrix} = A \begin{pmatrix} \xi_1 \\ \eta \end{pmatrix}, \quad \begin{pmatrix} \xi' \\ \eta_1' \end{pmatrix} = A \begin{pmatrix} \xi \\ \eta_1 \end{pmatrix}, \quad \begin{pmatrix} \eta_2' \\ \xi_3' \end{pmatrix} = A \begin{pmatrix} \eta_2 \\ \xi_3 \end{pmatrix}, \quad \begin{pmatrix} \eta_3' \\ \xi_2' \end{pmatrix} = A \begin{pmatrix} \eta_3 \\ \xi_2 \end{pmatrix},$$

$$\begin{pmatrix} x_1' \\ y_1' \end{pmatrix} = (\tau A) \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \quad \begin{pmatrix} x_2' \\ y_2' \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \quad \begin{pmatrix} x_3' \\ y_3' \end{pmatrix} = \begin{pmatrix} x_3 \\ y_3 \end{pmatrix}.$$

Then  $\varphi$  is an injective homomorphism:  $\varphi(SU(2)) \subset (E_7)^\sigma$ .

**PROPOSITION 1.4.4.**  $(E_7)^\sigma \cong (SU(2) \times Spin(12))/Z_2$ ,  $Z_2 = \{(E, 1), (-E, -\sigma)\}$ .

*Proof.* Let  $Spin(12) = (E_7)^{\kappa, \mu}$  (Lemma 1.4.2). We define a mapping  $\varphi_7 : SU(2) \times Spin(12) \rightarrow (E_7)^\sigma$  by

$$\varphi_7(A, \beta) = \varphi(A)\beta.$$

Since  $\varphi(A)$  and  $\beta$  are commutative,  $\varphi_7$  is a homomorphism. Furthermore,  $\varphi_7$  is onto and  $\text{Ker } \varphi_7 = \{(E, 1), (-E, -\sigma)\} = Z_2$ . Hence we have the required isomorphism (see [6] or [7] for details).  $\square$

Let  $SU(8) = \{A \in M(8, \mathbb{C}) \mid AA^* = E, \det A = 1\}$ ,  $\mathfrak{J}(4, H)^C = \{X \in M(4, H^C) \mid X^* = X\}$  and  $\mathfrak{S}(8, C)^C = \{S \in M(8, C^C) \mid {}^t S = -S\}$ . To define the following mapping  $\varphi_1 : SU(8) \rightarrow E_7$ , we use the  $C$ -linear mapping  $g : \mathfrak{J}^C \rightarrow \mathfrak{J}(4, H)^C$ ,

$$g(M + a) = \begin{pmatrix} \frac{1}{2}\text{tr}(M) & ia \\ ia^* & M - \frac{1}{2}\text{tr}(M)E \end{pmatrix}, \quad M + a \in \mathfrak{J}(3, H)^C \oplus (H^3)^C = \mathfrak{J}^C.$$

Now, we define the  $C$ -linear isomorphism  $\chi : \mathfrak{P}^C \rightarrow \mathfrak{S}(8, C)^C$  by

$$\chi(X, Y, \xi, \eta) = k\left(gX - \frac{\xi}{2}E\right)J + e_1 k\left(g(\gamma Y) - \frac{\eta}{2}E\right)J,$$

where the mapping  $k : M(4, H) \rightarrow M(8, C)$  is the naturally extended mapping of  $k$  defined in the section  $E_6$  and  $J = \text{diag}(J, J, J, J)$ ,  $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ .

**LEMMA 1.4.5.**  $(E_7)^{\tau\gamma} \cong SU(8)/Z_2$ ,  $Z_2 = \{E, -E\}$ .

*Proof.* We define a mapping  $\varphi_1 : SU(8) \rightarrow (E_7)^{\tau\gamma}$  by

$$\varphi_1(A)P = \chi^{-1}(A(\chi(P))^t A), \quad P \in \mathfrak{P}^C.$$

$\varphi_1$  is well-defined, a surjective homomorphism and  $\text{Ker } \varphi_1 = \{E, -E\}$ . Hence we have the required isomorphism (see [6] or [7] for details).  $\square$

We shall show that  $\gamma$  is conjugate to  $-\sigma$  in  $E_7$ . For this end, we first define an  $R$ -linear transformation  $\delta_1 : \mathfrak{C} \rightarrow \mathfrak{C}$  satisfying

$$1 \rightarrow 1, e_1 \rightarrow e_4, e_2 \rightarrow e_2, e_3 \rightarrow e_6, e_4 \rightarrow e_1, e_5 \rightarrow -e_5, e_6 \rightarrow e_3, e_7 \rightarrow -e_7,$$

then  $\delta_1 \in G_2 \subset F_4 \subset E_6 \subset E_7$ ,  $\delta_1^2 = 1$  and satisfies

$$\delta_1 \gamma \delta_1 = \gamma_1.$$

Next, we define a  $C$ -linear transformation  $\delta_2 : \mathfrak{P}^C \rightarrow \mathfrak{P}^C$  by  $\varphi_1(D)$ , where

$$D = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -e_1 & 0 & 0 & 0 & e_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -e_1 & 0 & 0 & 0 & e_1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -e_1 & 0 & 0 & 0 & e_1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -e_1 & 0 & 0 & 0 & e_1 \end{pmatrix} \in SU(8),$$

then  $\delta_2 \in E_7$  and  $\delta_2^{-1} \gamma_1 \delta_2 = -\sigma$ . Indeed, since  $\varphi_1(J) = \gamma_1$  and  $\varphi_1(e_1 I_4) = -\sigma$  ( $I_4 = \text{diag}(-1, -1, -1, -1, 1, 1, 1, 1) \in M(8, \mathbf{R})$ ), we have

$$\delta_2^{-1} \gamma_1 \delta_2 = \varphi_1(D^*) \varphi_1(J) \varphi_1(D) = \varphi_1(D^* J D) = \varphi_1(e_1 I_4) = -\sigma.$$

Now, let  $\delta = \delta_1 \delta_2$ . Then we have

$$\delta^{-1} \gamma \delta = -\sigma.$$

As a consequence, we obtain the following isomorphism

$$(E_7)^\gamma \cong (E_7)^{-\sigma} = (E_7)^\sigma \cong (SU(2) \times Spin(12))/\mathbf{Z}_2,$$

under the correspondence

$$\begin{array}{ccccc} SU(2) \times Spin(12) & \rightarrow & (E_7)^\sigma & \rightarrow & (E_7)^\gamma \\ (A, \beta) & \rightarrow & \varphi(A)\beta & \rightarrow & \delta(\varphi(A)\beta)\delta^{-1}. \end{array}$$

Instead of investigating the group  $(E_7)^{\gamma, \gamma'}$ , we shall study the group  $(E_7)^{\sigma, \sigma''}$ , where  $\sigma'' \in E_7$  is the involutive element defined by

$$\sigma'' = \delta^{-1} \gamma' \delta.$$

Since  $\delta_1 \gamma' \delta_1 = \gamma'$  and  $\gamma' = \varphi_1(e_1 I)(I = \text{diag}(1, -1, 1, -1, 1, -1, 1, -1) \in M(8, \mathbf{R}))$ , we have

$$\sigma'' = \delta^{-1} \gamma' \delta = \delta_2^{-1} \gamma' \delta_2 = \varphi_1(D^*) \varphi_1(e_1 I) \varphi_1(D) = \varphi_1(D^* e_1 I D) = \varphi_1(J''),$$

where  $J'' = D^* e_1 I D = \begin{pmatrix} 0 & e_1 E \\ e_1 E & 0 \end{pmatrix} \in SU(8)$  ( $E$  is the  $4 \times 4$  unit matrix).

Since  $J''^2 = -E$  ( $E$  is the  $8 \times 8$  unit matrix), we have  $\sigma''^2 = 1$ . The action of  $\sigma''$  on  $\mathfrak{P}^C$  is given by

$$\begin{aligned} & \sigma''(X, Y, \xi, \eta) \\ &= \left( \begin{pmatrix} \xi_1 & ie_4 x_3 & * \\ * & -\xi_2 & ix_1 e_4 \\ e_4 x_2 e_4 & * & \xi_3 \end{pmatrix}, \begin{pmatrix} \eta_1 & -ie_4 y_3 & * \\ * & -\eta_2 & -iy_1 e_4 \\ e_4 y_2 e_4 & * & \eta_3 \end{pmatrix}, -\xi, -\eta \right). \end{aligned} \quad (i)$$

If we use elements  $\sigma' \in E_6$  (indicated before) and  $\rho \in E_6$ :

$$\rho X = \overline{P} X P = \begin{pmatrix} -\xi_1 & -ie_4 x_3 & * \\ * & \xi_2 & ix_1 e_4 \\ e_4 x_2 e_4 & * & -\xi_3 \end{pmatrix}, \quad X \in \mathfrak{J}^C, P = \begin{pmatrix} ie_4 & & \\ & 1 & \\ & & ie_4 \end{pmatrix},$$

then  $\sigma''$  is also written as

$$\sigma''(X, Y, \xi, \eta) = -(\sigma' \rho X, \tau \sigma' \rho \tau Y, \xi, \eta).$$

From the form of (i), we see

$$\kappa \sigma'' = \sigma'' \kappa, \quad \mu \sigma'' = -\sigma'' \mu. \quad (ii)$$

Furthermore, we see that  $\sigma''$  leaves invariant the group  $\varphi(SU(2))$ . We shall determine elements  $\varphi(A)$ ,  $A \in SU(2)$  such that  $\sigma'' \varphi(A) \sigma'' = \varphi(A)$ . In the following,  $\varphi(SU(2))$  is often denoted by  $SU(2)$ .

**PROPOSITION 1.4.6.**  $(SU(2))^{\sigma''} \cong U(1)$ .

*Proof.* If  $A \in SU(2)$  satisfies  $\sigma'' \varphi(A) \sigma'' = \varphi(A)$ , then we can easily see that  $\varphi(A)$  is of the form  $\varphi\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right)$ ,  $a \in U(1) = \{a \in \mathbb{C} \mid a(\tau a) = 1\}$ . Hence we have

$$\begin{aligned} (SU(2))^{\sigma''} &= \{\varphi(A) \mid A \in SU(2), \sigma'' \varphi(A) \sigma'' = \varphi(A)\} \\ &= \left\{ \varphi\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right) \mid a \in U(1) \right\} \cong U(1). \end{aligned}$$

□

From (ii), we see that  $\sigma''$  leaves invariant the group  $Spin(12)$ :  $\sigma'' \beta \sigma'' \in Spin(12)$  for  $\beta \in Spin(12)$ . Now, we consider elements  $\beta \in Spin(12)$  such that  $\sigma'' \beta \sigma'' = \beta$ .

**LEMMA 1.4.7.**  $(Spin(12))^{\sigma''}/Z_2 \cong U(6)$ ,  $Z_2 = \{1, \sigma\}$ .

*Proof.* We define a  $C$ -vector space  $(V^C)^6$  by

$$\begin{aligned} (V^C)^6 &= (\mathfrak{P}^C)_{\kappa, \sigma''} = \{P \in \mathfrak{P}^C \mid \kappa P = P, \sigma'' P = P\} \\ &= \left\{ \left( \begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi_2 & x_1 \\ 0 & \bar{x}_1 & \xi_3 \end{pmatrix}, \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \eta \right) \mid x_1 \in \mathfrak{C}^C, \xi_k, \eta_1, \eta \in C, \sigma'' P = P \right\} \\ &= \left\{ \left( \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & \bar{x} & \xi \end{pmatrix}, \begin{pmatrix} \eta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, 0 \right) \mid x \in (\mathfrak{C}^C)_{ie_4}, \xi, \eta \in C \right\}, \end{aligned}$$

where  $(\mathfrak{C}^C)_{ie_4}$  is

$$\begin{aligned} (\mathfrak{C}^C)_{ie_4} &= \{x \in \mathfrak{C}^C \mid ixe_4 = x\} \\ &= \{(x_0 + x_1e_1 + x_2e_2 + x_3e_3) + i(x_0 + x_1e_1 + x_2e_2 + x_3e_3)e_4 \mid x_k \in C\}. \end{aligned}$$

$(V^C)^6$  has the norm  $\langle P, P \rangle = (\tau P, P)$ , that is, the norm of  $P \in (V^C)^6$  is given by

$$\begin{aligned} \langle P, P \rangle &= 2(\tau x)x + (\tau \xi)\xi + (\tau \eta)\eta \\ &= 4((\tau x_0)x_0 + (\tau x_1)x_1 + (\tau x_2)x_2 + (\tau x_3)x_3) + (\tau \xi)\xi + (\tau \eta)\eta. \end{aligned}$$

We define a unitary group  $U(6)$  by

$$U(6) = \{\alpha \in \text{Iso}_C((V^C)^6) \mid \langle \alpha P, \alpha P \rangle = \langle P, P \rangle\}.$$

Since  $\alpha \in (Spin(12))^{\sigma''}$  satisfies  $\kappa\alpha = \alpha\kappa$  and  $\sigma''\alpha = \alpha\sigma''$ ,  $\alpha$  leaves invariant the space  $(V^C)^6$  and preserves the norm  $\langle P, P \rangle$ , so  $\alpha$  induces an element of  $U(6)$ , hence we can define the mapping  $f : (Spin(12))^{\sigma''} \rightarrow U(6)$  by

$$f(\alpha) = \alpha|_{(V^C)^6}.$$

To show that  $f$  is onto, we use the following lemma.

**LEMMA 1.4.8.** The Lie algebra  $(\mathfrak{spin}(12))^{\sigma''}$  of the group  $(Spin(12))^{\sigma''}$  is given by

$$\begin{aligned} &(\mathfrak{spin}(12))^{\sigma''} \\ &= \left\{ \Phi \left( \begin{pmatrix} D_1 & & \\ -K_3 D_2 K_3 & D_2 & \\ & K_3 D_1 K_3 & \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -t_1 e_4 \\ 0 & -e_4 \bar{t}_1 & 0 \end{pmatrix} \right) + i \begin{pmatrix} \tau_1 & 0 & 0 \\ 0 & \tau_2 & t_1 \\ 0 & \bar{t}_1 & \tau_3 \end{pmatrix} \right\}, \end{aligned}$$

$$\left( \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha_2 & a_1 \\ 0 & \bar{a}_1 & 0 \end{pmatrix}, -\tau \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha_2 & a_1 \\ 0 & \bar{a}_1 & 0 \end{pmatrix}, -\frac{3}{2}i\tau_1 \right) \mid D_1, D_2 \in M(4, \mathbf{R}), {}^t D_1 = -D_1, \\ {}^t D_2 = K_3 D_2 K_3, t_1 \in \mathfrak{C}, \tau_k \in \mathbf{R}, \tau_1 + \tau_2 + \tau_3 = 0, \alpha_2 \in C, a_1 \in \mathfrak{C}^C, -ia_1 e_4 = a_1 \Big\},$$

where  $K_3 = \text{diag}(1, 1, -1, 1) \in M(4, \mathbf{R})$ . In particular, the dimension of  $(\mathfrak{spin}(12))^{\sigma''}$  is 36.

*Proof.* The definition of the Lie algebra  $(\mathfrak{spin}(12))^{\sigma''}$  is

$$(\mathfrak{spin}(12))^{\sigma''} = \{\Phi \in (\mathfrak{e}_7)^{\kappa, \mu} \mid \sigma'' \Phi \sigma'' = \Phi\}.$$

For  $\Phi(\phi, A, -\tau A, \nu) \in \mathfrak{e}_7$ , since

$$\sigma''(\Phi(\phi, A, -\tau A, \nu))\sigma'' = \Phi(\sigma' \rho \phi \rho \sigma', \sigma' \rho A, -\tau(\sigma' \rho A), \nu),$$

if  $\Phi(\phi, A, -\tau A, \nu) \in \mathfrak{spin}(12) = (\mathfrak{e}_7)^{\kappa, \mu}$ , then  $A$  is of the form  $\begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha_2 & a_1 \\ 0 & \bar{a}_1 & \alpha_3 \end{pmatrix}$ ,

$\alpha_k \in C, a_1 \in \mathfrak{C}^C$ . Together with the condition  $\sigma' \rho A = A$ , we see that  $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha_2 & a_1 \\ 0 & \bar{a}_1 & 0 \end{pmatrix}$ ,  $\alpha_2 \in C, a_1 \in \mathfrak{C}^C, -ia_1 e_4 = a_1$ . Next, let  $\phi = d + \tilde{A}_1(a) + i\tilde{T}$ ,  $d \in$

$$\mathfrak{so}(8), A_1(a) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & -\bar{a} & 0 \end{pmatrix}, a \in \mathfrak{C}, T = \begin{pmatrix} \tau_1 & 0 & 0 \\ 0 & \tau_2 & t_1 \\ 0 & \bar{t}_1 & \tau_3 \end{pmatrix} \tau_k \in \mathbf{R}, \tau_1 + \tau_2 + \tau_3 =$$

$0, t_1 \in \mathfrak{C}$ . We see that  $\sigma' \rho d \rho \sigma'$  belongs to  $\mathfrak{so}(8)$  and  $\sigma'(\tilde{A}_1(a) + i\tilde{T})\rho \sigma'$  has no part of  $\mathfrak{so}(8)$ . Hence the condition  $\sigma' \rho \phi \rho \sigma' = \phi$  implies  $\sigma' \rho d \rho \sigma' = d$  and  $\sigma' \rho(\tilde{A}_1(a) + i\tilde{T})\rho \sigma' = \tilde{A}_1(a) + i\tilde{T}$ . From the second condition  $\sigma' \rho(\tilde{A}_1(a) + i\tilde{T})\rho \sigma' = \tilde{A}_1(a) + i\tilde{T}$ , we see that  $a$  and  $t_1$  have the relation  $a = -t_1 e_4$ . Finally, we shall determine the form of  $d \in \mathfrak{so}(8)$  such that  $\sigma' \rho d \rho \sigma' = d$ . Denote  $d =$

$$\begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix} \in \mathfrak{so}(8), D_k \in M(4, \mathbf{R}). \text{ From the condition } \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix} \begin{pmatrix} x \\ iK_3 x \end{pmatrix} = \begin{pmatrix} y \\ iK_3 y \end{pmatrix}, x, y \in \mathbf{R}^4, \text{ we have the relation } D_3 x + iD_4 K_3 x = iK_3 D_3 x - K_3 D_2 K_3 x.$$

Hence  $D_4 = K_3 D_1 K_3, D_3 = -K_3 D_2 K_3$ . Since  $d \in \mathfrak{so}(8)$ , that is,  ${}^t d = -d$ ,  $D_1, D_2$  have the relations  ${}^t D_1 = -D_1, {}^t D_2 = K_3 D_2 K_3$ . Thus the Lie algebra  $(\mathfrak{spin}(12))^{\sigma''}$  is determined. The dimension of  $(\mathfrak{spin}(12))^{\sigma''}$  is  $16 + 2 + 8 + 10 = 36$ .  $\square$

We shall return to prove Lemma 1.4.7. It is easy to see that  $\text{Ker } f = \{1, \sigma\}$ . Since  $U(6)$  is connected and  $\dim((\mathfrak{spin}(12))^{\sigma''}) = 36 = \dim(\mathfrak{u}(6))$ ,  $f$  is onto. Hence we have the isomorphism  $(\text{Spin}(12))^{\sigma''}/\mathbb{Z}_2 \cong U(6)$ . Thus Lemma 1.4.7 is proved.  $\square$

Before we prove the following Proposition 1.4.9, we define an element  $w$  of  $\text{Spin}(12)$  by

$$w(X, Y, \xi, \eta) = \left( \begin{pmatrix} \omega^2 \xi_1 & \omega^2 \omega_4^2 x_3 & * \\ * & \omega^2 \xi_2 & x_1 \omega_4^2 \\ \omega_4 x_2 \omega_4 & * & \omega \xi_3 \end{pmatrix}, \begin{pmatrix} \omega \eta_1 & \omega \omega_4^2 y_3 & * \\ * & \omega \eta_2 & y_1 \omega_4^2 \\ \omega_4 y_2 \omega_4 & * & \omega^2 \eta_3 \end{pmatrix}, \omega \xi, \omega^2 \eta \right).$$

where  $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2} \in C$ ,  $\omega_4 = -\frac{1}{2} + \frac{\sqrt{3}}{2}e_4 \in \mathfrak{C}$ . Then  $w \in (\text{Spin}(12))^{\sigma''}$  and  $w^3 = 1$ .

**PROPOSITION 1.4.9.**  $(\text{Spin}(12))^{\sigma''} \cong (U(1) \times SU(6))/\mathbb{Z}_6$ ,  $\mathbb{Z}_6 = \{(1, 1), (-\sigma w^2, -\sigma w), (w, w^2), (-\sigma, -\sigma), (w^2, w), (-\sigma w, -\sigma w^2)\}$ .

*Proof.* The unitary group  $U(6)$  is decomposable as

$$U(6) = U_1(1)SU_1(6), \quad U_1(1) \cap SU_1(6) = \{zE \mid z \in C, z^6 = 1\},$$

where  $U_1(1) = \{e^{it}E \mid t \in \mathbb{R}\}$  which is the connected component of the center of  $U(6)$  and  $SU_1(6) = \{A \in U(6) \mid \det A = 1\}$ . On the other hand, the center of  $(\mathfrak{spin}(12))^{\sigma''}$  is

$$\left\{ \zeta(t) = \Phi \left( t \begin{pmatrix} 0 & K_3 \\ -K_3 & 0 \end{pmatrix} + \frac{2}{3}it \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, 0, 0, it \right) \mid t \in \mathbb{R} \right\},$$

hence the connected component  $U(1)$  of the center of  $(\text{Spin}(12))^{\sigma''}$  is given by

$$U(1) = \{z(t) = \exp(\zeta(t)) \mid t \in \mathbb{R}\}.$$

The action of  $z(t) \in U(1)$  on  $\mathfrak{P}^C$  is given by

$$z(t) \begin{pmatrix} X \\ Y \\ \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} e^{-it}\xi_1 & e^{-it}e^{-e_4 t}x_3 & * \\ * & e^{-it}\xi_2 & x_1 e^{-e_4 t} \\ e^{e_4 t}x_2 e^{e_4 t} & * & e^{it}\xi_3 \end{pmatrix} \\ \begin{pmatrix} e^{it}\eta_1 & e^{it}e^{-e_4 t}y_3 & * \\ * & e^{it}\eta_2 & y_1 e^{-e_4 t} \\ e^{e_4 t}y_2 e^{e_4 t} & * & e^{-it}\eta_3 \end{pmatrix} \\ e^{it}\xi \\ e^{-it}\eta \end{pmatrix}.$$



Since the restriction of the function  $e^{-e_4 t}$  on  $(\mathfrak{C}^C)_{ie_4}$  is

$$e^{-e_4 t} x = e^{it} x, \quad x \in (\mathfrak{C}^C)_{ie_4},$$

the restriction of  $z(t)$  on  $(V^6)^C$  is given by

$$\begin{aligned} z(t) & \left( \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & \bar{x} & \xi \end{pmatrix}, \begin{pmatrix} \eta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, 0 \right) \\ & = \left( \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e^{it} x \\ 0 & e^{it} \bar{x} & e^{it} \xi \end{pmatrix}, \begin{pmatrix} e^{it} \eta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, 0 \right). \end{aligned}$$

Hence  $f(z(t))$  is contained in  $U_1(1)$  and  $f$  induces an isomorphism  $f : U(1) \rightarrow U_1(1)$ . Next, we will find a subgroup  $SU(6)$  of  $(Spin(12))^{\sigma''}$  which is isomorphic to the group  $SU_1(6)$  under  $f$ . Consider the subgroup  $\widetilde{SU} = f^{-1}(SU_1(6))$  of  $(Spin(12))^{\sigma''}$ . Then  $\widetilde{SU}/Z_2 \cong SU_1(6)$ . Since  $SU_1(6)$  is simply connected,  $\widetilde{SU}$  is never connected. Let  $SU(6)$  be the connected component subgroup of  $\widetilde{SU}$  containing the identity 1, then  $SU(6)$  is the required one. Thus we have the following commutative diagram :

$$\begin{array}{ccc} U(1) \times SU(6) & \xrightarrow{h} & (Spin(12))^{\sigma''} \\ f \downarrow f & & \downarrow f \\ U_1(1) \times SU_1(6) & \xrightarrow{h_1} & U(6), \end{array}$$

where  $h, h_1$  are multiplication mappings in the groups, respectively. Evidently  $h$  is a surjective homomorphism. We shall find the kernel of  $h$ . Let  $(z, \alpha) \in \text{Ker } h$ . From the diagram above, we have  $f(z)f(\alpha) = f(h(z, \alpha)) = f(1) = 1$ . Hence we obtain  $\text{Ker } h = \{(1, 1), (-\sigma w^2, -\sigma w), (w, w^2), (-\sigma, -\sigma), (w^2, w), (-\sigma w, -\sigma w^2)\} = Z_6$ . Thus we have the required isomorphism  $(Spin(12))^{\sigma''} \cong (U(1) \times SU(6)) / Z_6$ .  $\square$

Now, we will determine the group structure of  $(E_7)^{\gamma, \gamma'} = ((E_7)^\gamma)^{\gamma'} = ((E_7)^{\gamma'})^\gamma = (E_7)^\gamma \cap (E_7)^{\gamma'}$ .

**THEOREM 1.4.10.**  $(E_7)^{\gamma, \gamma'} \cong (U(1) \times U(1) \times SU(6)) / (Z_2 \times Z_6) \times \{1, l_1\}$ ,  $Z_2 = \{(1, 1, 1), (-1, -\sigma, 1)\}$ ,  $Z_6 = \{(1, 1, 1), (1, -\sigma w^2, -\sigma w), (1, w, w^2), (1, -\sigma, -\sigma), (1, w^2, w), (1, -\sigma w, -\sigma w^2)\}$ .

*Proof.* Since  $(E_7)^{\gamma, \gamma'}$  is isomorphic to  $(E_7)^{\sigma, \sigma''}$ , we shall determine the group structure of  $(E_7)^{\sigma, \sigma''}$ . For  $\alpha \in (E_7)^{\sigma, \sigma''} \subset (E_7)^\sigma$ , there exist  $A \in SU(2)$  and

$\beta \in Spin(12)$  such that  $\alpha = \varphi(A)\beta$  (Proposition 1.4.4). From  $\sigma''\alpha\sigma'' = \alpha$ , we have  $\sigma''\varphi(A)\sigma''\sigma''\beta\sigma'' = \varphi(A)\beta$ . Hence

$$\begin{cases} \sigma''\varphi(A)\sigma'' = \varphi(A) \\ \sigma''\beta\sigma'' = \beta \end{cases} \quad \text{or} \quad \begin{cases} \sigma''\varphi(A)\sigma'' = -\varphi(A) \\ \sigma''\beta\sigma'' = -\sigma\beta. \end{cases}$$

In the former case, we have  $A \in U(1)$  (Proposition 1.4.6) and  $\beta \in (Spin(12))^{\sigma''}$ . Hence the group of the former case is

$$(U(1) \times (Spin(12))^{\sigma''})/\mathbf{Z}_2 \cong (U(1) \times U(1) \times SU(6))/(\mathbf{Z}_2 \times \mathbf{Z}_6)$$

(Proposition 1.4.9). We consider the latter case. For  $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ ,  $\varphi(J)$  is of the form

$$\varphi(J)(X, Y, \xi, \eta) = \left( \begin{pmatrix} \eta & x_3 & \bar{x}_2 \\ \bar{x}_3 & -\eta_3 & y_1 \\ x_2 & \bar{y}_1 & -\eta_2 \end{pmatrix}, \begin{pmatrix} \xi & y_3 & \bar{y}_2 \\ \bar{y}_3 & \xi_3 & -x_1 \\ y_2 & -\bar{x}_1 & \xi_2 \end{pmatrix}, -\eta_1, -\xi_1 \right),$$

and satisfies

$$\sigma''\varphi(J)\sigma'' = -\varphi(J).$$

To find an element  $l \in Spin(12)$  such that  $\sigma''l\sigma'' = -\sigma l$ , first consider  $\alpha_1 = \exp\left(\Phi\left(0, \frac{\pi}{2}, -\frac{\pi}{2}, 0\right)\right) \in E_7$ . The explicit form  $\alpha_1$  is given by

$$\alpha_1(X, Y, \xi, \eta) = \left( \begin{pmatrix} \eta & x_3 & \bar{x}_2 \\ \bar{x}_3 & -\eta_3 & y_1 \\ x_2 & \bar{y}_1 & -\eta_2 \end{pmatrix}, \begin{pmatrix} -\xi & y_3 & \bar{y}_2 \\ \bar{y}_3 & \xi_3 & -x_1 \\ y_2 & -\bar{x}_1 & \xi_2 \end{pmatrix}, \eta_1, \xi_1 \right),$$

and satisfies

$$\kappa\alpha_1 = -\alpha_1\kappa, \quad \mu\alpha_1 = -\alpha_1\mu, \quad \sigma''\alpha_1\sigma'' = -\sigma\alpha_1.$$

Next, for  $\lambda \in E_7$ , we have

$$\kappa\lambda = -\lambda\kappa, \quad \mu\lambda = -\lambda\mu, \quad \sigma''\lambda\sigma'' = \lambda\sigma_{13},$$

where  $\sigma_{13} \in F_4 \subset E_6 \subset F_7$  is defined by  $\sigma_{13}X = \begin{pmatrix} \xi_1 & -x_3 & \bar{x}_2 \\ -\bar{x}_3 & \xi_2 & -x_1 \\ x_2 & -\bar{x}_1 & \xi_3 \end{pmatrix}$ . Finally,

$\gamma \in G_2 \subset E_7$  satisfies

$$\kappa\gamma = \gamma\kappa, \quad \mu\gamma = \gamma\mu, \quad \sigma''\gamma\sigma'' = \sigma_{13}\sigma''.$$

Therefore for  $l = \gamma\lambda\alpha_1$ , we have

$$\kappa l = l\kappa, \quad \mu l = l\mu, \quad \sigma'' l \sigma'' = -\sigma l,$$

that is,  $l$  is the required one. Let  $l_2 = \varphi(J)l = \varphi(J)\gamma\lambda\alpha_1$ . Thus we have the required isomorphism  $(U(1) \times (Spin(12))^{\sigma''})/\mathbf{Z}_2 \times \{1, l_2\} \cong (E_7)^{\sigma, \sigma''}$ . The explicit form of  $l_2$  is given by

$$l_2(X, Y, \xi, \eta) = \left( \begin{pmatrix} -\eta_1 & \gamma y_3 & * \\ * & -\eta_2 & -\gamma y_1 \\ \gamma y_2 & * & -\eta_3 \end{pmatrix}, \begin{pmatrix} \xi_1 & -\gamma x_3 & * \\ * & \xi_2 & \gamma x_1 \\ -\gamma x_2 & * & \xi_3 \end{pmatrix}, \eta, \xi \right).$$

Putting  $l_1 = \delta l_2 \delta^{-1}$ , we have

$$(E_7)^{\gamma, \gamma'} \cong (U(1) \times U(1) \times SU(6))/(\mathbf{Z}_2 \times \mathbf{Z}_6) \times \{1, l_1\}.$$

□

*Remark.* We used the group  $(E_7)^\sigma$  instead of the group  $(E_7)^\gamma$ , and we determined the group structure of  $(E_7)^{\sigma, \sigma''}$  instead of the group  $(E_7)^{\gamma, \gamma'}$ . So the group  $(E_6)^{\gamma, \gamma'} \cong (U(1) \times U(1) \times SU(3) \times SU(3))/(\mathbf{Z}_2 \times \mathbf{Z}_3) \times \{1, \gamma_1\}$  is not subgroup of our group  $(E_7)^{\gamma, \gamma'} \cong (U(1) \times U(1) \times SU(6))/(\mathbf{Z}_2 \times \mathbf{Z}_6) \times \{1, l_1\}$ .

## 2. The second consideration

### 2.1 Group $G_2$

Any Cayley number  $x \in \mathfrak{C}$  can be expressed as

$$\begin{aligned} x &= x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4 + x_5 e_5 + x_6 e_6 + x_7 e_7 \quad (x_i \in \mathbf{R}) \\ &= (x_0 + x_1 e_1) + (x_2 + x_3 e_1) e_2 + (x_4 + x_5 e_1) e_4 + (x_6 + x_7 e_1) e_6 \\ &= a + m_1 e_2 + m_2 e_4 + m_3 e_6 \quad (a = x_0 + x_1 e_1, m_i = x_{2i} + x_{2i+1} e_1). \end{aligned}$$

We associate such  $x \in \mathfrak{C}$  with the element

$$a + \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}$$

of  $C \oplus C^3$ . In  $C \oplus C^3$ , we define the multiplication by

$$(a + \mathbf{m})(b + \mathbf{n}) = (ab - \langle \mathbf{m}, \mathbf{n} \rangle) + (a\mathbf{n} + \bar{b}\mathbf{m} - \overline{\mathbf{m} \times \mathbf{n}}),$$

where  $\langle \mathbf{m}, \mathbf{n} \rangle = {}^t \mathbf{m} \bar{\mathbf{n}}$  is the usual Hermite inner product and  $\mathbf{m} \times \mathbf{n}$  is the exterior product of  $\mathbf{m}$  and  $\mathbf{n}$ . Then  $C \oplus C^3$  is isomorphic to  $\mathfrak{C}$  as algebras. The involutive  $R$ -transformations  $\gamma, \gamma'$  and  $\gamma_1$  of  $C \oplus C^3 = \mathfrak{C}$  are given as

$$\gamma\left(a + \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}\right) = a + \begin{pmatrix} m_1 \\ -m_2 \\ -m_3 \end{pmatrix}, \quad \gamma'\left(a + \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}\right) = a + \begin{pmatrix} -m_1 \\ m_2 \\ -m_3 \end{pmatrix},$$

$$\gamma_1(a + \mathbf{m}) = \bar{a} + \bar{\mathbf{m}}.$$

Furthermore, we define an  $R$ -transformation  $w$  of  $\mathfrak{C} = C \oplus C^3$  by

$$w(a + \mathbf{m}) = a + \omega_1 \mathbf{m}, \quad a + \mathbf{m} \in C \oplus C^3 = \mathfrak{C},$$

where  $\omega_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}e_1 \in \mathfrak{C}$ . Then  $w \in G_2$  and  $w^3 = 1$ .

We consider the group  $G_{2,C}$  replaced with  $C$  in the place  $\mathfrak{C}$  in the definition of the group  $G_2$ . Then we have

$$G_{2,C} = \{\alpha \in \text{Iso}_R(C) \mid \alpha(xy) = (\alpha x)(\alpha y)\} = \{1, \varepsilon\} = \mathbf{Z}_2,$$

where  $\varepsilon$  is the complex conjugation of  $C$ :  $\varepsilon x = \bar{x}$ ,  $x \in C$ .

Before we consider the group  $(G_2)^{\gamma, \gamma'}$ , we study the subgroup  $(G_2)_{e_1}$  of  $G_2$ :

$$(G_2)_{e_1} = \{\alpha \in G_2 \mid \alpha e_1 = e_1\}.$$

**PROPOSITION 2.1.1.**  $(G_2)^w = (G_2)_{e_1} \cong SU(3)$ .

*Proof.* Let  $SU(3) = \{D \in M(3, C) \mid DD^* = E, \det D = 1\}$ . The mapping  $\psi_{2,w} : SU(3) \rightarrow (G_2)_{e_1}$  defined by

$$\psi_{2,w}(D)(a + \mathbf{m}) = a + D\mathbf{m}, \quad a + \mathbf{m} \in C \oplus C^3 = \mathfrak{C}$$

gives the required isomorphism  $SU(3) \cong (G_2)_{e_1}$  (see [7] for details. As for  $(G_2)^w = (G_2)_{e_1}$ , see [7], too).  $\square$

The group  $\mathbf{Z}_2 = \{1, \gamma_1\}$  acts on the group  $U(1) \times U(1)$  by

$$\gamma_1(p, q) = (\bar{p}, \bar{q}),$$

and let  $(U(1) \times U(1)) \cdot \mathbf{Z}_2$  be the semi-direct product of these groups under this action.

**THEOREM 2.1.2.**  $(G_2)^{\gamma, \gamma'} \cong (U(1) \times U(1)) \cdot \mathbf{Z}_2$ .

*Proof.* We define a mapping  $\psi_2 : (U(1) \times U(1)) \cdot \mathbf{Z}_2 \rightarrow (G_2)^{\gamma, \gamma'}$  by

$$\begin{aligned}\psi_2((p, q), 1)(a + \mathbf{m}) &= a + D(p, q)\mathbf{m}, \\ \psi_2((p, q), \gamma_1)(a + \mathbf{m}) &= \bar{a} + D(p, q)\bar{\mathbf{m}}, \quad a + \mathbf{m} \in \mathbf{C} \oplus \mathbf{C}^3 = \mathfrak{C},\end{aligned}$$

where  $D(p, q) = \text{diag}(p, q, \bar{p}\bar{q}) \in M(3, \mathbf{C})$ . We shall prove that  $\psi_2$  is well-defined. Since  $D(p, q) \in SU(3)$ , we have  $\psi_2((p, q), 1) \in G_2$  (Proposition 2.1.1), and  $\psi_2((p, q), \gamma_1) = \psi_2((p, q), 1)\gamma_1$  is also in  $G_2$ . Furthermore, since

$$\gamma = \psi_2((1, -1), 1), \quad \gamma' = \psi_2((-1, 1), 1),$$

$\psi_2((p, q), 1)$  commutes with  $\gamma$  and  $\gamma'$ . Moreover  $\gamma_1$  commutes with  $\gamma$  and  $\gamma'$ . Hence  $\psi_2$  is well-defined. It is easy to see that  $\psi_2$  is a homomorphism. We shall show that  $\psi_2$  is onto. Let  $\alpha \in (G_2)^{\gamma, \gamma'}$ . Since  $(\mathfrak{C})_{\gamma, \gamma'} = \{x \in \mathfrak{C} \mid \gamma x = x, \gamma' x = x\} = \mathbf{C}$ , the restriction of  $\alpha$  to  $\mathbf{C}$  belongs to  $G_{2, \mathbf{C}}$ . Hence we have

$$\alpha x = x \quad \text{or} \quad \alpha x = \bar{x}, \quad x \in \mathbf{C}.$$

In the former case, there exists  $D \in SU(3)$  such that  $\alpha = \psi_{2, w}(D)$  (Proposition 2.1.1). From the condition that  $\alpha$  commutes with  $\gamma$  and  $\gamma'$ ,  $D$  is of a diagonal form  $D(p, q)$ . Hence  $\alpha = \psi_{2, w}(D(p, q)) = \psi_2((p, q), 1)$ . In the latter case, since  $\gamma_1 e_1 = -e_1$ , we have  $\alpha\gamma_1 \in (G_2)_{e_1}$ . So  $\alpha\gamma_1$  is in the same situation as above. Thus that  $\psi_2$  is onto is shown.  $\text{Ker } \psi_2$  is trivial. Therefore we have the isomorphism  $(G_2)^{\gamma, \gamma'} \cong (U(1) \times U(1)) \cdot \mathbf{Z}_2$ .  $\square$

## 2.2 Group $F_4$

We associate an element  $\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix}$  of  $\mathfrak{J}$  with the element

$$\begin{pmatrix} \xi_1 & a_3 & \bar{a}_2 \\ \bar{a}_3 & \xi_2 & a_1 \\ a_2 & \bar{a}_1 & \xi_3 \end{pmatrix} + (\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3)$$

(where  $x_i = a_i + \mathbf{m}_i \in \mathbf{C} \oplus \mathbf{C}^3 = \mathfrak{C}$ ) of  $\mathfrak{J}(3, \mathbf{C}) \oplus M(3, \mathbf{C})$ . Hereafter,  $\mathfrak{J}(3, \mathbf{C})$  will be briefly denoted by  $\mathfrak{J}_{\mathbf{C}}$ . In  $\mathfrak{J}_{\mathbf{C}} \oplus M(3, \mathbf{C})$ , we define the multiplication  $\times$  by

$$(X + M) \times (Y + N) = \left( X \times Y - \frac{1}{2}(M^*N + N^*M) \right) - \frac{1}{2}(MY + NX + \overline{M \times N}),$$

where  $M \times N$  (for  $M = (m_1, m_2, m_3), N = (n_1, n_2, n_3) \in M(3, C)$ ) is defined by

$$M \times N = \begin{pmatrix} m_2 \times n_3 & m_3 \times n_1 & m_1 \times n_2 \\ + & + & + \\ n_2 \times m_3 & n_3 \times m_1 & n_1 \times m_2 \end{pmatrix} \in M(3, C).$$

Then  $\mathfrak{J}_C \oplus M(3, C)$  is isomorphic to  $\mathfrak{J}$  as Freudenthal algebras.

Using the inclusion  $G_2 \subset F_4$ , the  $R$ -linear transformations  $\gamma, \gamma', \gamma_1$  and  $w$  of  $C \oplus C^3 = \mathfrak{C}$  are naturally extended to  $R$ -linear transformations  $\gamma, \gamma', \gamma_1$  and  $w$  of  $\mathfrak{J}_C \oplus M(3, C) = \mathfrak{J}$  as

$$\begin{aligned} \gamma(X + M) &= X + \gamma(m_1, m_2, m_3) = X + (\gamma m_1, \gamma m_2, \gamma m_3), \\ \gamma'(X + M) &= X + \gamma'(m_1, m_2, m_3) = X + (\gamma' m_1, \gamma' m_2, \gamma' m_3), \\ \gamma_1(X + M) &= \overline{X} + \overline{M}, \\ w(X + M) &= X + \omega_1 M = X + (\omega_1 m_1, \omega_1 m_2, \omega_1 m_3). \end{aligned}$$

Before we consider the group  $(F_4)^{\gamma, \gamma'}$ , we study the group  $F_{4,C}$  replaced with  $C$  in the place  $\mathfrak{C}$  in the definition of the group  $F_4$ :

$$F_{4,C} = \{\alpha \in \text{Iso}_R(\mathfrak{J}_C) \mid \alpha(X \times Y) = \alpha X \times \alpha Y\}.$$

The group  $Z_2 = \{1, \gamma_1\}$  acts on the group  $SU(3)$  by

$$\gamma_1 D = \overline{D}, \quad D \in SU(3),$$

and let  $SU(3) \cdot Z_2$  be the semi-direct product of these groups under this action.

**LEMMA 2.2.1.**  $F_{4,C} \cong (SU(3)/Z_3) \cdot Z_2$ ,  $Z_3 = \{E, \omega_1 E, \omega_1^2 E\}$ .

*Proof.* We define a mapping  $\psi_{4,C} : SU(3) \cdot Z_2 \rightarrow F_{4,C}$  by

$$\psi_{4,C}(A, 1)X = AXA^*, \quad \psi_{4,C}(A, \gamma_1)X = A\overline{X}A^*, \quad X \in \mathfrak{J}_C.$$

Then  $\psi_{4,C}$  is well-defined, a surjective homomorphism and  $\text{Ker } \psi_{4,C} = (Z_3, 1)$ . Thus we have the required isomorphism (see [4] for details).  $\square$

**PROPOSITION 2.2.2.**  $(F_4)^w \cong (SU(3) \times SU(3))/Z_3$ ,  $Z_3 = \{(E, E), (\omega_1 E, \omega_1 E), (\omega_1^2 E, \omega_1^2 E)\}$ .

*Proof.* We define a mapping  $\psi_{4,w} : SU(3) \times SU(3) \rightarrow (F_4)^w$  by

$$\psi_{4,w}(D, A)(X + M) = AXA^* + DMA^*, \quad X + M \in \mathfrak{J}_C \oplus M(3, C) = \mathfrak{J}.$$

Then  $\psi_{4,w}$  is well-defined, a surjective homomorphism and  $\text{Ker } \psi_{4,w} = Z_3$ . Thus we have the required isomorphism (see [4] for details).  $\square$

The group  $Z_2 = \{1, \gamma_1\}$  acts on the group  $U(1) \times U(1) \times SU(3)$  by

$$\gamma_1(p, q, A) = (\bar{p}, \bar{q}, \bar{A}),$$

and let  $(U(1) \times U(1) \times SU(3)) \cdot Z_2$  be the semi-direct product of these groups under this action.

**THEOREM 2.2.3.**  $(F_4)^{\gamma, \gamma'} \cong ((U(1) \times U(1) \times SU(3))/Z_3) \cdot Z_2$ ,  $Z_3 = \{(1, 1, E), (\omega_1, \omega_1, \omega_1 E), (\omega_1^2, \omega_1^2, \omega_1^2 E)\}$ .

*Proof.* We define a mapping  $\psi_4 : (U(1) \times U(1) \times SU(3)) \cdot Z_2 \rightarrow (F_4)^{\gamma, \gamma'}$  by

$$\psi_4((p, q, A), 1)(X + M) = AXA^* + D(p, q)MA^*,$$

$$\psi_4((p, q, A), \gamma_1)(X + M) = A\bar{X}A^* + D(p, q)\bar{M}A^*, \quad X + M \in \mathfrak{J}_C \oplus M(3, C) = \mathfrak{J},$$

where  $D(p, q) = \text{diag}(p, q, \bar{p}\bar{q}) \in M(3, C)$ . We have to prove that  $\psi_4$  is well-defined. It is clear that  $\psi_4((p, q, A), 1) \in F_4$  (Proposition 2.2.2) and  $\psi_4((p, q, A), \gamma_1) = \psi_4((p, q, A), 1)\gamma_1 \in F_4$ . Furthermore, since

$$\gamma = \psi_4((1, -1, E), 1), \quad \gamma' = \psi_4((-1, 1, E), 1),$$

$\psi_4((p, q, A), 1)$  commutes with  $\gamma$  and  $\gamma'$ . Moreover  $\gamma_1$  commutes with  $\gamma$  and  $\gamma'$  in  $G_2 \subset F_4$ . Hence  $\psi_4$  is well-defined. It is easy to see that  $\psi_4$  is a homomorphism. We shall show that  $\psi_4$  is onto. Let  $\alpha \in (F_4)^{\gamma, \gamma'}$ . Since  $(\mathfrak{J})_{\gamma, \gamma'} = \{X \in \mathfrak{J} \mid \gamma X = X, \gamma' X = X\} = \mathfrak{J}_C$ , the restriction of  $\alpha$  to  $\mathfrak{J}_C$  belongs to  $F_{4,C}$ . Hence there exists  $A \in SU(3)$  such that

$$\alpha X = AXA^* \quad \text{or} \quad \alpha X = A\bar{X}A^*, \quad X \in \mathfrak{J}_C$$

(Lemma 2.2.1). In the former case, let  $\beta = \psi_{4,w}(E, A)^{-1}\alpha$ , then  $\beta|_{\mathfrak{J}_C} = 1$ , and so  $\beta \in G_2$ . Moreover  $\beta \in (G_2)^w = (G_2)_{e_1}$ . Hence there exists  $D \in SU(3)$  such that

$$\beta(X + M) = X + DM = \psi_{4,w}(D, E)(X + M), \quad X + M \in \mathfrak{J}_C \oplus M(3, C) = \mathfrak{J}$$

(Propositions 2.1.1, 2.2.2), that is,  $\beta = \psi_{4,w}(D, E)$ . Therefore we have  $\alpha = \psi_{4,w}(E, A)\psi_{4,w}(D, E) = \psi_{4,w}(D, A)$ . From the condition that  $\alpha$  commutes with  $\gamma$  and  $\gamma'$ ,  $D$  is of the form  $D(p, q)$ . Hence  $\alpha = \psi_{4,w}(D(p, q), A) = \psi_4((p, q, A), 1)$ . In the latter case, consider  $\alpha\gamma_1$ , then it is in the same situation as above. Thus that  $\psi_4$  is onto is shown.  $\text{Ker } \psi_4 = (Z_3, 1)$  is easily obtained. Therefore we have the isomorphism  $(F_4)^{\gamma, \gamma_1} \cong ((U(1) \times U(1) \times SU(3))/Z_3) \cdot Z_2$ .  $\square$

### 2.3 Group $E_6$

Note that in  $\mathfrak{J}^C = (\mathfrak{J}_C)^C \oplus M(3, C)^C$ , the multiplication  $\times$  is defined as that in  $\mathfrak{J} = \mathfrak{J}_C \oplus M(3, C)$ . Using the inclusion  $F_4 \subset E_6$ , the  $\mathbf{R}$ -linear transformations  $\gamma, \gamma', \gamma_1$  and  $w$  of  $\mathfrak{J}_C \oplus M(3, C) = \mathfrak{J}$  are naturally extended to  $C$ -linear transformations  $\gamma, \gamma', \gamma_1$  and  $w$  of  $(\mathfrak{J}_C)^C \oplus M(3, C)^C = \mathfrak{J}^C$ , respectively.

Before we consider the group  $(E_6)^{\gamma, \gamma'}$ , we study the group  $E_{6,C}$  replaced with  $C$  in the place  $\mathfrak{C}$  in the definition of the group  $E_6$ :

$$\begin{aligned} E_{6,C} &= \{\alpha \in \text{Iso}_C((\mathfrak{J}_C)^C) \mid \det \alpha X = \det X, \langle \alpha X, \alpha Y \rangle = \langle X, Y \rangle\} \\ &= \{\alpha \in \text{Iso}_C((\mathfrak{J}_C)^C) \mid \alpha X \times \alpha Y = \tau \alpha \tau(X \times Y), \langle \alpha X, \alpha Y \rangle = \langle X, Y \rangle\}. \end{aligned}$$

The group  $Z_2 = \{1, \gamma_1\}$  acts on the group  $SU(3) \times SU(3)$  by

$$\gamma_1(A, B) = (\bar{B}, \bar{A}).$$

and let the group  $(SU(3) \times SU(3)) \cdot Z_2$  be the semi-direct product of these groups under this action.

**LEMMA 2.3.1.**  $E_{6,C} \cong ((SU(3) \times SU(3))/Z_3) \cdot Z_2$ ,  $Z_3 = \{(E, E), (\omega_1 E, \omega_1 E), (\omega_1^2 E, \omega_1^2 E)\}$ .

*Proof.* We define the mapping  $h : M(3, C) \times M(3, C) \rightarrow M(3, C)^C$  by

$$h(A, B) = \frac{A + B}{2} + i \frac{A - B}{2} e_1.$$

Now, we define a mapping  $\psi_{6,C} : (SU(3) \times SU(3)) \cdot Z_2 \rightarrow E_{6,C}$  by

$$\begin{aligned} \psi_{6,C}((A, B), 1)X &= h(A, B)Xh(A, B)^*, \\ \psi_{6,C}((A, B), \gamma_1)X &= h(A, B)\bar{X}h(A, B)^*, \quad X \in (\mathfrak{J}_C)^C. \end{aligned}$$

Then  $\psi_{6,C}$  is well-defined, a surjective homomorphism and  $\text{Ker } \psi_{6,C} = (Z_3, 1)$ . Thus we have the required isomorphism (see [4] for details).  $\square$

**PROPOSITION 2.3.2.**  $(E_6)^w \cong (SU(3) \times SU(3) \times SU(3))/Z_3$ ,  $Z_3 = \{(E, E, E), (\omega_1 E, \omega_1 E, \omega_1 E), (\omega_1^2 E, \omega_1^2 E, \omega_1^2 E)\}$ .

*Proof.* We define a mapping  $\psi_{6,w} : SU(3) \times SU(3) \times SU(3) \rightarrow (E_6)^w$  by

$$\begin{aligned} \psi_{6,w}(D, A, B)(X + M) &= h(A, B)Xh(A, B)^* + DM\tau h(A, B)^*, \\ X + M &\in (\mathfrak{J}_C)^C \oplus M(3, C)^C = \mathfrak{J}^C. \end{aligned}$$

Then  $\psi_{6,w}$  is well-defined, a surjective homomorphism and  $\text{Ker } \psi_{6,w} = Z_3$ . Thus we have the required isomorphism (see [4] for details).  $\square$



The group  $Z_2 = \{1, \gamma_1\}$  acts on the group  $U(1) \times U(1) \times SU(3) \times SU(3)$  by

$$\gamma_1(p, q, A, B) = (\bar{p}, \bar{q}, \bar{B}, \bar{A}),$$

and let  $(U(1) \times U(1) \times SU(3) \times SU(3)) \cdot Z_2$  be the semi-direct product of these groups under this action.

**THEOREM 2.3.3.**  $(E_6)^{\gamma, \gamma'} \cong ((U(1) \times U(1) \times SU(3) \times SU(3))/Z_3) \cdot Z_2$ ,  $Z_3 = \{(1, 1, E, E), (\omega_1, \omega_1, \omega_1 E, \omega_1 E), (\omega_1^2, \omega_1^2, \omega_1^2 E, \omega_1^2 E)\}$ .

*Proof.* We define a mapping  $\psi_6 : (U(1) \times U(1) \times SU(3) \times SU(3)) \cdot Z_2 \rightarrow (E_6)^{\gamma, \gamma'}$  by

$$\begin{aligned} \psi_6((p, q, A, B), 1)(X + M) &= h(A, B)Xh(A, B)^* + D(p, q)M\tau h(A, B)^*, \\ \psi_6((p, q, A, B), \gamma_1)(X + M) &= h(A, B)\bar{X}h(A, B)^* + D(p, q)\bar{M}\tau h(A, B)^*, \\ X + M &\in (\mathfrak{J}_C)^C \oplus M(3, C)^C = \mathfrak{J}^C, \end{aligned}$$

where  $D(p, q) = \text{diag}(p, q, \bar{p}\bar{q}) \in M(3, C)$ . We have to prove that  $\psi_6$  is well-defined. It is clear that  $\psi_6((p, q, A, B), 1) \in E_6$  (Proposition 2.3.2) and  $\psi_6((p, q, A, B), \gamma_1) = \psi_6((p, q, A, B), 1)\gamma_1 \in E_6$ . Furthermore, since

$$\gamma = \psi_6((1, -1, E, E), 1), \quad \gamma' = \psi_6((-1, 1, E, E), 1),$$

$\psi_6((p, q, A, B), 1)$  commutes with  $\gamma$  and  $\gamma'$ . Moreover  $\gamma_1$  commutes with  $\gamma$  and  $\gamma'$  in  $G_2 \subset F_4 \subset E_6$ . Hence  $\psi_6$  is well-defined. It is easy to see that  $\psi_6$  is a homomorphism. We shall show that  $\psi_6$  is onto. Let  $\alpha \in (E_6)^{\gamma, \gamma'}$ . Since  $(\mathfrak{J}^C)^{\gamma, \gamma'} = \{X \in \mathfrak{J}^C \mid \gamma X = X, \gamma' X = X\} = (\mathfrak{J}_C)^C$ , the restriction of  $\alpha$  to  $(\mathfrak{J}_C)^C$  belongs to  $E_{6,C}$ . Hence there exist  $A, B \in SU(3)$  such that

$$\alpha X = h(A, B)Xh(A, B)^* \quad \text{or} \quad \alpha X = h(A, B)\bar{X}h(A, B)^*, \quad X \in (\mathfrak{J}_C)^C$$

(Lemma 2.3.1). In the former case, let  $\beta = \psi_{6,w}(E, A, B)^{-1}\alpha$ , then  $\beta|(\mathfrak{J}_C)^C = 1$ , and so  $\beta \in G_2$ . Moreover  $\beta \in (G_2)^w = (G_2)_{e_1}$ . Hence there exists  $D \in SU(3)$  such that

$$\begin{aligned} \beta(X + M) &= X + DM = \psi_{6,w}(D, E, E)(X + M), \\ X + M &\in (\mathfrak{J}_C)^C \oplus M(3, C)^C = \mathfrak{J}^C \end{aligned}$$

(Propositions 2.1.1, 2.3.2), that is,  $\beta = \psi_{6,w}(D, E, E)$ . Therefore we have  $\alpha = \psi_{6,w}(E, A, B)\psi_{6,w}(D, E, E) = \psi_{6,w}(D, A, B)$ . From the condition that  $\alpha$  commutes with  $\gamma$  and  $\gamma'$ ,  $D$  is of the form  $D(p, q)$ . Hence  $\alpha = \psi_{6,w}(D(p, q), A, B) = \psi_6((p, q, A, B), 1)$ . In the latter case, consider  $\alpha\gamma_1$ , then it is in the same situation as above. Thus that  $\psi_6$  is onto is shown.  $\text{Ker } \psi_6 = (Z_3, 1)$  is easily obtained. Therefore we have the isomorphism  $(E_6)^{\gamma, \gamma'} \cong ((U(1) \times U(1) \times SU(3) \times SU(3))/Z_3) \cdot Z_2$ .  $\square$

## 2.4 Group $E_7$

We identify  $(\mathfrak{P}_C)^C \oplus (M(3, C)^C \oplus M(3, C)^C)$  with  $\mathfrak{P}^C$  (using the identification  $(\mathfrak{J}_C)^C \oplus M(3, C)^C$  with  $\mathfrak{J}^C$ ) by

$$(X, Y, \xi, \eta) + (M, N) = (X + M, Y + N, \xi, \eta).$$

We often denote any element of  $(\mathfrak{P}_C)^C$  by  $P_C$ .

Using the inclusion  $E_6 \subset E_7$ , the  $C$ -linear transformations  $\gamma, \gamma', \gamma_1$  and  $w$  of  $(\mathfrak{J}_C)^C \oplus M(3, C)^C = \mathfrak{J}^C$  are naturally extended to  $C$ -linear transformations  $\gamma, \gamma', \gamma_1$  and  $w$  of  $(\mathfrak{P}_C)^C \oplus (M(3, C)^C \oplus M(3, C)^C) = \mathfrak{P}^C$  as

$$\begin{aligned} \gamma((X, Y, \xi, \eta) + (M, N)) &= (X, Y, \xi, \eta) + (\gamma M, \gamma N), \\ \gamma'((X, Y, \xi, \eta) + (M, N)) &= (X, Y, \xi, \eta) + (\gamma' M, \gamma' N), \\ \gamma_1((X, Y, \xi, \eta) + (M, N)) &= (\bar{X}, \bar{Y}, \xi, \eta) + (\bar{M}, \bar{N}), \\ w((X, Y, \xi, \eta) + (M, N)) &= (X, Y, \xi, \eta) + (\omega_1 M, \omega_1 N). \end{aligned}$$

Before we consider the group  $(E_7)^{\gamma, \gamma'}$ , we study the group  $E_{7, C}$  replaced with  $C$  in the place  $\mathfrak{C}$  in the definition of the group  $E_7$ :

$$E_{7, C} = \{\alpha \in \text{Iso}_C((\mathfrak{P}_C)^C) \mid \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q, \langle \alpha P, \alpha Q \rangle = \langle P, Q \rangle\}.$$

We define the mapping  $h' : C^C \rightarrow C$  by

$$h'(a + bi) = a + be_1, \quad a, b \in C.$$

Now, let  $\Lambda^3(C^6)$  be the third exterior product of  $C$ -vector space  $C^6$  and we define the  $C$ - $C$ -linear isomorphism  $f_C : (\mathfrak{P}_C)^C \rightarrow \Lambda^3(C^6)$  by

$$f_C \left( \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix}, \begin{pmatrix} \eta_1 & y_3 & \bar{y}_2 \\ \bar{y}_3 & \eta_2 & y_1 \\ y_2 & \bar{y}_1 & \eta_3 \end{pmatrix}, \xi, \eta \right) = \sum_{i < j < k} x_{ijk} e_i \wedge e_j \wedge e_k$$

( $\{e_1, e_2, \dots, e_6\}$  is the canonical basis of  $C^6$  and  $x_{ijk} \in C$  are skew-symmetric tensors:  $x_{i'j'k'} = \text{sgn} \begin{pmatrix} i & j & k \\ i' & j' & k' \end{pmatrix} x_{ijk}$ ), where

$$\begin{aligned} x_{156} &= h'(\xi_1), & x_{164} &= h'(x_3), & x_{145} &= h'(\bar{x}_2), \\ x_{256} &= h'(\bar{x}_3), & x_{264} &= h'(\xi_2), & x_{245} &= h'(x_1), \\ x_{356} &= h'(x_2), & x_{364} &= h'(\bar{x}_1), & x_{345} &= h'(\xi_3), \\ x_{423} &= h'(\eta_1), & x_{431} &= h'(y_3), & x_{412} &= h'(\bar{y}_2), \end{aligned}$$

$$\begin{aligned}
x_{523} &= h'(\bar{y}_3), & x_{531} &= h'(\eta_2), & x_{512} &= h'(y_1), \\
x_{623} &= h'(y_2), & x_{631} &= h'(\bar{y}_1), & x_{612} &= h'(\eta_3), \\
x_{123} &= h'(\xi), \\
x_{456} &= h'(\eta).
\end{aligned}$$

Furthermore, we define the  $C$ - $C$ -linear mapping  $k : M(3, C)^C \oplus M(3, C)^C \rightarrow M(6, C)$  by

$$k(M, N) = k(M_1 + iM_2, N_1 + iN_2) = \begin{pmatrix} -N_2 - N_1 e_1 & M_2 + M_1 e_1 \\ M_2 - N_1 e_1 & N_2 + N_1 e_1 \end{pmatrix},$$

where  $M_i, N_i \in M(3, C)$ , then the inverse mapping  $k^{-1} : M(6, C) \rightarrow M(3, C)^C \oplus M(3, C)^C$  of  $k$  is given by

$$\begin{aligned}
& k^{-1} \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \\
&= \left( \frac{(M_{21} - M_{12})e_1}{2} + i \frac{M_{21} + M_{12}}{2}, \frac{(M_{22} + M_{11})e_1}{2} + i \frac{M_{22} - M_{11}}{2} \right),
\end{aligned}$$

where  $M_{ij} \in M(3, C)$ .

The group  $SU(6)$  acts on  $\Lambda^3(C^6)$ , that is, the action of  $A \in SU(6)$  on  $a \wedge b \wedge c \in \Lambda^3(C^6)$  is defined by

$$A(a \wedge b \wedge c) = Aa \wedge Ab \wedge Ac.$$

The group  $Z_2 = \{1, \gamma_1\}$  acts on the group  $SU(6)$  by

$$\gamma_1 A = \overline{(\text{Ad } J_3)A}, \quad J_3 = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix},$$

and let  $SU(6) \cdot Z_2$  be the semi-direct product of these groups under this action.

**LEMMA 2.4.1.**  $E_{7,C} \cong (SU(6)/Z_3) \cdot Z_2$ ,  $Z_3 = \{E, \omega_1 E, \omega_1^2 E\}$ .

*Proof.* We define a mapping  $\psi_{7,C} : SU(6) \cdot Z_2 \rightarrow E_{7,C}$  by

$$\begin{aligned}
\psi_{7,C}(A, 1)P_C &= f_C^{-1}(A(f_C P_C)), \\
\psi_{7,C}(A, \gamma_1)P_C &= f_C^{-1}(A(f_C \bar{P}_C)), \quad P_C \in (\mathfrak{P}_C)^C.
\end{aligned}$$

Then  $\psi_{7,C}$  is well-defined, a surjective homomorphism and  $\text{Ker } \psi_{7,C} = (Z_3, 1)$ . Thus we have the required isomorphism (see [4] for details).  $\square$

We define the  $C$ - $C$ -linear isomorphism  $f : \mathfrak{P}^C \rightarrow \Lambda^3(C^6) \oplus M(6, C)$  by

$$\begin{aligned} f(P_C + (M, N)) &= f_C P_C + k(M, N), \\ P_C + (M, N) &\in (\mathfrak{P}_C)^C \oplus (M(3, C))^C \oplus M(3, C)^C = \mathfrak{P}^C. \end{aligned}$$

The group  $SU(3) \times SU(6)$  acts on  $\Lambda^3(C^6) \oplus M(6, C)$  by

$$(D, A) \left( \sum (a \wedge b \wedge c) + \widetilde{M} \right) = \sum (Aa \wedge Ab \wedge Ac) + D\widetilde{M}A^*,$$

where  $D\widetilde{M}$  means  $\begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = \begin{pmatrix} DM_{11} & DM_{12} \\ DM_{21} & DM_{22} \end{pmatrix}$ ,  $M_{ij} \in M(3, C)$ .

**PROPOSITION 2.4.2.**  $(E_7)^w \cong (SU(3) \times SU(6)) / \mathbf{Z}_3$ ,  $\mathbf{Z}_3 = \{(E, E), (\omega_1 E, \omega_1 E), (\omega_1^2 E, \omega_1^2 E)\}$ .

*Proof.* We define a mapping  $\psi_{7,w} : SU(3) \times SU(6) \rightarrow (E_7)^w$  by

$$\psi_{7,w}(D, A)P = f^{-1}((D, A)(fP)), \quad P \in \mathfrak{P}^C.$$

Then  $\psi_{7,w}$  is well-defined, a surjective homomorphism and  $\text{Ker } \psi_{7,w} = \mathbf{Z}_3$ . Thus we have the required isomorphism (see [4] for details).  $\square$

The group  $\mathbf{Z}_2 = \{1, \gamma_1\}$  acts on the group  $U(1) \times U(1) \times SU(6)$  by

$$\gamma_1(p, q, A) = (\bar{p}, \bar{q}, (\text{Ad } J_3)A),$$

and let  $(U(1) \times U(1) \times SU(6)) \cdot \mathbf{Z}_2$  be the semi-direct product of these groups under this action.

**THEOREM 2.4.3.**  $(E_7)^{\gamma, \gamma'} \cong ((U(1) \times U(1) \times SU(6)) / \mathbf{Z}_3) \cdot \mathbf{Z}_2$ ,  $\mathbf{Z}_3 = \{(1, 1, E), (\omega_1, \omega_1, \omega_1 E), (\omega_1^2, \omega_1^2, \omega_1^2 E)\}$ .

*Proof.* We define a mapping  $\psi_7 : (U(1) \times U(1) \times SU(6)) \cdot \mathbf{Z}_2 \rightarrow (E_7)^{\gamma, \gamma'}$  by

$$\begin{aligned} \psi_7((p, q, A), 1)P &= f^{-1}((D(p, q), A)(fP)), \\ \psi_7((p, q, A), \gamma_1)P &= f^{-1}((D(p, q), A)(f\gamma_1 P)), \quad P \in \mathfrak{P}^C, \end{aligned}$$

where  $D(p, q) = \text{diag}(p, q, \bar{p}\bar{q})$ . We have to prove that  $\psi_7$  is well-defined. It is clear that  $\psi_7((p, q, A), 1) \in E_7$  (Proposition 2.4.2) and  $\psi_7((p, q, A), \gamma_1) = \psi_7((p, q, A), 1)\gamma_1 \in E_7$ . Furthermore, since

$$\gamma = \psi_7((1, -1, E), 1), \quad \gamma' = \psi_7((-1, 1, E), 1),$$

$\psi_7((p, q, A), 1)$  commutes with  $\gamma$  and  $\gamma'$ . Moreover  $\gamma_1$  commutes with  $\gamma$  and  $\gamma'$  in  $G_2 \subset F_4 \subset E_6 \subset E_7$ . Hence  $\psi_7$  is well-defined. We see that  $\psi_7$  is a homomorphism (see [4]). We shall show  $\psi_7$  is onto. Let  $\alpha \in (E_7)^{\gamma, \gamma'}$ . Since  $(\mathfrak{P}^C)_{\gamma, \gamma'} = \{P \in \mathfrak{P}^C \mid \gamma P = P, \gamma' P = P\} = (\mathfrak{P}_C)^C$ , the restriction of  $\alpha$  to  $(\mathfrak{P}_C)^C$  belongs to  $E_{7,C}$ . Hence there exists  $A \in SU(6)$  such that

$$\alpha P_C = f_C^{-1}(A(f_C P_C)) \quad \text{or} \quad \alpha P_C = f_C^{-1}(A(f_C \overline{P_C})), \quad P_C \in (\mathfrak{P}_C)^C$$

(Lemma 2.4.1). In the former case, let  $\beta = \psi_{7,w}(E, A)^{-1}\alpha$ , then  $\beta|(\mathfrak{P}_C)^C = 1$ , and so  $\beta \in G_2$ . Moreover,  $\beta \in (G_2)^w = (G_2)_{e_1}$ . Hence there exists  $D \in SU(3)$  such that

$$\begin{aligned} \beta(P_C + (M, N)) &= P_C + D(M, N) = P_C + (DM, DN) \\ &= \psi_{7,w}(D, E)(P_C + (M, N)), \quad P_C + (M, N) \in \mathfrak{P}^C \end{aligned}$$

(Propositions 2.1.1, 2.4.2), that is,  $\beta = \psi_{7,w}(D, E)$ . Hence we have

$$\alpha = \psi_{7,w}(E, A)\beta = \psi_{7,w}(E, A)\psi_{7,w}(D, E) = \psi_{7,w}(D, A).$$

From the condition that  $\alpha$  commutes with  $\gamma, \gamma'$ ,  $D$  is of the form  $D(p, q)$ . Hence  $\alpha = \psi_{7,w}(D(p, q), A) = \psi_7((p, q, A), 1)$ . In the latter case, consider  $\alpha\gamma_1$ , then it is in the same situation as above. Thus that  $\psi_7$  is onto is shown.  $\text{Ker } \psi_7 = (\mathbf{Z}_3, 1)$  is easily obtained. Therefore we have the isomorphism  $(E_7)^{\gamma, \gamma'} \cong ((U(1) \times U(1) \times SU(6))/\mathbf{Z}_3) \cdot \mathbf{Z}_2$ .  $\square$

*Remark.* Instead of  $D(p, q) = \text{diag}(p, q, \overline{pq})$ , if we use  $\text{diag}(q^2, p\overline{q}, \overline{pq})$ , then the kernel of each  $\psi_i$  is  $\mathbf{Z}_2 \times \mathbf{Z}_3$ . Consequently we have the results of the first consideration.

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