YOKOHAMA MATHEMATICAL
JOURNAL VOL. 53, 2006

SUBMULTIPLICATIVE MOMENTS OF THE SUPREMUM OF A MARKOV-MODULATED RANDOM WALK

By

MIKHAIL SGIBNEV

(Received February 4, 2004; Revised June 21, 2006)

Abstract. Let M_∞ be the supremum of a random walk $\{S_n\}$ defined on a finite Markov chain $\{\kappa_n\}$ and let $\varphi(x)$, $x \geq 0$, be a submultiplicative function: $\varphi(x+y) \leq \varphi(x)\varphi(y)$. Conditions are given for $\mathsf{E}\varphi(M_\infty \mid \kappa_0 = i) < \infty$.

Introduction

Let $\{\kappa_n\}$ be a finite Markov chain with state space $\{1, \dots, N\}$ and transition matrix $\mathbf{P} = (p_{ij})$, where $p_{ij} = \mathsf{P}(\kappa_n = j \mid \kappa_{n-1} = i)$, $n = 1, 2, \dots$. It will be assumed throughout that $\{\kappa_n\}$ is an ergodic (irreducible, aperiodic and positive recurrent) Markov chain with stationary distribution $\{\pi_1, \dots, \pi_N\}$, where $\pi_i > 0$, $i = 1, \dots, N$.

A random walk $\{S_n\}$ governed by the Markov chain $\{\kappa_n\}$ is defined as follows. Let probability distributions F_{ij}, $i, j = 1, \dots, N$, be given. For each pair (i, j), let $\{X_m(i,j)\}_{m=1}^\infty$ be a sequence of independent identically distributed random variables with common distribution F_{ij}. Suppose the sequences $\{X_m(i,j)\}_{m=1}^\infty$, $i, j = 1, \dots, N$, and $\{\kappa_n\}$ are mutually independent. We set $S_0 = 0$, $S_n = S_{n-1} + X_n(\kappa_{n-1}, \kappa_n)$ for $n \geq 1$, and $M_\infty = \sup_{n \geq 0} S_n$.

Let $\varphi(x)$, $x \in \mathbb{R}$, be a submultiplicative function, i.e. $\varphi(x)$ is a finite, positive, Borel measurable function such that

$$\varphi(0) = 1, \quad \varphi(x+y) \leq \varphi(x)\varphi(y), \qquad x, y \in \mathbb{R}. \tag{1}$$

It is well known [5, Section 7.6] that

$$-\infty < r_-(\varphi) := \lim_{x \to -\infty} \frac{\log \varphi(x)}{x} = \sup_{x<0} \frac{\log \varphi(x)}{x}$$
$$\leq \inf_{x>0} \frac{\log \varphi(x)}{x} = \lim_{x \to \infty} \frac{\log \varphi(x)}{x} := r_+(\varphi) < \infty. \tag{2}$$

2000 Mathematics Subject Classification: 60J05, 60G50
Key words and phrases: random walk, Markov-modulation, supremum, first positive sum, submultiplicative moment, Banach algebra

Here are some examples of such functions on $[0,\infty)$: $\varphi(x) = (1+x)^r$, $r > 0$; $\varphi(x) = \exp(cx^\beta)$ with $c > 0$ and $\beta \in (0,1)$; $\varphi(x) = \exp(rx)$, $r \in \mathbb{R}$. Moreover, if $R(x)$, $x \in \mathbb{R}_+$, is a positive, ultimately nondecreasing regularly varying function at infinity with a nonnegative exponent β (i.e. $R(tx)/R(x) \to t^\beta$ for $t > 0$ as $x \to \infty$ [4, Section VIII.8]), then there exist a nondecreasing submultiplicative function $\varphi(x)$ and a point $x_0 \in (0,\infty)$ such that $c_1 R(x) \le \varphi(x) \le c_2 R(x)$ for all $x \ge x_0$, where c_1 and c_2 are some positive constants [12, Proposition]. The product of a finite number of submultiplicative functions is again a submultiplicative function.

Consider the collection $S(\varphi)$ of all complex-valued measures θ defined on the σ-algebra $\mathcal{B}$ of Borel subsets of $\mathbb{R}$ and such that

$$\|\theta\|_\varphi := \int_{\mathbb{R}} \varphi(x)|\,\theta|(\mathrm{d}x) < \infty;$$

here $|\theta|$ stands for the total variation of θ. The collection $S(\varphi)$ is a Banach algebra with norm $\|\cdot\|_\varphi$ by the usual operations of addition and scalar multiplication of measures, the product of two elements ν and θ of $S(\varphi)$ is defined as their convolution $\nu * \theta$ [5, Section 4.16]. The unit element of $S(\varphi)$ is the Dirac measure δ_0, i.e. the atomic measure of unit mass at the origin. The Laplace transform of an element $\theta \in S(\varphi)$ is defined by $\hat{\theta}(s) := \int_{\mathbb{R}} \exp(sx)\,\theta(\mathrm{d}x)$. By (2), this integral converges absolutely with respect to $|\theta|$ for all s in the strip $\{s \in \mathbb{C} : r_-(\varphi) \le \Re s \le r_+(\varphi)\}$. Obviously, $\hat{\delta}_0(s) \equiv 1$.

In what follows, we shall assume that $\varphi(x) \equiv 1$ on $(-\infty, 0)$. Then, by (1), $\varphi(x)$ is nondecreasing and, by (2), $r_+(\varphi) \ge 0$. Conversely, if a nondecreasing function $\varphi(x)$, $x \ge 0$, satisfies (1) for all x, $y \ge 0$ and $r_+(\varphi) \ge 0$, then putting $\varphi(x) \equiv 1$ on $(-\infty, 0)$ we obtain a submultiplicative function on the whole real line $\mathbb{R}$.

We shall assume throughout that $M_\infty < \infty$ almost surely. In particular, this is the case if the "stationary" expectation of S_1 is negative, i.e. $\mathsf{E}_\pi S_1 := \sum_{i,j=1}^N \pi_i p_{ij} E X_1(i,j) < 0$. Denote $\mathsf{P}_i(\cdot) = \mathsf{P}(\cdot \mid \kappa_0 = i)$ and let E_i, $i = 1, \dots, N$, stand for the corresponding expectations.

The aim of the present note is to derive conditions for $\mathsf{E}_i \varphi(M_\infty) < \infty$ and related results, where $\varphi(x)$, $x \ge 0$, is a nondecreasing submultiplicative function. In the case of usual random walks, ordinary moments of the supremum, i.e. $\mathsf{E}M_\infty^k$, were studied in [8] and the submultiplicative moments $\mathsf{E}\varphi(M_\infty)$ in [12] (see also the references therein).

1. Preliminaries

Denote by $S(\gamma', \gamma)$ the Banach algebra $S(\varphi)$ for the following choice of the submultiplicative function $\varphi(x)$: $\varphi(x) := \exp(\gamma' x)$ for $x < 0$ and $\varphi(x) := \exp(\gamma x)$

for $x \geq 0$; here $0 \leq \gamma' \leq \gamma$.

Suppose a matrix $\mathbf{B} = (B_{ij})$ is made up of elements of $S(\varphi)$. Then $\hat{\mathbf{B}}(s)$ will denote the matrix $(\hat{B}_{ij}(s))$ of the corresponding Laplace transforms, and we shall write $\hat{\mathbf{B}}(s) \in \hat{S}(\varphi)$.

Let $\psi(x)$, $x \in \mathbb{R}$, be a submultiplicative function such that $r_-(\psi) = \gamma'$ and $r_+(\psi) = \gamma$. In view of (2), it is clear that $S(\psi) \subset S(\gamma', \gamma)$. Denote by $\mathcal{H}_1$ and $\mathcal{H}$ the collections of all homomorphisms of $S(\psi)$ and $S(\gamma', \gamma)$ into $\mathbb{C}$ respectively. Looking at the structure of an arbitrary $h_1 \in \mathcal{H}_1$ [11, Theorem 2], we see that $h_1 = h|_{S(\psi)}$ for a uniquely determined $h \in \mathcal{H}$. Conversely, for each $h \in \mathcal{H}$, $h|_{S(\psi)} \in \mathcal{H}_1$. It follows from the general theory of Banach algebras that an element $\nu \in S(\psi)$ is invertible in $S(\psi)$ if and only if $h_1(\nu) \neq 0$ for all $h_1 \in \mathcal{H}_1$. Also, for $\nu \in S(\gamma', \gamma)$, there exists an inverse $\nu^{-1} \in S(\gamma', \gamma)$ if and only if $h(\nu) \neq 0$ for all $h \in \mathcal{H}$. Suppose now that $\nu \in S(\psi)$ is invertible in $S(\gamma', \gamma)$. Then, in view of the above, $\nu^{-1} \in S(\psi)$. The following lemma says that this property remains valid in the matrix case.

LEMMA 1. *Let $\mathbf{U}$ be an $N \times N$ matrix whose entries are elements of $S(\psi)$. Suppose that $\hat{\mathbf{U}}(s)$ is invertible in $\hat{S}(\gamma', \gamma)$, i.e. the entries of $[\hat{\mathbf{U}}(s)]^{-1}$ are elements of $\hat{S}(\gamma', \gamma)$. Then $[\hat{\mathbf{U}}(s)]^{-1} \in \hat{S}(\psi)$.*

Proof. The function $\det \hat{\mathbf{U}}(s)$ is a linear combination of products of N factors. These factors are the Laplace transforms of elements of the matrix $\mathbf{U} \in S(\psi)$. Hence $\det \hat{\mathbf{U}}(s)$ is the Laplace transform $\hat{\alpha}(s)$ of some measure $\alpha \in S(\psi)$. Denote by $\hat{\mathbf{M}}(s)$ the adjugate matrix of $\hat{\mathbf{U}}(s)$. By the same reason, $\hat{\mathbf{M}}(s) \in \hat{S}(\psi)$. Since $[\hat{\mathbf{U}}(s)]^{-1} \in \hat{S}(\gamma', \gamma)$, $\det\{[\hat{\mathbf{U}}(s)]^{-1}\}$ is the Laplace transform of some $\beta \in S(\gamma', \gamma)$. We have $\alpha * \beta = \delta_0$, i.e. α is invertible in $S(\gamma', \gamma)$. By the discussion preceding the lemma, α is invertible in $S(\psi)$: $\beta = \alpha^{-1} \in S(\psi)$. We have $[\hat{\mathbf{U}}(s)]^{-1} = \hat{\mathbf{M}}(s)\hat{\beta}(s) \in \hat{S}(\psi)$. ∎

Let $\mathbf{A}$ denote the $N \times N$ matrix $(p_{ij}F_{ij})$ and δ_{ij} the Kronecker delta. Put

$$\mathbf{U}(s) := \left(\frac{s+1}{s}\delta_{1i} + \delta_{ij}(1 - \delta_{1j}) \right),$$

$\mathbf{J} := \operatorname{diag}(\pi_1, \ldots, \pi_N)$ and $\mathbf{B}(s) := \mathbf{U}(s)\mathbf{J}$. Then

$$\mathbf{B}(s) = \begin{pmatrix} \frac{s+1}{s}\pi_1 & \frac{s+1}{s}\pi_2 & \ldots & \frac{s+1}{s}\pi_N \\ 0 & \pi_2 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \ldots & \pi_N \end{pmatrix}.$$

Let $\mathbf{I}$ be the $N \times N$ identity matrix. Set $\eta(x) := \inf\{n \geq 1 : S_n > x\}$. Define $\chi(x) := S_{\eta(x)} - x$ and $\overline{s}_n := \max_{1 \leq m \leq n} S_m$.

Denote by $\lambda(\gamma)$ the maximal positive eigenvalue of the matrix $\hat{\mathbf{A}}(\gamma)$.
We shall need the following result [1, Theorem 2].

THEOREM 2. *Let $\mathbf{A} \in S(0,\gamma)$ for some $\gamma \geq 0$ and let $\lambda(\gamma) < 1$ in case of $\gamma > 0$. Suppose the expectations $\mathsf{E}X_1(i,j)$, i, $j = 1, \dots, N$, are finite and $\mathsf{E}_\pi S_1 \in (-\infty, 0)$. Assume the F_{ij} are absolutely continuous for all i, $j = 1, \dots, N$. Then*

$$\mathbf{B}(s)[\mathbf{I} - \hat{\mathbf{A}}(s)] = [\mathbf{B}(s)\hat{\mathbf{A}}_-(s)]\hat{\mathbf{A}}_+(s), \qquad 0 \leq \Re s \leq \gamma,$$

where

$$\hat{\mathbf{A}}_-(s) = \mathbf{I} - \left(\sum_{n=1}^{\infty} \int_{-\infty}^{0} \mathrm{e}^{sx}\, \mathsf{P}_i(\overline{s}_{n-1} < S_n \in \mathrm{d}x, \kappa_n = j) \right),$$

$$\hat{\mathbf{A}}_+(s) = \mathbf{I} - \left(\int_0^{\infty} \mathrm{e}^{sx}\, \mathsf{P}_i(\chi(0) \in \mathrm{d}x, \kappa_{\eta(0)} = j) \right); \tag{3}$$

moreover, the matrices $\mathbf{B}(s)\hat{\mathbf{A}}_-(s)$ and $\hat{\mathbf{A}}_+(s)$ have inverses in $\hat{S}(0,\gamma)$.

In general, the factorization $\mathbf{I} - \hat{\mathbf{A}}(s) = \hat{\mathbf{A}}_-(s)\hat{\mathbf{A}}_+(s)$ holds regardless of whether the F_{ij} are absolutely continuous or not [3, Theorem 4.1] (in the latter case, the matrix $\mathbf{B}(s)\hat{\mathbf{A}}_-(s)$ may not have an inverse in $\hat{S}(0,\gamma)$). Notice also that the invertibility of $\mathbf{A}_+$ in $S(0,0)$ is valid without the requirement that the F_{ij} be absolutely continuous. Actually, $\mathbf{G}_+ := \delta_0\mathbf{I} - \mathbf{A}_+$ is a matrix of nonnegative measures such that the spectral radius of $\mathbf{G}_+(\mathbb{R})$ is less than 1, due to the fact that $\mathsf{E}_\pi S_1 < 0$ [3, Proposition 4.2]. So we have $\sum_{m=0}^{\infty}[\mathbf{G}_+(\mathbb{R})]^m < \infty$ [6, Corollary 5.6.13], i.e. $\sum_{m=0}^{\infty} \mathbf{G}_+^{m*}$ is a finite matrix measure; here $\mathbf{G}_+^{m*}$ is the m-fold convolution of the matrix measure $\mathbf{G}_+$. It is easily checked that $\mathbf{A}_+^{-1} = \sum_{m=0}^{\infty} \mathbf{G}_+^{m*}$.

Further, suppose $\hat{\mathbf{A}}(\gamma) < \infty$, $\gamma > 0$. Choose $\gamma' \in (0,\gamma)$. The matrix $\mathbf{I} - \hat{\mathbf{A}}(s)$ admits the right canonical factorization $\mathbf{I} - \hat{\mathbf{A}}(s) = \hat{\mathbf{A}}_-(s)\hat{\mathbf{A}}_+(s)$ for all $\gamma' \leq \Re s \leq \gamma$, where the matrices $\hat{\mathbf{A}}_-(s)$ and $\hat{\mathbf{A}}_+(s)$ have the same meaning as before and possess inverses in $\hat{S}(\gamma',\gamma)$ [2, Proposition 1] (see also [9]).

The following relation is a consequence of [10, Theorem 2.2] (see [2]):

$$\mathbf{I} - (\mathsf{P}_i(M_\infty > 0, \kappa_{\eta(0)} = j)) - \left(\int_{0+}^{\infty} \mathrm{e}^{sx}\, \mathrm{d}\mathsf{P}_i(M_\infty > x, \kappa_{\eta(x)} = j) \right)$$
$$= [\hat{\mathbf{A}}_+(s)]^{-1}\hat{\mathbf{A}}_+(0), \qquad \Re s = 0. \tag{4}$$

Let $\mathbf{W}$ denote the $N \times N$ matrix (W_{ij}), where the measures W_{ij} are defined by the relations $W_{ij}((-\infty,0)) := 0$,

$$W_{ij}((x,\infty)) := \mathsf{P}_i(M_\infty > x, \kappa_{\eta(x)} = j), \qquad x > 0,$$

and $W_{ij}(\{0\}) := \delta_{ij} - \mathsf{P}_i(M_\infty > 0, \kappa_{\eta(0)} = j)$, i, $j = 1, \ldots, N$. It follows that (4) may be rewritten as

$$\hat{\mathbf{W}}(s) = [\hat{\mathbf{A}}_+(s)]^{-1}\hat{\mathbf{A}}_+(0), \qquad \Re s = 0. \tag{5}$$

2. Main results

Let ν be a finite complex-valued measure. Define

$$T\nu(A) := \int_A n_1(x)\,\mathrm{d}x, \qquad A \in \mathcal{B},$$

where $n_1(x) := -\nu((-\infty, x])$ for $x < 0$ and $n_1(x) := \nu((x,\infty))$ for $x \geq 0$. If $\int_{\mathbb{R}} |x|\,|\nu|(\mathrm{d}x) < \infty$, then $T\nu$ is a finite complex-valued measure and its Laplace transform $(T\nu)^\wedge(s)$ is equal to $[\hat{\nu}(s) - \hat{\nu}(0)]/s$, $\Re s = 0$. If $\mathbf{B} = (B_{ij})$ is a matrix whose entries are finite complex-valued measures, then $T\mathbf{B}$ will denote the matrix (TB_{ij}).

THEOREM 3. *Let $\varphi(x)$, $x \geq 0$, be a nondecreasing submultiplicative function. Suppose $\mathsf{E}_\pi S_1 \in (-\infty, 0)$ and $W_{ij} \in S(\varphi)$ for all i, j. If $\gamma := r_+(\varphi) > 0$, assume additionally that $\lambda(\gamma) < 1$. Then $p_{ij} \int_0^\infty \varphi(x)\,TF_{ij}(\mathrm{d}x) < \infty$ for all i, j.*

THEOREM 4. *Let $\varphi(x)$, $x \geq 0$, be a nondecreasing submultiplicative function. If $\gamma = r_+(\varphi) > 0$, assume that $\lambda(\gamma) < 1$. Suppose $p_{ij} \int_0^\infty \varphi(x)\,TF_{ij}(\mathrm{d}x) < \infty$ for all i, j and $\mathsf{E}_\pi S_1 \in [-\infty, 0)$. Then $\mathsf{E}_i\varphi(M_\infty) < \infty$ for all i. Suppose that all the F_{ij} are absolutely continuous, $p_{ij} \int_0^\infty \varphi(x)\,TF_{ij}(\mathrm{d}x) < \infty$ for all i, j, and $\mathsf{E}_\pi S_1 \in (-\infty, 0)$. Then $W_{ij} \in S(\varphi)$ for all i, j.*

Proof of Theorem 3. By (5), $[\hat{\mathbf{A}}_+(s)]^{-1} \in \hat{S}(\varphi) \subset \hat{S}(0,\gamma) \subset \hat{S}(0,0)$. If $\gamma > 0$, then, by the discussion after Theorem 2, $\mathbf{A}_+ \in S(\gamma', \gamma)$ for every $\gamma' \in (0, \gamma)$. Since $\mathbf{A}_+$ is a finite matrix measure concentrated on $[0,\infty)$, we have $\mathbf{A}_+ \in S(0,\gamma)$. Applying Lemma 1, we obtain $\mathbf{A}_+ \in S(\varphi)$. It follows from $\mathsf{E}_\pi S_1 \in (-\infty, 0)$ that $\mathbf{B}(s)[\mathbf{I} - \hat{\mathbf{A}}(s)] \in \hat{S}(0,0)$, since the $(1,j)$-entry of $\mathbf{B}(s)[\mathbf{I} - \hat{\mathbf{A}}(s)]$ is equal to

$$\begin{aligned} \frac{s+1}{s}\left[\pi_j - \sum_{i=1}^N \pi_i p_{ij} \hat{F}_{ij}(s)\right] &= \frac{s+1}{s} \sum_{i=1}^N \pi_i p_{ij} \left[1 - \hat{F}_{ij}(s)\right] \\ &= \sum_{i=1}^N \pi_i p_{ij} \left[1 - \hat{F}_{ij}(s) - (TF_{ij})^\wedge(s)\right]. \end{aligned} \tag{6}$$

We have

$$\mathbf{B}(s)\hat{\mathbf{A}}_-(s) = \mathbf{B}(s)[\mathbf{I} - \hat{\mathbf{A}}(s)][\hat{\mathbf{A}}_+(s)]^{-1} \in \hat{S}(0,0).$$

Since the entries of $\mathbf{B}(s)\hat{\mathbf{A}}_-(s)$ are the Laplace transforms of finite measures concentrated on $(-\infty, 0]$, it is clear that $\mathbf{B}(s)\hat{\mathbf{A}}_-(s) \in \hat{S}(\varphi)$. Therefore,

$$\mathbf{B}(s)[\mathbf{I} - \hat{\mathbf{A}}(s)] = [\mathbf{B}(s)\hat{\mathbf{A}}_-(s)]\hat{\mathbf{A}}_+(s) \in \hat{S}(\varphi).$$

Since all the π_i are positive, relation (6) implies $p_{ij}TF_{ij} \in S(\varphi)$. Q.E.D.

Proof of Theorem 4. Let F be an arbitrary distribution. Then $TF \in S(\varphi)$ implies $F \in S(\varphi)$ [12, proof of Theorem 2]. Hence $\mathbf{B}(s)[\mathbf{I} - \hat{\mathbf{A}}(s)] \in \hat{S}(\varphi)$. Now suppose that all the F_{ij} are absolutely continuous and $\mathsf{E}_\pi S_1 \in (-\infty, 0)$. Applying Theorem 2, we have $[\hat{\mathbf{A}}_+(s)]^{-1} = \{\mathbf{B}(s)[\mathbf{I} - \hat{\mathbf{A}}(s)]\}^{-1}[\mathbf{B}(s)\hat{\mathbf{A}}_-(s)]$. By Lemma 1, $\{\mathbf{B}(s)[\mathbf{I} - \hat{\mathbf{A}}(s)]\}^{-1} \in \hat{S}(\varphi)$. Further, $\mathbf{B}(s)\hat{\mathbf{A}}_-(s)$, being a matrix of Laplace transforms of measures on $(-\infty, 0]$, is an element of $\hat{S}(\varphi)$. Hence $[\hat{\mathbf{A}}_+(s)]^{-1} \in \hat{S}(\varphi)$, which implies $\mathbf{W} \in S(\varphi)$.

The general case is considered as follows. Let $\{Y_m(i,j)\}_{m=1}^\infty$, $i, j = 1, \ldots, N$, be sequences of independent identically distributed random variables with uniform distribution on $[0, h]$, which are independent of all $\{X_m(i,j)\}_{m=1}^\infty$ and $\{\kappa_n\}_{n=0}^\infty$. Consider the random variables $X'_m(i,j) := X_m(i,j)$ if $X_m(i,j) \geq b$ and $X'_m(i,j) := b$ if $X_m(i,j) < b$ for a sufficiently remote negative level b. We now form a new random walk $\{S_n^*\}$ just in the same way as $\{S_n\}$ upon replacing the $X_m(i,j)$ by $X_m^*(i,j) := X'_m(i,j) + Y_m(i,j)$. Clearly, $M_\infty^* := \sup_{n \geq 0} S_n^* \geq M_\infty$. By choosing $|b|$ sufficiently large and h sufficiently small, we can achieve that $\mathsf{E}_\pi S_1^* \in (-\infty, 0)$ and $\lambda^*(\gamma) < 1$ (the latter follows from the fact that the maximal eigenvalue of a nonnegative matrix depends analytically on its entries; the superscript $*$ denotes the corresponding quantities for the new random walk $\{S_n^*\}$). Moreover, the underlying matrix $\mathbf{A}^*$ for $\{S_n^*\}$ possesses the following property: $T\mathbf{A}^* \in S(\varphi)$. In fact, choose an arbitrary element $G = p_{ij}F_{ij}^*$ with $p_{ij} > 0$, where F_{ij}^* stands for the common distribution of the random variables $X_m^*(i,j)$. Then

$$\begin{aligned}\frac{1}{p_{ij}} \int_0^\infty \varphi(x)\, G((x,\infty))\, \mathrm{d}x &\leq \int_0^\infty \varphi(x) F_{ij}((x-h,\infty))\, \mathrm{d}x \\ &\leq h\varphi(h) + \varphi(h) \int_0^\infty \varphi(x) F_{ij}((x,\infty))\, \mathrm{d}x < \infty.\end{aligned}$$

By the already proven, $\mathsf{E}_i \varphi(M_\infty^*) < \infty$, and hence $\mathsf{E}_i \varphi(M_\infty) < \infty$. Q.E.D.

Suppose now that $\varphi(x)/\exp(\gamma' x)$, $x \geq 0$, is nondecreasing for some $\gamma' \in (0, r_+(\varphi)]$. This assumption is not a very restrictive one since, in view of (2),

$\varphi(x)/\exp[r_+(\varphi)x] \geq 1$ for all $x \geq 0$. In this case we can somewhat strengthen the assertions of Theorems 3 and 4, admitting the possibility $\mathsf{E}_\pi S_1 = -\infty$ in both implications.

THEOREM 5. *Let $\varphi(x)$, $x \geq 0$, be a submultiplicative function such that $\gamma = r_+(\varphi) > 0$ and $\varphi(x)/\exp(\gamma' x)$ is nondecreasing for some $\gamma' \in (0, \gamma]$. Suppose $\mathsf{E}_\pi S_1 \in [-\infty, 0)$ and $\lambda(\gamma) < 1$. Then $\mathbf{W} \in S(\varphi)$ if and only if $\mathbf{A} \in S(\varphi)$. The relation $\mathbf{W} \in S(\varphi)$ clearly implies $\mathsf{E}_i\varphi(M_\infty) < \infty$ for all i.*

Proof. Put $\psi(x) := \exp(\gamma' x)$ for $x < 0$ and $\psi(x) := \varphi(x)$ for $x \geq 0$. Then $\psi(x)$, $x \in \mathbb{R}$, is obviously a submultiplicative function with $r_-(\psi) = \gamma'$ and $r_+(\psi) = \gamma$. Suppose first $\mathbf{W} \in S(\varphi)$. For a finite measure ν, we have $\nu \in S(\psi) \Leftrightarrow \nu \in S(\varphi)$. It follows that $\mathbf{W} \in S(\psi)$. By (5), $[\hat{\mathbf{A}}_+(s)]^{-1} \in \hat{S}(\psi)$. By Lemma 1, $\hat{\mathbf{A}}_+(s) \in \hat{S}(\gamma', \gamma) \Rightarrow \hat{\mathbf{A}}_+(s) \in \hat{S}(\psi)$. Since $\mathbf{A}_-$ is concentrated on $(-\infty, 0]$, $\mathbf{A}_- \in S(\psi)$. Hence $\mathbf{A} = \delta_0\mathbf{I} - \mathbf{A}_- * \mathbf{A}_+ \in S(\psi)$, i.e. $\mathbf{A} \in S(\varphi)$.

We now prove the converse assertion. Suppose $\mathbf{A} \in S(\varphi)$. Since $\mathbf{A}_-$ is concentrated on $(-\infty, 0]$, $\mathbf{A}_- \in S(\psi)$. There exists $\mathbf{A}_-^{-1} \in S(\gamma', \gamma)$ (see the corresponding discussion after Theorem 2). By Lemma 1, $\mathbf{A}_-^{-1} \in S(\psi)$. It follows that $\mathbf{A}_+ = \mathbf{A}_-^{-1} * (\delta_0\mathbf{I} - \mathbf{A}) \in S(\psi)$. Again by Lemma 1, we have $\mathbf{A}_+^{-1} \in S(\psi)$. By (5), $\mathbf{W} \in S(\psi)$. ■

Remark 1. Examining the proofs above, we see that assertions similar to Theorems 3, 4 and 5 are also valid for submultiplicative moments of the first positive sum $\chi(0)$ (see (3)). We only need to replace $\mathsf{E}_i\varphi(M_\infty)$ by $\mathsf{E}_i\varphi(\chi(0))$ in their statements throughout.

Remark 2. The exact tail behaviour of M_∞ and $\chi(0)$ for Markov-modulated random walks has been studied in [1] in the context of $\mathcal{S}(\gamma)$-distributions, $\gamma \geq 0$. The subexponential tail behaviour of M_∞ in the case of Markov-modulation has been considered in [7].

Remark 3. If $\lambda(\gamma) = 1$ or if $\lambda(\gamma) > 1$, $\gamma > 0$, then the distribution of M_∞ is highly influenced by the roots of the characteristic equation $\det(\mathbf{I} - \hat{\mathbf{A}}(s)) = 0$, which lie in the strip $\{s \in \mathbb{C} : 0 < \Re s \leq \gamma\}$, and Theorems 3 – 5 do not hold. The cases $\lambda(\gamma) = 1$ and $\lambda(\gamma) > 1$ for some $\gamma > 0$ will be dealt with in another paper.

Acknowledgement. I wish to express gratitude to a referee for a careful reading of the manuscript.

References

[1] G. Alsmeyer and M. Sgibnev, On the tail behaviour of the supremum of a random walk defined on a Markov chain, *Yokohama Math. J.*, **46** (1999), 139-159.

[2] K. Arndt, Asymptotic properties of the distribution of the supremum of a random walk on a Markov chain, *Theory Probab. Appl.*, **25** (1980), 309–324.

[3] S. Asmussen, Aspects of matrix Wiener-Hopf factorisation in applied probability, *Math. Scientist.*, **14** (1989), 101–116.

[4] W. Feller *An Introduction to Probability Theory and Its Applications II*, Wiley, New York, 1966.

[5] E. Hille and R.S. Phillips *Functional Analysis and Semi-Groups*, AMS Colloquium Publications **31**, Providence, 1957.

[6] R. A. Horn and C. R. Johnson *Matrix Analysis*, Cambridge University Press, Cambridge, 1986.

[7] P.R. Jelenković and A.A. Lazar, Subexponential asymptotics of a Markov-modulated random walk with queueing applications, *J. Appl. Probab.*, **35** (1998), 325–347.

[8] J. Kiefer and J. Wolfowitz, On the characteristics of the general queueing process, with applications to random walk, *Ann. Math. Statist.*, **27** (1956), 147–161.

[9] H.D. Miller, A matrix factorization problem in the theory of random variables defined on a finite Markov chain, *Proc. Cambr. Phil. Soc.*, **58** (1962), 268–285.

[10] E.L. Presman, Factorization methods and boundary problems for sums of random variables given on a Markov chain, *Math. USSR Izv.*, **3** (1969), 815–852.

[11] B.A. Rogozin and M.S. Sgibnev, Banach algebras of measures on the line. *Siberian Math. J.*, **21** (1980), 265–273.

[12] M.S. Sgibnev, Submultiplicative moments of the supremum of a random walk with negative drift, *Statistics and Probability Letters*, **32** (1997), 377-384.

S.L. Sobolev Institute of Mathematics,
Siberian Branch of the Russian Academy of Sciences,
Novosibirsk 90, 630090
RUSSIA.
E-mail address: `sgibnev@math.nsc.ru`