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AN EXISTENCE THEOREM FOR A SURFACE IN S^n WITH A GIVEN MAP AS ITS GAUSS MAP

By

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Abstract. The purpose of this paper is to study the existence of a surface in the n- dimensional Euclidean unit sphere S^n with prescribed Gauss map. For a given C^∞-mapping G from a torus T^2 into the complex quadric Q_{n-1}, we show that there exists a conformal immersion $X:\hat{T}^2 \longrightarrow S^n$ such that the Gauss map of the surface $S=(\hat{T}^2, S^n, X)$ is $G\circ\pi$ where $\pi:\hat{T}^2 \longrightarrow T^2$ is a covering map. Let G be a C^∞-mapping from a connected Riemann surface M into Q_{n-1}. Under a certain condition for G we also show that there exists a surface defined by a C^∞-conformal immersion X from M to the n-dimensional real projective space RP^n with the property that a neighborhood of each point of $X(M)$ is covered by a surface in S^n with prescribed Gauss map G. By using this result we give a characterization of certain tori immersed in RP^n.

1. Introduction

By a surface S in an n-dimensional ($n \geq 3$) Riemannian manifold $\hat{M}$ we mean a triple $(M, \hat{M}, X)$ consisting of a connected Riemann surface M, the ambient space $\hat{M}$ and a C^∞-conformal immersion $X: M \longrightarrow \hat{M}$.

Let $S=(M, R^{n+1}, X)$ be a surface in the $(n+1)$-dimensional Euclidean space R^{n+1}. For each $u\in M$ we can assign an oriented tangent plane to $X(M)$ at $X(u)$ to a unique point of the complex quadric Q_{n-1} in the n-dimensional complex projective space CP^n. This induces the (generalized) Gauss map $G: M \longrightarrow Q_{n-1}$. It is very important to study the property of the Gauss map of surfaces in a Euclidean space. There are many studies on the Gauss map from several points of view (see the bibliography in [1]). For a given C^∞-mapping G from a Riemann surface M into Q_{n-1}, it is an interesting problem to find a surface in a Euclidean space or a Euclidean sphere such that the Gauss map is G. For a simply-connected Riemann surface M and a C^∞-mapping $G: M \longrightarrow Q_{n-1}$ with certain conditions, Hoffman and Osserman showed that there exists a surface in R^{n+1} with the Gauss map G ([2]). Let $S=(M, S^n, X)$ be a surface in the

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n-dimensional Euclidean unit sphere S^n. By regarding it as a surface in R^{n+1}, we obtain the Gauss map of S. In this paper we investigate the existence of a surface in S^n with prescribed Gauss map. It seems to us that the existence of a surface in S^n with prescribed Gauss map cannot be shown directly by using the results due to Hoffman and Osserman ([2]). In this paper, by using different methods from theirs, we obtain an existence theorem for S^n and RP^n.

Let $G : M \longrightarrow Q_{n-1}$ be a C^∞-mapping. For each $u \in M$ $G(u)$ corresponds to a unique oriented 2-plane $\hat{G}(u)$ in R^{n+1} passing through the origin. On an open set U of M we have a C^∞-mapping

$$E = (E_1, E_2, E_3, \cdots, E_{n+1}) : U \longrightarrow SO(n+1)$$

such that for each $u \in U$ $E^T(u) := (E_1(u), E_2(u))$ is an orthonormal frame of $\hat{G}(u)$ giving the orientation of it. By using this E, we give sufficient conditions (3.3), (3.11) and (3.12) for the existence of a surface in S^n with prescribed Gauss map.

The main results of this paper are Theorems 3.2 and 3.3. Let $G : M \longrightarrow Q_{n-1}$ be a C^∞-mapping which satisfies the conditions (3.3), (3.11) and (3.12). We show in Theorem 3.2 that if M is a torus T^2, then there exist a covering space $(\hat{T}^2, T^2, \hat{\pi})$ over T^2 and a surface $S = (\hat{T}^2, S^n, X)$ such that the Gauss map is $G \circ \hat{\pi}$. In Theorem 3.3 we also show that there exists a surface $S = (M, RP^n, X)$ in the n-dimensional real projective space RP^n with the property that a neighborhood of each point of $X(M)$ is covered by a surface in S^n with prescribed Gauss map G.

We show in Section 4 that if we impose certain conditions on G in Theorem 3.3, then M is a torus.

We note that there are Riemann surfaces admitting C^∞-mappings $G : M \longrightarrow Q_{n-1}$ with the conditions (3.3), (3.11) and (3.12). In the last section we give examples of C^∞-mappings $G : T^2 \longrightarrow Q_{n-1}$ with the conditions in Theorem 3.3.

2. The Gauss Map

We assume in this paper that manifolds and apparatus on them are of class C^∞ and that manifolds satisfy the second countability axiom, unless otherwise stated.

Let M be a connected Riemann surface and $S = (M, R^n, X)$ a surface in the Euclidean n-space R^n. $X : M \longrightarrow R^n$ is said to be conformal if for any complex coordinate system $(U,\ z = u_1 + \sqrt{-1}u_2)$ of M it satisfies

$$\left\langle \frac{\partial X}{\partial z}, \frac{\partial X}{\partial \bar{z}} \right\rangle = 0$$

where $< \, , \, >$ denotes the canonical Hermitian product on C^n.

We define the Gauss map of a surface $S = (M, R^n, X)$ in R^n following Hoffman and Osserman ([1]). Let Q_{n-2} be the complex quadric in the $(n-1)$-dimensional complex projective space CP^{n-1}, which is defined as

$$Q_{n-2} = \{[w] \in CP^{n-1} | {w_1}^2 + \cdots + {w_n}^2 = 0\}.$$

Q_{n-2} is diffeomorphic to the Grassmaniann manifold

$$\tilde{G}(2, n) = SO(n)/SO(2) \times SO(n-2)$$

consisting of all oriented 2-planes in R^n passing through the origin. For a complex coordinate system $(U, z = u_1 + \sqrt{-1}\,u_2)$ of M and a C^∞-mapping $A : U \longrightarrow R^k$, we put

$$A_z = \frac{\partial A}{\partial z} = \frac{1}{2}\left(\frac{\partial A}{\partial u_1} - \sqrt{-1}\,\frac{\partial A}{\partial u_2}\right), \quad A_{\bar{z}} = \frac{\partial A}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial A}{\partial u_1} + \sqrt{-1}\,\frac{\partial A}{\partial u_2}\right).$$

For each $p \in M$, by identifying the tangent vectors

$$dX_p\left(\left(\frac{\partial}{\partial u_1}\right)_p\right), \quad dX_p\left(\left(\frac{\partial}{\partial u_2}\right)_p\right)$$

with

$$\frac{\partial X}{\partial u_1}(p), \quad \frac{\partial X}{\partial u_2}(p)$$

respectively by parallel translations, each tangent plane of $X(M)$ corresponds to a unique point of Q_{n-2} where $z = u_1 + \sqrt{-1}u_2$ is a local coordinate function about p. Thus the generalized Gauss map of $S = (M, R^n, X)$ can be defined as

$$G : M \longrightarrow Q_{n-2} \quad \left(p \longmapsto \left[\frac{\partial X}{\partial \bar{z}}(p)\right]\right).$$

In case of $n = 3$, the generalized Gauss map can be regarded as the classical Gauss map. For simplicity, the generalized Gauss map will be called the Gauss map in this paper.

Let $G : M \longrightarrow Q_{n-1}$ be a C^∞-mapping. For each $u \in M$, we denote by $\hat{G}(u)$ the oriented 2-plane in R^{n+1} passing through the origin which corresponds to $G(u)$. Let $(U, z = u_1 + \sqrt{-1}u_2)$ be a complex coordinate system of M. For $u \in U$ we express R^{n+1} as the direct sum

$$R^{n+1} = \hat{G}(u) \oplus \hat{G}^{\perp}(u)$$

where $\hat{G}^{\perp}(u)$ denotes the orthogonal complement to $\hat{G}(u)$. We set

$$P(M,G) = \bigcup_{u \in M} \hat{G}(u).$$

We denote by V the smallest linear subspace in R^{n+1} containing $P(M,G)$. Let $V^{\perp}$ be the orthogonal complement of V in R^{n+1}. In the following we denote by k the dimension of V.

The following results will be used later.

LEMMA 2.1 ([3]). *Let M be a connected Riemann surface and $G : M \longrightarrow Q_{n-1}$ a C^{∞}-mapping. Let V and $V^{\perp}$ be as defined above. Let $S = (M, S^n, X)$ be a surface in S^n such that the Gauss map is G. If $3 \leq k \leq n$, then the following holds:*
(1) *There exists a surface $\hat{S} = (M, S^n, \hat{X})$ in S^n such that the Gauss map coincides with G and*

$$\hat{X}(M) \subset V \cap S^n.$$

(2) *If the Gauss map of a surface $S_Y = (M, S^n, Y)$ in S^n is G, then Y can be expressed as*

$$Y = c\hat{X} + tq$$

where c and t are constants such that

$$c = \epsilon\sqrt{1-t^2}, \quad \epsilon = \pm 1, \quad |t| < 1,$$

and $q \in V^{\perp} \cap S^n$.

LEMMA 2.2 ([3]). *Let M, G and $S = (M, S^n, X)$ be as in Lemma 2.1. Let $S_Y = (M, S^n, Y)$ be a surface in S^n such that the Gauss map is G. If $k = n+1$, then $Y = \pm X$.*

Let $S = (M, S^n, X)$ be a surface in S^n. We take a complex coordinate system (U, z) of M. We have

$$\frac{\partial X}{\partial z} \cdot X = 0 \tag{2.1}$$

on U. We set

$$E_1 = \frac{1}{\lambda}\left(\frac{\partial X}{\partial z} + \frac{\partial X}{\partial \bar{z}}\right), \quad E_2 = \frac{\sqrt{-1}}{\lambda}\left(\frac{\partial X}{\partial z} - \frac{\partial X}{\partial \bar{z}}\right) \tag{2.2}$$

where

$$\lambda = \sqrt{2}\left\langle \frac{\partial X}{\partial z}, \frac{\partial X}{\partial z} \right\rangle^{\frac{1}{2}}.$$

By taking U sufficiently small, we obtain a C^∞-mapping $E : U \longrightarrow SO(n+1)$ such that $E = (E_1, E_2, \cdots, E_{n+1})$. Then X can be expressed as

$$X(u) = \sum_{\alpha=3}^{n+1} X^\alpha(u) E_\alpha(u), \quad u \in U. \tag{2.3}$$

Let $\mathscr{A}(m)$ be the set of all $m \times m$ complex skew symmetric matrices. We set

$$\frac{\partial E_k}{\partial z} = \sum_{l=1}^{n+1} a_k^l E_l \quad (1 \leq k \leq n+1) \tag{2.4}$$

and define a C^∞-mapping $A : U \longrightarrow \mathscr{A}(n+1)$ as $A(u) = \big(a_k^l(u)\big)$. We express A in the form

$$A = \left(\begin{array}{c|c} A_1^1 & -{}^t A_1^2 \\ \hline A_1^2 & B \end{array} \right)$$

where $A_1^1(u) \in \mathscr{A}(2)$ and $B(u) \in \mathscr{A}(n-1)$ $(u \in U)$. Here ${}^t A_1^2$ stands for the transposed matrix of A_1^2. We set

$$A_1^1 = \begin{pmatrix} 0 & \nu \\ -\nu & 0 \end{pmatrix}, \quad A_1^2 = (a^1\, a^2), \quad B = \begin{pmatrix} L & D \\ -{}^t D & J \end{pmatrix},$$

$$D = (a^n\, a^{n+1}), \quad J = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$$

where $J(u) \in \mathscr{A}(2)$ and $L(u) \in \mathscr{A}(n-3)$ $(u \in U)$. Then we have

$$\frac{\partial X}{\partial z} = \sum_{k=1}^{2} \left(\sum_{\alpha=3}^{n+1} X^\alpha a_\alpha^k \right) E_k + \sum_{\alpha=3}^{n+1} \left(\frac{\partial X^\alpha}{\partial z} + \sum_{\beta=3}^{n+1} X^\beta a_\beta^\alpha \right) E_\alpha.$$

By (2.2) we get

$$\sum_{\alpha=3}^{n+1} X^\alpha a_\alpha^1 = \frac{1}{2}\lambda, \quad \sum_{\alpha=3}^{n+1} X^\alpha a_\alpha^2 = -\frac{\sqrt{-1}}{2}\lambda, \tag{2.5}$$

$$\frac{\partial X^\alpha}{\partial z} = -\sum_{\beta=3}^{n+1} X^\beta a_\beta^\alpha \qquad (3 \leq \alpha \leq n+1). \tag{2.6}$$

We set $\hat{X} = {}^t(X^3, \cdots, X^{n+1})$. It follows from (2.3), (2.5) and (2.6) that

$$\frac{\partial E}{\partial z} = EA, \quad {}^t a^1 \hat{X} = \frac{1}{2}\lambda, \quad {}^t a^2 \hat{X} = -\frac{\sqrt{-1}}{2}\lambda, \quad \frac{\partial \hat{X}}{\partial z} = -B\hat{X}.$$

Since

$$\frac{\partial^2 E}{\partial z \partial \bar{z}} = \frac{\partial^2 E}{\partial \bar{z} \partial z},$$

we get

$$\frac{\partial A}{\partial \bar{z}} - \frac{\partial \bar{A}}{\partial z} = [A, \bar{A}] \tag{2.7}$$

where $[A, \bar{A}] = A\bar{A} - \bar{A}A$. Then (2.7) is rewritten as

$$\frac{\partial A_1^1}{\partial \bar{z}} - \frac{\partial \bar{A}_1^1}{\partial z} = [A_1^1, \bar{A}_1^1] - [{}^t A_1^2, \bar{A}_1^2], \tag{2.8}$$

$$\frac{\partial A_1^2}{\partial \bar{z}} - \frac{\partial \bar{A}_1^2}{\partial z} = [A_1^2, \bar{A}_1^1] + [B, \bar{A}_1^2], \tag{2.9}$$

$$\frac{\partial B}{\partial \bar{z}} - \frac{\partial \bar{B}}{\partial z} = [B, \bar{B}] - [A_1^2, {}^t \bar{A}_1^2]. \tag{2.10}$$

By (2.2) and (2.4) we have

$$\begin{aligned}
\frac{\partial^2 X}{\partial z \partial \bar{z}} &= \frac{1}{2}\frac{\partial}{\partial \bar{z}}(\lambda(E_1 - \sqrt{-1}E_2)) \\
&= \frac{1}{2}\left(\frac{\partial \lambda}{\partial \bar{z}} - \sqrt{-1}\lambda \bar{a}_2^1\right)E_1 + \frac{1}{2}\left(-\sqrt{-1}\frac{\partial \lambda}{\partial \bar{z}} + \lambda \bar{a}_1^2\right)E_2 \\
&\quad + \frac{1}{2}\lambda \sum_{\alpha=3}^{n+1} \bar{a}_1^\alpha E_\alpha - \frac{\sqrt{-1}}{2}\lambda \sum_{\alpha=3}^{n+1} \bar{a}_2^\alpha E_\alpha.
\end{aligned}$$

Since

$$\frac{\partial^2 X}{\partial z \partial \bar{z}} = \frac{\partial^2 X}{\partial \bar{z} \partial z},$$

we have

$$\begin{aligned}
&\left(\frac{\partial \lambda}{\partial \bar{z}} - \sqrt{-1}\lambda \bar{a}_2^1\right)E_1 + \left(-\sqrt{-1}\frac{\partial \lambda}{\partial \bar{z}} + \lambda \bar{a}_1^2\right)E_2 + \lambda \sum_{\alpha=3}^{n+1} \bar{a}_1^\alpha E_\alpha - \sqrt{-1}\lambda \sum_{\alpha=3}^{n+1} \bar{a}_2^\alpha E_\alpha \\
&= \left(\frac{\partial \lambda}{\partial z} + \sqrt{-1}\lambda a_2^1\right)E_1 + \left(\sqrt{-1}\frac{\partial \lambda}{\partial z} + \lambda a_1^2\right)E_2 + \lambda \sum_{\alpha=3}^{n+1} a_1^\alpha E_\alpha + \sqrt{-1}\lambda \sum_{\alpha=3}^{n+1} a_2^\alpha E_\alpha.
\end{aligned}$$

As $E_1, \cdots, E_{n+1}$ are linearly independent, we obtain

$$\frac{\partial\lambda}{\partial\bar{z}} - \sqrt{-1}\lambda\bar{a}_2^1 = \frac{\partial\lambda}{\partial z} + \sqrt{-1}\lambda a_2^1, \tag{2.11}$$

$$\frac{\partial\lambda}{\partial\bar{z}} + \sqrt{-1}\lambda\bar{a}_1^2 = -\frac{\partial\lambda}{\partial z} + \sqrt{-1}\lambda a_1^2, \tag{2.12}$$

$$\bar{a}_1^\alpha = a_1^\alpha, \quad -\bar{a}_2^\alpha = a_2^\alpha \qquad (3 \leq \alpha \leq n+1). \tag{2.13}$$

The partial differential equations (2.11) and (2.12) are equivalent to the following equation

$$\frac{\partial\lambda}{\partial z} = -\sqrt{-1}\lambda\nu. \tag{2.14}$$

We now suppose that X^n and X^{n+1} are written as

$$X^n = f_1Y + h_1, \quad X^{n+1} = f_2Y + h_2 \tag{2.15}$$

where

$$f_k = (f_k^3, \cdots, f_k^{n-1}) \ (k = 1, 2), \quad Y = {}^t(X^3, \cdots, X^{n-1}),$$

and h_k $(k = 1, 2)$ are C^∞-functions on U. By substituting (2.15) into (2.6), we get

$$\frac{\partial X^n}{\partial z} = ({}^ta^n - bf_2)Y - bh_2, \tag{2.16}$$

$$\frac{\partial X^{n+1}}{\partial z} = ({}^ta^{n+1} + bf_1)Y + bh_1, \tag{2.17}$$

$$\frac{\partial X^\alpha}{\partial z} = -\sum_{\beta=3}^{n-1} a_\beta^\alpha X^\beta - a_n^\alpha(f_1Y + h_1) - a_{n+1}^\alpha(f_2Y + h_2) \tag{2.18}$$

where $3 \leq \alpha \leq n-1$. These equations imply

$$\frac{\partial Y}{\partial z} = -(L + Df)Y - Dh \tag{2.19}$$

where f and h are C^∞-mappings defined as

$$f : U \longrightarrow \mathscr{M}(2, n-3) \quad \left(u \longmapsto \begin{pmatrix} f_1(u) \\ f_2(u) \end{pmatrix}\right),$$

$$h : U \longrightarrow \mathscr{M}(2, 1) \quad \left(u \longmapsto \begin{pmatrix} h_1(u) \\ h_2(u) \end{pmatrix}\right).$$

Here $\mathscr{M}(s,t)$ denotes the set of all $s\times t$ complex matrices and $f_j(u)\in R^{n-3}$, $h_j(u)\in R$ $(j=1,2)$. It follows from (2.15), (2.16), (2.17) and (2.19) that

$$\frac{\partial h}{\partial z}+\frac{\partial f}{\partial z}Y-f(L+Df)Y-fDh=({}^tD-Jf)Y-Jh.$$

This is written as

$$\left(\frac{\partial f}{\partial z}-f(L+Df)-{}^tD+Jf\right)Y+\left(\frac{\partial h}{\partial z}-fDh+Jh\right)=0. \tag{2.20}$$

If f and h are solutions of the following system of partial differential equations:

$$\frac{\partial f}{\partial z}=f(L+Df)+{}^tD-Jf, \tag{2.21}$$

$$\frac{\partial h}{\partial z}=fDh-Jh, \tag{2.22}$$

then (2.20) holds. By the Frobenius theorem ([4]), a necessary and sufficient condition for the existence of a solution of the system of the partial differential equations (2.21) and (2.22) can be expressed as

$$\frac{\partial\bar{D}}{\partial z}-\frac{\partial D}{\partial\bar{z}}+[L,\bar{D}]+[D,\bar{J}]=0, \tag{2.23}$$

$$\frac{\partial\bar{L}}{\partial z}-\frac{\partial L}{\partial\bar{z}}+[L,\bar{L}]-[D,{}^t\bar{D}]=0, \tag{2.24}$$

$$\frac{\partial\bar{J}}{\partial z}-\frac{\partial J}{\partial\bar{z}}-[{}^tD,\bar{D}]=0. \tag{2.25}$$

These conditions (2.23), (2.24) and (2.25) are equivalent to

$$\frac{\partial B}{\partial\bar{z}}-\frac{\partial\bar{B}}{\partial z}=[B,\bar{B}]. \tag{2.26}$$

Let f and h be solutions of the system of partial differential equations (2.21) and (2.22). Then a necessary and sufficient condition for the existence of a solution of the partial differential equation (2.19) is (2.26).

3. Existence of a surface in S^n with a given Gauss map

Throughout this section let M be a connected Riemann surface and $G: M\longrightarrow Q_{n-1}$ a C^∞-mapping. Let $(U, z=u_1+\sqrt{-1}u_2)$ be a complex coordinate system of M. For G, by taking U sufficiently small if necessary, there exist a C^∞-mapping

$$E=(E_1,E_2,E_3,\cdots,E_{n+1}):U\longrightarrow SO(n+1) \tag{3.1}$$

such that for each $u \in U$ $E^T(u) := (E_1(u), E_2(u))$ is an orthonormal frame in $\hat{G}(u)$ giving the orientation of it. In the following we call a pair $((U, z), E^T)$ a local expression of G and put $E = (E^T, E^N)$.

Let $((U, z = u_1 + \sqrt{-1}u_2), E^T)$ be a local expression of G. We set in U

$$\frac{\partial E_i}{\partial u_j}(u) = E_{ij}(u) + F_{ij}(u) \quad (i, j = 1, 2) \tag{3.2}$$

where $E_{ij}(u) \in \hat{G}(u)$ and $F_{ij}(u) \in \hat{G}^{\perp}(u)$. We consider the condition

$$E_{11} = -E_{12},\ E_{21} = -E_{22},\ F_{11} = F_{22},\ F_{21} = F_{12},\ F_{11} \wedge F_{21} \neq 0. \tag{3.3}$$

From now on let T denotes a 2×2 matrix in the form

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

unless otherwise stated. We set

$$E_{ij} = E^T P_{ij}, \quad F_{ij} = E^N Q_{ij} \quad (i, j = 1, 2) \tag{3.4}$$

where $P_{ij}(u) \in \mathscr{M}(2, 1)$ and $Q_{ij}(u) \in \mathscr{M}(n-1, 1)$. By using (3.4), we can rewrite (3.3) as

$$P_{11} = -P_{12},\ P_{21} = -P_{22},\ Q_{11} = Q_{22},\ Q_{21} = Q_{12},\ Q_{11} \wedge Q_{21} \neq 0. \tag{3.5}$$

In the following we set

$$P_1 = P_{11}, \quad P_2 = P_{21}, \quad Q_1 = Q_{11}, \quad Q_2 = Q_{21}.$$

We define C^∞-mappings

$$P : U \longrightarrow \mathscr{A}(2), \quad Q : U \longrightarrow \mathscr{M}(2, n-1) \tag{3.6}$$

by

$$P(u) = \frac{1}{2}\begin{pmatrix} P_1(u) + \sqrt{-1}P_1(u) & P_2(u) + \sqrt{-1}P_2(u) \end{pmatrix}, \tag{3.7}$$

$$Q(u) = \frac{1}{2}\begin{pmatrix} {}^tQ_1(u) \\ {}^tQ_2(u) \end{pmatrix} - \frac{\sqrt{-1}}{2}T\begin{pmatrix} {}^tQ_1(u) \\ {}^tQ_2(u) \end{pmatrix} \tag{3.8}$$

respectively. Then we have

$$\frac{\partial E}{\partial z} = EA \tag{3.9}$$

where $A(u) \in \mathscr{A}(n+1)$. A can be expressed in the form

$$A = \left(\begin{array}{c|c} P & -Q \\ \hline {}^tQ & B \end{array}\right) \tag{3.10}$$

where $B(u) \in \mathscr{A}(n-1)$. Let us consider the following conditions

$$\frac{\partial P}{\partial z} = \frac{1}{2}\left(TQ\,{}^tQ - Q\,{}^tQT\right), \quad \frac{\partial P}{\partial \bar{z}} = \frac{\sqrt{-1}}{2}\left(TQ\,{}^tQ - Q\,{}^tQT\right), \tag{3.11}$$

$$\frac{\partial Q}{\partial z} = PQ + QB, \qquad \frac{\partial Q}{\partial \bar{z}} = -\bar{P}Q + Q\bar{B}. \tag{3.12}$$

We note that for a torus with an affine structure, the conditions (3.11) and (3.12) are independent of the choice of complex coordinate systems.

We shall show that the conditions (3.11) and (3.12) are invariant under the choice of orthonormal frame fields.

PROPOSITION 3.1. *Let $((U, z), E^T)$ be a local expression of G such that C^∞-mappings P and Q defined by* (3.7) *and* (3.8) *satisfy the conditions* (3.11) *and* (3.12). *Then the conditions* (3.11) *and* (3.12) *are independent of the choice of orthonormal frame fields E^N on U which are orthogonal to E^T in R^{n+1}.*

Proof. Let $E = (E_1, E_2, \cdots, E_{n+1})$ and $F = (F_1, F_2, \cdots, F_{n+1})$ be different orthonormal frame fields of R^{n+1} on U such that

$$E_i = F_i \quad (i = 1, 2).$$

By (3.9) we have

$$\frac{\partial E}{\partial z} = EA, \quad \frac{\partial F}{\partial z} = F\hat{A} \tag{3.13}$$

where $A(u)$, $\hat{A}(u) \in \mathscr{A}(n+1)$. By (3.10) A and $\hat{A}$ can be expressed in the form

$$A = \left(\begin{array}{c|c} P & -Q \\ \hline {}^tQ & B \end{array}\right), \quad \hat{A} = \left(\begin{array}{c|c} P & -\hat{Q} \\ \hline {}^t\hat{Q} & \hat{B} \end{array}\right) \tag{3.14}$$

where $B(u)$, $\hat{B}(u) \in \mathscr{A}(n-1)$. We put

$$\mathscr{O}(m) = \{\ \Omega_m \in \mathscr{M}(m, m) \mid {}^t\Omega_m \Omega_m = I_m\ \}.$$

Here I_m denotes the unit matrix of degree m. Since E and F are orthonormal frames, there exists a C^∞- mapping $\Omega_{n+1} : U \longrightarrow \mathscr{O}(n+1)$ such that

$$E = F\Omega_{n+1}, \quad \Omega_{n+1} = \left(\begin{array}{c|c} I_2 & 0 \\ \hline 0 & \Omega_{n-1} \end{array}\right).$$

This and (3.13) imply

$$\frac{\partial \Omega_{n+1}}{\partial z} = \Omega_{n+1} A - \hat{A}\Omega_{n+1}. \tag{3.15}$$

Since $E_i = F_i$ $(i = 1, 2)$, by using (3.14) and (3.15) we get

$$\frac{\partial \Omega_{n-1}}{\partial z} = \Omega_{n-1} B - \hat{B}\Omega_{n-1}, \tag{3.16}$$

$$Q = \hat{Q}\Omega_{n-1}. \tag{3.17}$$

We now assume for $\hat{A}$ that

$$\frac{\partial P}{\partial z} = \frac{1}{2}(T\hat{Q}^t\hat{Q} - \hat{Q}^t\hat{Q}T), \quad \frac{\partial P}{\partial \bar{z}} = \frac{\sqrt{-1}}{2}(T\hat{Q}^t\hat{Q} - \hat{Q}^t\hat{Q}T), \tag{3.18}$$

$$\frac{\partial \hat{Q}}{\partial z} = P\hat{Q} + \hat{Q}\hat{B}, \qquad \frac{\partial \hat{Q}}{\partial \bar{z}} = -\bar{P}\hat{Q} + \hat{Q}\bar{\hat{B}}. \tag{3.19}$$

It follows from (3.17) and (3.18) that

$$\frac{\partial P}{\partial z} = \frac{1}{2}(TQ^t\Omega_{n-1}{\Omega_{n-1}}^tQ - Q^t\Omega_{n-1}{\Omega_{n-1}}^tQT) = \frac{1}{2}(TQ^tQ - Q^tQT),$$
$$\frac{\partial P}{\partial \bar{z}} = \frac{\sqrt{-1}}{2}(TQ^t\Omega_{n-1}{\Omega_{n-1}}^tQ - Q^t\Omega_{n-1}{\Omega_{n-1}}^tQT) = \frac{\sqrt{-1}}{2}(TQ^tQ - Q^tQT).$$

Furthermore, using (3.16), (3.17) and (3.19), we obtain

$$\frac{\partial Q}{\partial z} = \frac{\partial \hat{Q}}{\partial z}\Omega_{n-1} + \hat{Q}\frac{\partial \Omega_{n-1}}{\partial z} = P\hat{Q}\Omega_{n-1} + \hat{Q}\Omega_{n-1}B = PQ + QB,$$
$$\frac{\partial Q}{\partial \bar{z}} = \frac{\partial \hat{Q}}{\partial \bar{z}}\Omega_{n-1} + \hat{Q}\frac{\partial \Omega_{n-1}}{\partial \bar{z}} = -\bar{P}\hat{Q}\Omega_{n-1} + \hat{Q}\Omega_{n-1}\bar{B} = -\bar{P}Q + Q\bar{B}.$$

Thus the conditions (3.11) and (3.12) are independent of the choice of orthonormal frame fields E^N. We complete the proof.

Let $E = (E^T, E^N) = (E_1, E_2, E_3, \cdots, E_{n+1})$ be as above. Since

$$\frac{\partial^2 E}{\partial z \partial \bar{z}} = \frac{\partial^2 E}{\partial \bar{z} \partial z},$$

by (3.9) and (3.10) we have

$$\frac{\partial B}{\partial \bar{z}} - \frac{\partial \bar{B}}{\partial z} = [B, \bar{B}]. \tag{3.20}$$

This is a necessary and sufficient condition for the existence of solutions of the system of the partial differential equations (3.11) and (3.12). We express B in the form

$$B = \begin{pmatrix} L & D \\ -{}^tD & J \end{pmatrix} \tag{3.21}$$

where $J(u) \in \mathscr{A}(2)$ and $L(u) \in \mathscr{A}(n-3)$.

Since rank $Q = 2$ by assumption, we can choose a minor 2×2 matrix K from Q so that K is invertible on a sufficiently small open neighborhood $U_1 (\subset U)$ of a point $m_0 \in U$. Without loss of generality we may assume that Q is expressed as

$$Q = \begin{pmatrix} H & -K \end{pmatrix} = \begin{pmatrix} {}^tH_1 & -{}^tK_1 \\ {}^tH_2 & -{}^tK_2 \end{pmatrix} \tag{3.22}$$

where $H : U_1 \longrightarrow \mathscr{M}(2, n-3)$ and $K : U_1 \longrightarrow \mathscr{M}(2,2)$ are C^∞-mappings.

Let

$$\Lambda : U_1 \longrightarrow \mathscr{M}(2,1) \quad \left(u \longmapsto \frac{1}{2} \begin{pmatrix} \lambda(u) \\ -\sqrt{-1}\lambda(u) \end{pmatrix} \right)$$

be a C^∞-mapping such that

$$\frac{\partial \Lambda}{\partial z} = P\Lambda, \quad \frac{\partial \Lambda}{\partial \bar{z}} = -\bar{P}\Lambda \tag{3.23}$$

on U_1. It should be noted that a solution Λ of the partial differential equation (3.23) exists on a sufficiently small open neighborhood $U_2 (\subset U_1)$ of m_0 because (3.11) holds. For simplicity, we let $U = U_2$.

We define C^∞-mappings f and h on U by

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = K^{-1}H, \quad h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = K^{-1}\Lambda. \tag{3.24}$$

We shall show that f and h are solutions of the system of the partial differential equations (2.21) and (2.22). The equation (3.12) are rewritten as

$$\frac{\partial H}{\partial z} = PH + HL + K{}^tD, \quad \frac{\partial H}{\partial \bar{z}} = -P\bar{H} + H\bar{L} + K{}^t\bar{D}, \tag{3.25}$$

$$\frac{\partial K}{\partial z} = PK - HD + KJ, \quad \frac{\partial K}{\partial \bar{z}} = -\bar{P}K - H\bar{D} + K\bar{J}. \tag{3.26}$$

By taking differentiation of the both sides of (3.24) and by using (3.24), (3.25) and (3.26), we have

$$\begin{aligned} \frac{\partial f}{\partial z} &= \frac{\partial K^{-1}}{\partial z} H + K^{-1}\frac{\partial H}{\partial z} \\ &= -K^{-1}\frac{\partial K}{\partial z}K^{-1}H + K^{-1}\frac{\partial H}{\partial z} \\ &= K^{-1}HDK^{-1}H - JK^{-1}H + K^{-1}HL + {}^tD \end{aligned} \tag{3.27}$$

and

$$\frac{\partial h}{\partial z} = K^{-1}HDK^{-1}\Lambda - JK^{-1}\Lambda. \tag{3.28}$$

Substituting (3.24) into the right hands of the equations (2.21) and (2.22), these are written as

$$\frac{\partial f}{\partial z} = K^{-1}H(L + DK^{-1}H) + {}^tD - JK^{-1}H, \tag{3.29}$$
$$\frac{\partial h}{\partial z} = K^{-1}HDK^{-1}\Lambda - JK^{-1}\Lambda. \tag{3.30}$$

Hence f and h are solutions of the system of the partial differential equations (2.21) and (2.22).

Now we shall show the following theorem.

THEOREM 3.1. *Let M be a connected Riemann surface and $G : M \longrightarrow Q_{n-1}$ a C^∞-mapping. Assume that a local expression $((U, z = u_1 + \sqrt{-1}u_2), E^T)$ of G satisfies the following conditions :*
(1) The tangent component E_{ij} and the normal component F_{ij} of $\frac{\partial E_i}{\partial u_j}$ in R^{n+1} satisfy the condition (3.3)*;*
(2) P and Q defined by (3.7) *and* (3.8) *satisfy the conditions* (3.11) *and* (3.12)*.*
Then there exists a surface $S = (U_0, S^n, \Psi)$ with the Gauss map $G|_{U_0}$ where U_0 is an open set in M such that $U_0 \subset U$.

Proof. We will use the notations stated above. We take $m_0 \in U$. Let

$$\Lambda = \frac{1}{2}\begin{pmatrix} \lambda \\ -\sqrt{-1}\lambda \end{pmatrix} : U_1 \longrightarrow \mathscr{M}(2,1)$$

be a C^∞-mapping which is a solution of the partial differential equation (3.23) where $U_1(\subset U)$ is an open neighborhood of m_0. Since rank $Q = 2$ by assumption, we can choose a minor 2×2 matrix K from Q so that K is invertible on a sufficiently small simply connected open neighborhood $U_0(\subset U_1)$ of m_0. Without loss of generality, we may assume that Q is expressed as in the form (3.22). We define C^∞-mappings f and h on U_0 by (3.24). We note here that f and h are solutions of the system of the partial differential equations (2.21) and (2.22) as showed above. Let us consider the following partial differential equation

$$\frac{\partial Y}{\partial z} = -(L + Df)Y - Dh \tag{3.31}$$

on U_0. Since (3.20) holds, this partial differential equation has a solution

$$Y = {}^t(\Psi^3, \cdots, \Psi^{n-1}) : U_0 \longrightarrow R^{n-3}$$

where we took U_0 sufficiently small. Define a C^∞-mapping

$$Z = \begin{pmatrix} \Psi^n \\ \Psi^{n+1} \end{pmatrix} : U_0 \longrightarrow R^2$$

by $Z = fY + h = K^{-1}HY + K^{-1}\Lambda$. We set

$$\tilde{\Psi} = \begin{pmatrix} Y \\ Z \end{pmatrix}.$$

By using (3.24) and (3.31) we have

$$\frac{\partial Z}{\partial z} = \frac{\partial f}{\partial z}Y + f\frac{\partial Y}{\partial z} + \frac{\partial h}{\partial z} = ({}^tD - JK^{-1}H)Y - JK^{-1}\Lambda.$$

Let us define a C^∞-mapping $\Psi : U_0 \longrightarrow R^{n+1}$ by

$$\Psi = \sum_{\alpha=3}^{n+1} \Psi^\alpha E_\alpha = E^N \tilde{\Psi}.$$

By differentiating it with respect to z, we get

$$\begin{aligned}
\frac{\partial \Psi}{\partial z} &= \frac{\partial E^N}{\partial z}\tilde{\Psi} + E^N \frac{\partial \tilde{\Psi}}{\partial z} \\
&= (-E^TQ + E^NB)\tilde{\Psi} + E^N \begin{pmatrix} \frac{\partial Y}{\partial z} \\ \\ \frac{\partial Z}{\partial z} \end{pmatrix} \\
&= \left(-E^T \begin{pmatrix} H & -K \end{pmatrix} + E^N \begin{pmatrix} L & D \\ -{}^tD & J \end{pmatrix}\right) \begin{pmatrix} Y \\ Z \end{pmatrix} \\
&\qquad\qquad + E^N \begin{pmatrix} -(L + DK^{-1}H)Y - DK^{-1}\Lambda \\ ({}^tD - JK^{-1}H)Y - JK^{-1}\Lambda \end{pmatrix} \\
&= -E^T \begin{pmatrix} H & -K \end{pmatrix} \begin{pmatrix} Y \\ Z \end{pmatrix} \\
&\qquad + E^N \begin{pmatrix} LY + DK^{-1}\Lambda + DK^{-1}HY \\ -{}^tDY + JK^{-1}\Lambda + JK^{-1}HY \end{pmatrix} \\
&\qquad + E^N \begin{pmatrix} -(L + DK^{-1}H)Y - DK^{-1}\Lambda \\ ({}^tD - JK^{-1}H)Y - JK^{-1}\Lambda \end{pmatrix} \\
&= -E^T(HY - KZ) \\
&= -E^T(HY - K(K^{-1}HY + K^{-1}\Lambda)) \\
&= E^T\Lambda \\
&= \frac{\lambda}{2}\left(E_1 - \sqrt{-1}E_2\right).
\end{aligned} \tag{3.32}$$

Hence Ψ is conformal and by (3.32) we have

$$\Psi \cdot \frac{\partial \Psi}{\partial z} = 0$$

on U_0. This implies that $< \Psi, \Psi >^{\frac{1}{2}}$ is constant on U_0. Since we can take Ψ such that $< \Psi(m_0), \Psi(m_0) >^{\frac{1}{2}} = 1$ at m_0, $< \Psi, \Psi >^{\frac{1}{2}} = 1$ holds on U_0. Then the surface $S = (U_0, S^n, \Psi)$ has the Gauss map $G|_{U_0}$. This completes the proof.

Let M be the Gaussian plane C. Assume that G satisfies the conditions (1), (2) in Theorem 3.1 about any point of C. In the case where $k = n+1$, by using Theorem 3.1, Lemma 2.2 and the monodromy theorem, we can show that there exists a surface $S = (C, S^n, X)$ with the Gauss map G. Next let $3 \leq k \leq n$. It follows from Theorem 3.1 and Lemma 2.1 that for each point $m_0 \in M$ there exist an open neighborhood U_0 of m_0 and a surface $S = (U_0, V \cap S^n, X)$ with the Gauss map $G|_{U_0}$. Then we can apply the argument above to the $(k-1)$-dimensional unit sphere $V \cap S^n$. Hence we have the following.

PROPOSITION 3.2. *Let $G : C \longrightarrow Q_{n-1}$ be a C^∞-mapping such that $k \geq 3$. Assume that a local expression of G about any point of C satisfies the conditions* (1), (2) *in Theorem* 3.1. *Then there exists a surface $S = (C, S^n, X)$ with the Gauss map G.*

From now on, we denote by N and Z the set of all natural numbers and integers respectively. For a_1, $a_2 > 0$, let φ_1 and φ_2 be translations on C defined by

$$\varphi_1(z) = u_1 + a_1 + \sqrt{-1}u_2, \quad \varphi_2(z) = u_1 + \sqrt{-1}(u_2 + a_2)\ (z = u_1 + \sqrt{-1}u_2 \in C).$$

We denote by $\Gamma(a_1, a_2)$ the transformation group on C generated by φ_1 and φ_2. For k_1, $k_2 \in N$, let $\Gamma(k_1a_1, k_2a_2)$ be the transformation group on C generated by translations ${\varphi_1}^{k_1}$ and ${\varphi_2}^{k_2}$ where

$${\varphi_1}^{k_1}(z) = u_1 + k_1a_1 + \sqrt{-1}u_2, \quad {\varphi_2}^{k_2}(z) = u_1 + \sqrt{-1}(u_2 + k_2a_2).$$

We denote by $T^2(k_1a_1, k_2a_2)$ the torus $C/\Gamma(k_1a_1, k_2a_2)$. For the natural projection $\pi_{k_1k_2} : C \longrightarrow T^2(k_1a_1, k_2a_2)$, we put $[z] = \pi_{k_1k_2}(z)$ $(z \in C)$.

Under the notations above, we have the following.

THEOREM 3.2. *Let $G : T^2(a_1, a_2) \longrightarrow Q_{n-1}$ be a C^∞-mapping such that $k \geq 3$. Assume that a local expression of G about any point of $T^2(a_1, a_2)$ satisfies the following conditions:*

(1) *The tangent component E_{ij} and the normal component F_{ij} of $\frac{\partial E_i}{\partial u_j}$ in R^{n+1}*

satisfy the condition (3.3);
(2) P *and* Q *defined by* (3.7) *and* (3.8) *satisfy the conditions* (3.11) *and* (3.12). *Then there exist a covering space* $(\hat{T}^2, T^2(a_1, a_2), \hat{\pi})$ *over* $T^2(a_1, a_2)$ *and a surface* $S = (\hat{T}^2, S^n, X)$ *such that the Gauss map is* $G \circ \hat{\pi}$.

The proof of Theorem 3.2 is similar to the one of Theorem 5.2 in [3]. For completeness of this paper we give it.

Proof. For k_1, $k_2 \in N$, we define C^∞-mappings

$$\tilde{\pi}_{k_1 k_2} : T^2(k_1 a_1, k_2 a_2) \longrightarrow T^2(a_1, a_2), \quad G_{k_1 k_2} : T^2(k_1 a_1, k_2 a_2) \longrightarrow Q_{n-1}$$

as

$$\tilde{\pi}_{k_1 k_2}([z]) = \pi_{11}(z) \quad (z \in C), \quad G_{k_1 k_2} = G \circ \tilde{\pi}_{k_1 k_2}$$

respectively. Let $\tilde{G} : C \longrightarrow Q_{n-1}$ be the C^∞-mapping defined by $G \circ \pi_{11}$. This mapping is $\Gamma(a_1, a_2)$-invariant. $\tilde{G}$ satisfies the conditions in Proposition 3.2, since $\tilde{\pi}_{k_1 k_2}$ is holomorphic. Then there exists a surface $\tilde{S}_1 = (C, S^n, \tilde{X}_1)$ such that the Gauss map is $\tilde{G}$. We define conformal immersions $\tilde{X}_j : C \longrightarrow S^n$ $(j = 2, 3)$ by $\tilde{X}_2 = \tilde{X}_1 \circ \varphi_1$ and $\tilde{X}_3 = \tilde{X}_1 \circ \varphi_2$. Then the surfaces $\tilde{S}_j = (C, S^n, \tilde{X}_j)$ have the same Gauss map $\tilde{G}$.

We first consider the case where $k = n + 1$. Since $\tilde{S}_1$, $\tilde{S}_2$ and $\tilde{S}_3$ have the same Gauss map $\tilde{G}$, by Lemma 2.2, we have

$$\tilde{X}_1 = \pm \tilde{X}_1 \circ \varphi_1 = \pm \tilde{X}_1 \circ \varphi_2.$$

Then the following four cases are possible:
(1) $\tilde{X}_1 = \tilde{X}_1 \circ \varphi_1 = \tilde{X}_1 \circ \varphi_2$, (2) $\tilde{X}_1 = -\tilde{X}_1 \circ \varphi_1 = \tilde{X}_1 \circ \varphi_2$, (3) $\tilde{X}_1 = \tilde{X}_1 \circ \varphi_1 = -\tilde{X}_1 \circ \varphi_2$, (4) $\tilde{X}_1 = -\tilde{X}_1 \circ \varphi_1 = -\tilde{X}_1 \circ \varphi_2$.
For each case we can show that the claim of the theorem holds ([3]).

Next we consider the case where $3 \leq k \leq n$. By Lemma 2.1 there exist surfaces $\tilde{S}_j{}' = (C, S^n, \tilde{X}_j{}')$ $(j = 2, 3)$. Then we can apply the argument above for $k = n + 1$ to the $(k-1)$-dimensional unit sphere $V \cap S^n$. We complete the proof of Theorem 3.2.

We denote by π the canonical projection from S^n to the n-dimensional real projective space RP^n.

THEOREM 3.3. *Let* M *be a connected Riemann surface and* $G : M \longrightarrow Q_{n-1}$ *a* C^∞*-mapping such that* $k \geq 3$. *Assume that a local expression of* G *about any point of* M *satisfies the conditions* (1), (2) *in Theorem* 3.1. *Then there exists a surface* $S = (M, RP^n, X)$ *with the property that a neighborhood of each point of* $X(M)$ *is covered by a surface in* S^n *with the Gauss map* G.

Proof. We need to consider two cases: $k = n+1$ and $3 \le k \le n$. Let $k = n+1$. By hypothesis a local existence of a surface in S^n with the Gauss map G follows from Theorem 3.1. Let (U_0, z) be a local coordinate system on M such that U_0 is connected. From Lemma 2.2, if there exist surfaces $\hat{S}_X = (U_0, S^n, \hat{X})$ and $\hat{S}_Y = (U_0, S^n, \hat{Y})$ such that $\hat{S}_X$ and $\hat{S}_Y$ have the same Gauss map, then $\hat{Y} = \pm\hat{X}$. We define C^∞-mappings $X : U_0 \longrightarrow RP^n$ and $Y : U_0 \longrightarrow RP^n$ as $X = \hat{X} \circ \pi$ and $Y = \hat{Y} \circ \pi$ respectively. Then surfaces $S_X = (U_0, RP^n, X)$ and $S_Y = (U_0, RP^n, Y)$ have the property that neighborhoods of each point of $X(U_0)$ and $Y(U_0)$ are covered by surfaces in S^n with the same Gauss map. Since $\hat{Y} = \pm\hat{X}$, we have $X = Y$. Let $\hat{S}_1 = (U_1, S^n, \hat{X}_1)$ and $\hat{S}_2 = (U_2, S^n, \hat{X}_2)$ be surfaces with the same Gauss map such that $W = U_1 \cap U_2 \neq \emptyset$. We define surfaces $S_1 = (U_1, RP^n, X_1)$ and $S_2 = (U_2, RP^n, X_2)$ where $X_1 = \hat{X}_1 \circ \pi$ and $X_2 = \hat{X}_2 \circ \pi$. By the argument above we have $X_1 = X_2$ on W and hence there exists a surface $S_3 = (U_3, RP^n, X_3)$ with the property stated above where $U_3 = U_1 \cup U_2$, $X_3|_{U_1} = X_1$ and $X_3|_{U_2} = X_2$. By using the same argument as in the proof of Theorem 6.1 in [3], we can show the existence of a surface $S = (M, RP^n, X)$ with the property stated above. In case of $3 \le k \le n$, by Lemma 2.1 the proof can be reduced to the case $k = n+1$. Hence Theorem 3.3 holds.

Next we consider the another condition for G. Let $((U, z), E^T)$ be a local expression of G. Let us consider the following conditions:

$$E_{11} = -E_{12},\ E_{21} = -E_{22},\ F_{11} \wedge F_{22} \neq 0,\ F_{21} = F_{12} = 0. \tag{3.33}$$

Let T_1 and T_2 denote 2×2 matrices in the form

$$T_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

We set

$$E_{ij} = E^T P_{ij}, \quad F_{ij} = E^N Q_{ij} \quad (i, j = 1, 2) \tag{3.34}$$

where $P_{ij}(u) \in \mathscr{M}(2,1)$ and $Q_{ij}(u) \in \mathscr{M}(n-1,1)$. By using (3.34), we can rewrite (3.33) as

$$P_{11} = -P_{12},\ P_{21} = -P_{22},\ Q_{11} \wedge Q_{22} \neq 0,\ Q_{21} = Q_{12} = 0. \tag{3.35}$$

We put

$$P_1 = P_{11}, \quad P_2 = P_{21}, \quad Q_1 = Q_{11}, \quad Q_2 = Q_{22}.$$

Let us define C^{∞}-mappings

$$P : U \longrightarrow \mathscr{A}(2), \quad Q : U \longrightarrow \mathscr{M}(2, n-1) \tag{3.36}$$

by

$$P(u) = \frac{1}{2}\left(\ P_1(u) + \sqrt{-1}P_1(u) \quad P_2(u) + \sqrt{-1}P_2(u) \ \right), \tag{3.37}$$

$$Q(u) = \frac{1}{2}T_1 \begin{pmatrix} {}^tQ_1(u) \\ {}^tQ_2(u) \end{pmatrix} - \frac{\sqrt{-1}}{2}T_2 \begin{pmatrix} {}^tQ_1(u) \\ {}^tQ_2(u) \end{pmatrix}. \tag{3.38}$$

We have

$$\frac{\partial E}{\partial z} = EA, \quad A = \left(\begin{array}{c|c} P & -Q \\ \hline {}^tQ & B \end{array}\right) \in \mathscr{A}(n+1), \quad B(u) \in \mathscr{A}(n-1).$$

If a local expression $((U, z = u_1 + \sqrt{-1}u_2), E^T)$ of G about any point of M satisfies the following conditions:
(1) The tangent component E_{ij} and the normal component F_{ij} of $\frac{\partial E_i}{\partial u_j}$ in R^{n+1} satisfy the condition (3.33);
(2) P and Q defined by (3.37) and (3.38) satisfy the conditions

$$\frac{\partial P}{\partial z} = (T_2 Q\,{}^tQ T_1 - T_1 Q\,{}^tQ T_2), \ \frac{\partial P}{\partial \bar{z}} = \sqrt{-1}(T_2 Q\,{}^tQ T_1 - T_1 Q\,{}^tQ T_2), \tag{3.39}$$

$$\frac{\partial Q}{\partial z} = PQ + QB, \qquad \frac{\partial Q}{\partial \bar{z}} = -\bar{P}Q + Q\bar{B}, \tag{3.40}$$

then we obtain the same results as Theorems 3.2 and 3.3.

We note that the conditions (3.39) and (3.40) are independent of the choice of orthonormal frame fields E^N on U which are orthogonal to E^T in R^{n+1}.

4. Conformal immersions of a torus into RP^n

Let M be a connected Riemann surface. In this section we use same notations as in Section 3. Let $G : M \longrightarrow Q_{n-1}$ be a C^{∞}-mapping satisfying the conditions in Theorem 3.3. Then we have a surface $S = (M, RP^n, X)$ with the property that a neighborhood of each point of $X(M)$ is covered by a surface in S^n with the Gauss map G. In the following we also regard M as a Riemannian manifold with the metric induced by X. We take a complex coordinate system $(U, z = u_1 + \sqrt{-1}u_2)$ of M such that $X : U \longrightarrow RP^n$ is an embedding and $X(U)$ is covered by a surface in S^n with the Gauss map G. By taking U sufficiently small, there exists a conformal embedding $\hat{X} : U \longrightarrow S^n$ such that $X = \pi \circ \hat{X}$ and $X(U)$ is isometric to $\hat{X}(U)$. Moreover, by taking U sufficiently small if necessary, we may assume that the assumption in Theorem 3.3 holds on U. Then we have

$$\frac{\partial \hat{X}}{\partial z} = \frac{\lambda}{2}(E_1 - \sqrt{-1}E_2) \tag{4.1}$$

where λ is a positive function in U. We define an orthonormal frame field $\hat{E} = (\hat{E}_1, \cdots, \hat{E}_{n+1})$ along $\hat{X}(U)$ by $\hat{E}_i = (\hat{X}, E_i)$ $(1 \leq i \leq n+1)$ where $\hat{X}$ denotes the base point of $\hat{E}_i$. Then we have

$$\frac{\partial E}{\partial z} = EA$$

where $E = (E_1, E_2, \cdots, E_{n+1})$. We define C^∞-mappings

$$A_i = (a^\alpha_{i\beta}) : U \longrightarrow \mathscr{A}(n+1) \quad (i = 1, 2) \tag{4.2}$$

as

$$A_1 = A + \bar{A}, \quad A_2 = \sqrt{-1}(A - \bar{A}).$$

Let $\bar{\nabla}$ be the standard connection of R^{n+1} and ∇ the connection on U induced by the embedding $\hat{X}$. Then from (3.9) and (4.1) we have on U

$$\bar{\nabla}_{\hat{E}_i}\hat{E}_j = \nabla_{\hat{E}_i}\hat{E}_j + \sum_{\alpha=3}^{n+1} h^\alpha(\hat{E}_i, \hat{E}_j)\hat{E}_\alpha \quad (i, j = 1, 2)$$

where

$$h^\alpha(\hat{E}_i, \hat{E}_j) = -\frac{1}{\lambda}a^j_{i\alpha}.$$

Here $a^j_{i\alpha}$ is the (j, α) component of A_i in (4.2). Let $K(M)$ be the Gauss curvature of M. By (3.3) we have

$$\begin{aligned} K(M) &= \sum_{\alpha=3}^{n+1} \det \begin{pmatrix} h^\alpha(\hat{E}_1, \hat{E}_1) & h^\alpha(\hat{E}_1, \hat{E}_2) \\ h^\alpha(\hat{E}_2, \hat{E}_1) & h^\alpha(\hat{E}_2, \hat{E}_2) \end{pmatrix} \\ &= \frac{1}{\lambda^2}\sum_{\alpha=3}^{n+1} \det \begin{pmatrix} a^1_{1\alpha} & a^2_{1\alpha} \\ a^1_{2\alpha} & a^2_{2\alpha} \end{pmatrix} \\ &= \frac{1}{\lambda^2}\sum_{\alpha=3}^{n+1}((a^1_{1\alpha})^2 - (a^2_{1\alpha})^2). \end{aligned} \tag{4.3}$$

Under the notations above, we have the following.

THEOREM 4.1. *Let M be a compact, connected Riemann surface. Let $G : M \longrightarrow Q_{n-1}$ be a C^∞-mapping such that $k \geq 3$. Assume that a local expression $((U, z = u_1 + \sqrt{-1}u_2), E^T)$ of G about any point of M satisfies the following conditions:*
(1) The tangent component E_{ij} and the normal component F_{ij} of $\frac{\partial E_i}{\partial u_j}$ in R^{n+1} satisfy the condition (3.3);
(2) P and Q defined by (3.7) and (3.8) satisfy the conditions

$$TQ^tQ = Q^tQT, \tag{4.4}$$

$$\frac{\partial Q}{\partial z} = PQ + QB, \quad \frac{\partial Q}{\partial \bar{z}} = -\bar{P}Q + Q\bar{B}. \tag{4.5}$$

Then M is a torus.

Proof. It follows from Theorem 3.3 that there exists a surface with the property stated above. Let $((U, z = u_1 + \sqrt{-1}u_2), E^T)$ be a local expression of G. Since the conditions (4.4) and (4.5) are independent of the choice of orthonormal frame fields E^N on U which are orthogonal to E^T in R^{n+1}, we take an orthonormal frame field $F = (F_1, \cdots, F_{n+1})$ in R^{n+1} so that

$$F_1 = E_1, \quad F_2 = E_2, \quad F_{n+1} = \hat{X}$$

where $X = \pi \circ \hat{X}$. Then from (4.4) and (4.5) we have

$$\frac{\partial F}{\partial z} = F\hat{A}$$

where

$$\hat{A} = (\hat{a}^j_\alpha) = \left(\begin{array}{c|c} P & -\hat{Q} \\ \hline {}^t\hat{Q} & \hat{B} \end{array}\right).$$

The components of $\hat{A}$ satisfy

$$T\hat{Q}^t\hat{Q} = \hat{Q}^t\hat{Q}T, \tag{4.6}$$

$$\frac{\partial \hat{Q}}{\partial z} = P\hat{Q} + \hat{Q}\hat{B}, \quad \frac{\partial \hat{Q}}{\partial \bar{z}} = -\bar{P}\hat{Q} + \hat{Q}\bar{\hat{B}}. \tag{4.7}$$

Since $\hat{X}$ satisfies (4.1), we get

$$\hat{a}^1_{n+1} = \frac{1}{2}\lambda, \quad \hat{a}^2_{n+1} = -\frac{\sqrt{-1}}{2}\lambda.$$

These are equivalent to

$$\hat{a}^2_{1\,n+1} = \hat{a}^1_{2\,n+1} = 0, \quad \hat{a}^1_{1\,n+1} = \hat{a}^2_{2\,n+1} = \lambda.$$

By using this and (4.3), we have

$$K(M) = \frac{1}{\lambda^2}\Big(\sum_{\alpha=3}^{n}((\hat{a}^1_{1\,\alpha})^2 - (\hat{a}^2_{1\,\alpha})^2) + \lambda^2\Big) = 1 + \frac{1}{\lambda^2}\sum_{\alpha=3}^{n}((\hat{a}^1_{1\,\alpha})^2 - (\hat{a}^2_{1\,\alpha})^2). \tag{4.8}$$

It follows from (4.6) that

$$0 = T\hat{Q}^t\hat{Q} - \hat{Q}^t\hat{Q}T = \sum_{\alpha=3}^{n+1}((\hat{a}^1_{1\,\alpha})^2 - (\hat{a}^2_{1\,\alpha})^2) = \sum_{\alpha=3}^{n}((\hat{a}^1_{1\,\alpha})^2 - (\hat{a}^2_{1\,\alpha})^2) + \lambda^2. \tag{4.9}$$

From (4.8) and (4.9) we obtain $K(M) = 0$. Hence, by Gauss-Bonnet Theorem, the Euler characteristic of M equals zero. Thus M is a torus. We complete the proof.

5. Examples

We will give examples of C^∞-mappings $G : M \longrightarrow Q_{n-1}$ with the conditions in Theorems 3.3. Let Γ be the transformation group on C generated by translations φ_1 and φ_2 such that

$$\varphi_1(z) = u_1 + 2\pi + \sqrt{-1}u_2,\ \varphi_2(z) = u_1 + \sqrt{-1}\,(u_2 + 2\pi)\,(z = u_1 + \sqrt{-1}u_2 \in C).$$

We define a torus T^2 as C/Γ.

Example 5.1. Let $G : T^2 \longrightarrow Q_2$ be a C^∞-mapping such that

$$G(z) = \left[\begin{pmatrix} -\sin u_1 \cos u_2 \\ -\sin u_1 \sin u_2 \\ \cos u_1 \cos u_2 \\ \cos u_1 \sin u_2 \end{pmatrix} + \sqrt{-1}\begin{pmatrix} -\cos u_1 \sin u_2 \\ \cos u_1 \cos u_2 \\ -\sin u_1 \sin u_2 \\ \sin u_1 \cos u_2 \end{pmatrix}\right].$$

This G satisfies the conditions (3.3), (3.11) and (3.12).

Example 5.2. We define a C^∞-mapping $G : T^2 \longrightarrow Q_2$ as

$$G(z) = [{}^t(-\sin u_1,\ \cos u_1,\ -\sqrt{-1}\sin u_2,\ \sqrt{-1}\cos u_2)].$$

Then G satisfies the conditions (3.33), (3.39) and (3.40).

By using Example 5.1 we can give another examples of G with the conditions (3.3), (3.11) and (3.12). We shall show it. Let M be a connected Riemann surface and $G : M \longrightarrow Q_{n-1}$ a C^∞-mapping. Assume that a local expression $((U, z = u_1 + \sqrt{-1}u_2), E^T)$ of G about any point of M satisfies the condition (3.3) and such that C^∞-mappings P and Q defined by (3.7) and (3.8) satisfy the conditions (3.11) and (3.12). Let $E = (E^T, E^N) = (E_1, E_2, E_3, \cdots, E_{n+1})$ be an orthonormal frame field in R^{n+1}. We define a C^∞-mapping $\tilde{G} : M \longrightarrow Q_{2n}$ such that for each $u \in U$ $F^T(u) := (F_1(u), F_2(u))$ is an orthonormal frame in $\tilde{\tilde{G}}(u)$ where

$$F_i : U \longrightarrow R^{2(n+1)} \setminus \{0\} \quad (i = 1, 2)$$

are C^∞-mappings defined as

$$F_i(u) = \frac{1}{\sqrt{2}} \begin{pmatrix} E_i(u) \\ E_i(u) \end{pmatrix} \quad (u \in U).$$

Let $F := (F_1, F_2, \cdots, F_{2(n+1)})$ be an orthonormal frame field on U where

$$F = \frac{1}{\sqrt{2}} \begin{pmatrix} E^T & E & E^N \\ E^T & -E & E^N \end{pmatrix}.$$

Then we have

$$\frac{\partial F}{\partial z} = F\tilde{A}$$

where $\tilde{A}(u) \in \mathscr{A}\big(2(n+1)\big)$. $\tilde{A}$ can be expressed in the form

$$\tilde{A} = \left(\begin{array}{c|c|c} P & 0 & -Q \\ \hline 0 & \hat{B} & 0 \\ \hline {}^tQ & 0 & B \end{array}\right)$$

where $\hat{B}(u) \in \mathscr{A}(n+1)$. We set

$$\tilde{P} = P, \quad \tilde{Q} = \begin{pmatrix} 0 & Q \end{pmatrix}, \quad \tilde{B} = \left(\begin{array}{c|c} \hat{B} & 0 \\ \hline 0 & B \end{array}\right).$$

Since (3.11) and (3.12) hold, we have

$$\frac{\partial \tilde{P}}{\partial z} = \frac{\partial P}{\partial z} = \frac{1}{2}(T\tilde{Q}^t\tilde{Q} - \tilde{Q}^t\tilde{Q}T),$$
$$\frac{\partial \tilde{P}}{\partial z} = \frac{\partial P}{\partial \bar{z}} = \frac{\sqrt{-1}}{2}(T\tilde{Q}^t\tilde{Q} - \tilde{Q}^t\tilde{Q}T),$$
$$\frac{\partial \tilde{Q}}{\partial z} = \begin{pmatrix} 0 & \frac{\partial Q}{\partial z} \end{pmatrix} = (0 \quad PQ + QB) = \tilde{P}\tilde{Q} + \tilde{Q}\tilde{B},$$
$$\frac{\partial \tilde{Q}}{\partial \bar{z}} = \begin{pmatrix} 0 & \frac{\partial Q}{\partial \bar{z}} \end{pmatrix} = -\bar{\tilde{P}}\tilde{Q} + \tilde{Q}\bar{\tilde{B}}.$$

Hence we have a C^∞-mapping $\tilde{G} : M \longrightarrow Q_{2n}$ with the same conditions as G.

By induction, we have the following.

PROPOSITION 5.1. *Let M be a connected Riemann surface. If a C^∞-mapping $G : M \longrightarrow Q_{n-1}$ satisfies the conditions in Theorem 3.3, there exists a C^∞-mapping*

$$\tilde{G} : M \longrightarrow Q_{l-1}, \quad l = m(n+1) - 1, \quad m \geq 2\,(m \in N)$$

with the same conditions as G.

For the conditions (3.33), (3.39) and (3.40) of G, we have the same results as Proposition 5.1.

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References

[1] D.A. Hoffman & R. Osserman, The geometry of the generalized Gauss map, *Memoirs of American Mathematical Society*, **236** (1980).

[2] D. A. Hoffman & R. Osserman, The Gauss map of surfaces in R^n, *J. Differential Geometry*, **18**(1983), 733–754.

[3] A. Tanaka, Surfaces in S^n with prescribed Gauss map, *to appear in Tokyo Journal of Mathematics.*

[4] F.W. Warner, *Foundations of Differentiable Manifolds and Lie Groups*, Scott, Foresman, Chicago, 1971.

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