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NONLINEAR STRONG ERGODIC THEOREMS FOR ASYMPTOTICALLY NONEXPANSIVE SEMIGROUPS WITH COMPACT DOMAINS

By

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Abstract. In this paper, we prove a nonlinear ergodic theorem for a commutative semigroup of asymptotically nonexpansive mappings from a compact convex subset of a strictly convex Banach space into itself. Using this result, we obtain two nonlinear ergodic theorems proved by Yoshimoto [15].

1. Introduction

Let C be a compact convex subset of a Banach space E and let S be a commutative semigroup with identity. A family $\mathcal{S} = \{T(s) : s \in S\}$ of mappings from C into itself is said to be an asymptotically nonexpansive semigroup on C with Lipschitz constants $\{k(s) : s \in S\}$ if the following are satisfied:

(i) $\|T(s)x - T(s)y\| \leq k(s)\|x - y\|$ for all $x, y \in C$;
(ii) $T(t+s)x = T(t)T(s)x$ for all $t, s \in S$ and $x \in C$;
(iii) $k(s) \geq 1$ for all $s \in S$ and $\limsup_s k(s) = 1$.

Such a semigroup $\mathcal{S}$ is called a one-parameter asymptotically nonexpansive semigroup if $S = \mathbb{R}^+$ and moreover, $\mathcal{S}$ satisfies the following:

(iv) $T(0)x = x$ for all $x \in C$;
(v) for each $x \in C$, the mapping $t \mapsto T(t)x$ is continuous;
(vi) $t \mapsto k(t)$:$[0,\infty) \to [0,\infty)$ is continuous.

An asymptotically nonexpansive semigroup $\mathcal{S}$ is called a nonexpansive semigroup on C if $k(s) = 1$ for all $s \in S$. In 2000, Atsushiba, Lau and Takahashi [2] proved a strong ergodic theorem for a commutative semigroup of nonexpansive mappings from a compact convex subset of a strictly convex Banach space into itself. On the other hand, recently Yoshimoto [15] obtained two strong ergodic theorems for a one-parameter asymptotically nonexpansive semigroup on a compact convex

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subset of a strictly convex Banach space.

In this paper, motivated by Atsushiba, Lau and Takahashi [2] and Yoshimoto [15], we prove a nonlinear strong ergodic theorem for an asymptotically nonexpansive semigroup with compact domain in a strictly convex Banach space. This theorem is used to prove the results obtained by Yoshimoto [15].

2. Preliminaries

Throughout this paper, we assume that a Banach space E is real and S is a commutative semigroup with identity. Then, $(S, \leq)$ is a directed system when the binary relation $\leq$ is defined by $a \leq b$ if and only if there is $c \in S$ such that $a + c = b$. Let E^* be the dual space of E. Then, the duality mapping J on E is defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. By the Hahn-Banach theorem, $J(x)$ is nonempty; see [14] for more details. We denote by $\mathbb{N}$ and $\mathbb{R}^+$ the sets of all positive integers and nonnegative real numbers, respectively. We also denote by $B(S)$ the Banach space of all bounded real valued functions on S with supremum norm. For each $s \in S$ and $f \in B(S)$, we define an element $r_s f$ of $B(S)$ by $(r_s f)(t) = f(t+s)$ for all $t \in S$. Let D be a subspace of $B(S)$ containing constants and let X^* be its dual. Then, an element μ of D^* is said to be a mean on X if $\|\mu\| = \mu(1) = 1$. We sometimes use $\mu_t(f(t))$ instead of $\mu(f)$ or $\int f d\mu$ for $\mu \in D^*$ and $f \in D$. Let D be r_s−invariant, i.e., $r_s(D) \subset D$ for each $s \in S$. Then, a mean μ on D is said to be invariant if $\mu(r_s f) = \mu(f)$ for all $f \in D$ and $s \in S$.

The following definition which was introduced by Takahashi [13] is crucial in the nonlinear ergodic theory for abstract semigroups: see [6, 7, 8, 13, 14] for more details. Let f be a function of S into E such that the weak closure of $\{f(t) : t \in S\}$ is weakly compact. Let D be a subspace of $B(S)$ containing constants which is r_s−invariant for every $s \in S$. Assume that for each $x^* \in E^*$, the function $t \mapsto \langle f(t), x^* \rangle$ is in D. Then, for any $\mu \in D^*$ there exists a unique element f_μ such that

$$\langle f_\mu, x^* \rangle = \int \langle f(t), x^* \rangle d\mu(t)$$

for all $x^* \in E^*$. If μ is a mean on D, then f_μ is contained in $\overline{\text{co}}\{f(t) : t \in S\}$.

A Banach space E is said to be strictly convex if $\frac{\|x+y\|}{2} < 1$ for $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. Let C be a subset of E, let T be a mapping from C into itself and let $\varepsilon > 0$. We denote by $F_\varepsilon(T)$ the set $\{x \in C : \|x - Tx\| \leq \varepsilon\}$. Let $K > 0$. We denote by $Lip(C, K)$ the set of all mappings from C into itself satisfying $\|Tx-Ty\| \leq K\|x-y\|$. We denote by Γ the set of all strictly increasing, continuous convex functions $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ with $\gamma(0) = 0$. We denote by $\overline{\mathrm{co}}C$ is the closure of the convex hull of C. We denote by $F(\mathcal{S})$ the set of all common fixed points of $\mathcal{S}$, i.e., $\bigcap_{s\in S}\{x \in C : T(s)x = x\}$.

3. Lemmas

In this section, we obtain some lemmas which are used to prove our main theorem(Threorem 4.1). The following lemma was obtained by Bruck [4].

LEMMA 3.1. *Let C be a nonempty compact convex subset of a strictly convex Banach space E. Then, for each $n \in \mathbb{N}$, there exists $\gamma_n \in \Gamma$ such that for each $K > 0$ and $T \in Lip(C, K)$,*

$$\left\|T\left(\sum_{i=1}^{n} \lambda_i x_i\right) - \sum_{i=1}^{n} \lambda_i T x_i\right\| \leq K\gamma_n{}^{-1}\left(\max_{1\leq i,j\leq n}\left(\|x-y\| - \frac{1}{K}\|Tx_i - Tx_j\|\right)\right)$$

holds for every $\{x_i\}_{i=1}^n$ in C, $\{\lambda_i\}$ in $\mathbb{R}^+$ with $\sum_{i=1}^n \lambda_i = 1$.

Following ideas in Atsushiba, Lau and Takahashi [2] and Nakajo and Takahashi [12], we can prove the following lemma.

LEMMA 3.2. *Let C be a nonempty compact convex subset of a strictly convex Banach space E and let $\mathcal{S} = \{T(t) : t \in S\}$ be an asymptotically nonexpansive semigroup on C with Lipschitz constants $\{k(t) : t \in S\}$. Let $x \in C$. Then, for any finite mean μ on S and $\varepsilon > 0$, there exist $w_0 = w_0(\mu, \varepsilon)$ and $h_0 = h_0(\mu, \varepsilon)$ such that*

$$\left\|\int T(h+s+w)xd\mu(s) - T(h)\int T(s+w)xd\mu(s)\right\| < \varepsilon$$

for any $h \geq h_0$ and $w \geq w_0$.

Proof. Let μ be a finite mean on S and suppose

$$\mu = \sum_{i=1}^{n} a_i\delta_{s_i} \quad (a_i \geq 0, \sum_{i=1}^{n} a_i = 1).$$

From Lemma 3.1, there exists $\gamma_n \in \Gamma$ such that

$$\begin{aligned}&\left\|\int T(h+s+w)xd\mu(s) - T(h)\int T(s+w)xd\mu(s)\right\| \\ &= \left\|\sum_{i=1}^{n} a_i T(h+s_i+w)x - T(h)\sum_{i=1}^{n} a_i T(s_i+w)x\right\|\end{aligned}$$

$$\leq k(h)\gamma_n^{-1}\Big(\max_{1\leq i,j\leq n}\Big\{\|T(w+s_i)x-T(w+s_j)x\| - \frac{1}{k(h)}\|T(h+w+s_i)x-T(h+w+s_j)x\|\Big\}\Big)$$

for all $h, w \in S$. Since $\gamma_n \in \Gamma$ and $\limsup_h k(h) = 1$, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $k(h)\gamma_n^{-1}(\delta) < \varepsilon$ for all $h \in S$. For i, j with $1 \leq i, j \leq n$, we put $r_{i,j} = \inf_{w\in S} \|T(w+s_i)x - T(w+s_j)x\|$. Then, there exists $w_1 \in S$ such that

$$r_{i,j} \leq \|T(w_1+s_i)x - T(w_1+s_j)x\| \leq r_{i,j} + \frac{\delta}{4}.$$

Moreover, from $\limsup_h k(h) = 1$, there exists $h_1 \in S$ such that

$$k(h) \leq \frac{r_{i,j} + \frac{\delta}{2}}{\|T(w_1+s_i)x - T(w_1+s_j)x\|}$$

for every $h \geq h_1$. Therefore, for any $h \geq h_1$, we have

$$\begin{aligned}&\|T(h+w_1+s_i)x - T(h+w_1+s_j)x\| \\ &\leq k(h)\|T(w_1+s_i)x - T(w_1+s_j)x\| < r_{i,j} + \frac{\delta}{2}.\end{aligned}$$

Put $w_2 = h_1 + w_1$. Then, we obtain

$$r_{i,j} \leq \|T(w+s_i)x - T(w+s_j)x\| \leq r_{i,j} + \frac{\delta}{2}$$

for each $w \geq w_2$. Similarly, there exists $h_2 \in S$ such that

$$\frac{1}{k(h)}\|T(h+s+w_i)x - T(h+s+w_j)x\| \geq r_{i,j} - \frac{\delta}{2}$$

for every $h \geq h_2$. Therefore, for $w \geq w_1$ and $h \geq h_2$, we obtain

$$0 \leq \|T(w+s_i)x - T(w+s_j)x\| - \frac{1}{k(h)}\|T(h+w+s_i)x - T(h+w+s_j)x\| \leq \delta.$$

Put $w_* = \sum_{i,j} w_2(i,j)$ and $h_* = \sum_{i,j} h_2(i,j)$. Then, for any $w \geq w_*$ and $h \geq h_*$, we have

$$\max_{1\leq i,j\leq n}\Big\{\|T(w+s_i)x - T(w+s_j)x\| - \frac{1}{k(h)}\|T(h+w+s_i)x - T(h+w+s_j)x\|\Big\} \leq \delta.$$

This completes the proof. □

The following two lemmas were proved by Nakajo and Takahashi [12].

LEMMA 3.3. *Let C be a nonempty compact convex subset of a strictly convex Banach space E. For any $\varepsilon > 0$, there exists $\delta > 0$ such that $\overline{\text{co}}F_\delta(T) \subset F_\varepsilon(T)$ holds for any $T \in Lip(C, 1+\delta)$.*

LEMMA 3.4. *Let C be a nonempty compact convex subset of a strictly convex Banach space E. Then, for any $\varepsilon > 0$, there exist $\delta > 0$ and $N_0 \in \mathbb{N}$ such that for any $l \in \mathbb{N}$ and any mapping T from C into itself satisfying $T^l \in Lip(C, 1+\delta)$, there holds*

$$\left\| \frac{1}{m} \sum_{i=0}^{m-1} T^i x - T^l \left(\frac{1}{m} \sum_{i=0}^{m-1} T^i x \right) \right\| \leq \varepsilon$$

for all $m \in \mathbb{N}$ with $m - 1 \geq N_0$ and $x \in C$.

As in the proof of [2], we obtain the following lemma.

LEMMA 3.5. *Let C be a nonempty compact convex subset of a strictly convex Banach space E, let $\mathcal{S} = \{T(t) : t \in S\}$ be an asymptotically nonexpansive semigroup on C with Lipschitz constants $\{k(t) : t \in S\}$ and let $x \in C$. Let $\{\mu_\alpha : \alpha \in I\}$ be a net of finite means on S such that*

$$\lim_\alpha \|\mu_\alpha - r_t{}^* \mu_\alpha\| = 0$$

for every $t \in S$. Then, for any $\varepsilon > 0$ and $p \in S$, there exists $\alpha_0 \in I$ and $t_0 \in S$ such that

$$\left\| \int T(s+p)x d\mu_\alpha(s) - T(t) \int T(s+p)x d\mu_\alpha(s) \right\| < \varepsilon$$

for all $\alpha \geq \alpha_0$ and $t \geq t_0$.

Proof. Let $\varepsilon > 0$. From Lemma 3.3, there exist $\delta > 0$ and $t_0 \in S$ such that for any $t \geq t_0$,

$$T(t) \in Lip(C, 1+\delta) \quad \text{and} \quad \overline{\text{co}}F_\delta(T(t)) \subset F_{\varepsilon/3}(T(t)).$$

From Lemma 3.4, there exist $\delta_1 \in S$ and $n_1 \in \mathbb{N}$ such that for any $n \geq n_1$,

$$k(t) \leq 1+\delta \quad \text{and} \quad \left\| \frac{1}{n} \sum_{i=1}^{n} T(t)^i T(s)x - T(t) \frac{1}{n} \sum_{i=1}^{n} T(t)^i T(s)x \right\| \leq \delta.$$

Put $t_* = t_1 + t_2$. Then, for $t \geq t_*$, we have

$$\overline{\text{co}}F_\delta(T(t)) \subset F_{\varepsilon/3}(T(t))$$

and

$$\begin{aligned}
&\left\| \frac{1}{n} \sum_{i=1}^{n} T(it+s)x - T(t) \left(\frac{1}{n} \sum_{i=1}^{n} T(it+s)x \right) \right\| \\
=&\left\| \frac{1}{n} \sum_{i=1}^{n} (T(t))^i T(s)x - T(t) \left(\frac{1}{n} \sum_{i=1}^{n} (T(t))^i T(s)x \right) \right\| < \delta
\end{aligned}$$

for every $n \geq n_1$. So, it follows that

$$\frac{1}{n} \sum_{i=1}^{n} T(it+s)x \in F_\delta(T(t)) \subset \overline{\mathrm{co}} F_\delta(T(t))$$

for every $t \geq t_*$, $s \in S$ and $n \geq n_1$. Let $n \geq n_1$ and $t \geq t_*$. Then, we have, for $p \in S$ and $\alpha \in I$,

$$\begin{aligned}
&\left\| \int T(s+p)x d\mu_\alpha(s) - T(t) \int T(s+p)x d\mu_\alpha(s) \right\| \\
\leq &\left\| \int T(s+p)x d\mu_\alpha(s) - \int \frac{1}{n} \sum_{i=1}^{n} T(it+s+p)x d\mu_\alpha(s) \right\| \\
&+ \left\| \int \frac{1}{n} \sum_{i=1}^{n} T(it+s+p)x d\mu_\alpha(s) - T(t) \left(\int \frac{1}{n} \sum_{i=1}^{n} T(it+s+p)x d\mu_\alpha(s) \right) \right\| \\
&+ \left\| T(t) \left(\int \frac{1}{n} \sum_{i=1}^{n} T(it+s+p)x d\mu_\alpha(s) \right) - T(t) \left(\int T(s+p)x d\mu_\alpha(s) \right) \right\| \\
\leq &(2+\delta) \left\| \int T(s+p)x d\mu_\alpha(s) - \int \frac{1}{n} \sum_{i=1}^{n} T(it+s+p)x d\mu_\alpha(s) \right\| \\
&+ \left\| \int \frac{1}{n} \sum_{i=1}^{n} T(it+s+p)x d\mu_\alpha(s) - T(t) \left(\int \frac{1}{n} \sum_{i=1}^{n} T(it+s+p)x d\mu_\alpha(s) \right) \right\| \\
= &(2+\delta) I_1 + I_2
\end{aligned}$$

and

$$\begin{aligned}
I_1 &= \left\| \int T(s+p)x d\mu_\alpha(s) - \int \frac{1}{n} \sum_{i=1}^{n} T(it+s+p)x d\mu_\alpha(s) \right\| \\
&\leq \frac{1}{n} \sum_{i=1}^{n} \left\| \int T(s+p)x d\mu_\alpha(s) - \int T(it+s+p)x d\mu_\alpha(s) \right\| \\
&= \frac{1}{n} \sum_{i=1}^{n} \left\| \int T(s+p)x d(\mu_\alpha - r_{it}{}^* \mu_\alpha)(s) \right\| \\
&\leq \frac{1}{n} \sum_{i=1}^{n} \sup_{z \in C} \|z\| \|\mu_\alpha - r_{it}{}^* \mu_\alpha\|.
\end{aligned}$$

From the assumption of the net $\{\mu_\alpha : \alpha \in I\}$, there exists $\alpha_1 \in I$ such that $\|\mu_\alpha - r_{it}{}^*\mu_\alpha\| < \frac{2\varepsilon}{3(2+\delta)\sup_{z\in C}\|z\|}$ for every $\alpha \geq \alpha_1$ and $i \in \{1, 2, \ldots, n\}$. So, $I_1 < 2\varepsilon/3(2+\delta)$ for every $\alpha \geq \alpha_1$ and $p \in S$. Next we prove that there exists $\alpha_2 \in I$ such that $\int (1/n)\sum_{i=1}^n T(it+s+p)x d\mu_\alpha(s) \in \overline{\text{co}} F_\delta(T(t))$ for every $p \in S$, $t \geq t_*$ and $\alpha \geq \alpha_2$. If not, we have, for each $\alpha_2 \in I$,

$$\int \frac{1}{n}\sum_{i=1}^n T(it + s + p')x d\mu_{\alpha'}(s) \notin \overline{\text{co}} F_\delta(T(t'))$$

for some $p' \in S$, $t' \geq t_*$ and $\alpha' \geq \alpha_2$. From the separation theorem, there exists $y_0{}^* \in E^*$ such that

$$\int \left\langle \frac{1}{n}\sum_{i=1}^n T(it' + s + p')x, y_0{}^* \right\rangle d\mu_{\alpha'}(s) < \inf\{\langle z, y_0{}^*\rangle : z \in \overline{\text{co}} F_\delta(T(t'))\}.$$

Then, we obtain

$$\begin{aligned}
\inf\{\langle z, y_0{}^* : z \in \overline{\text{co}} F_\delta(T(t'))\} &\leq \inf_{s\in S}\left\langle \frac{1}{n}\sum_{i=1}^n T(it' + s + p')x, y_0{}^*\right\rangle \\
&\leq \int \left\langle \frac{1}{n}\sum_{i=1}^n T(it' + s + p')x, y_0{}^*\right\rangle d\mu_{\alpha'}(s) \\
&< \inf\{\langle z, y_0{}^*\rangle : z \in \overline{\text{co}} F_\delta(T(t'))\}.
\end{aligned}$$

This is a contradiction. Hence, there exists $\alpha_2 \in I$ such that

$$\int \frac{1}{n}\sum_{i=i}^n T(it + s + p)x d\mu_\alpha(s) \in \overline{\text{co}} F_\delta(T(t)) \subset F_{\frac{\varepsilon}{3}}(T(t))$$

for every $p \in S$, $t \geq t_*$ and $\alpha \geq \alpha_2$. Then, we obtain $I_2 < \varepsilon/3$ for every $p \in S$, $t \geq t_*$ and $\alpha \geq \alpha_2$. Let $\alpha_0 \in I$ with $\alpha_0 \geq \alpha_1 + \alpha_2$. Then, we obtain

$$\left\| \int T(s+p)x d\mu_\alpha(s) - T(t)\left(\int T(s+p)x d\mu_\alpha(s)\right)\right\| \leq (2+\delta)I_1 + I_2 < \varepsilon$$

for every $\alpha \geq \alpha_0$, $t \geq t_*$ and $p \in S$. This completes the proof. □

Remark. We can prove that $F(\mathcal{S})$ is nonempty. In fact, let $x \in C$ and put $x_\alpha = \int T(s + p_\alpha)x d\mu_\alpha(s)$ for $\alpha \in I$. From the compactness of C, there exists a subnet $\{x_{\alpha_\beta}\}$ such that x_{α_β} converges strongly to some x_0 in C. Since

$$\limsup_\alpha \limsup_t \sup_{p\in S}\left\| \int T(s+p)x d\mu_\alpha(s) - T(t)\int T(s+p)x d\mu_\alpha(s)\right\| = 0,$$

we have

$$\begin{aligned}0 &= \limsup_t \limsup_\alpha \|x_\alpha - T(t)x_\alpha\| \\ &= \limsup_t \limsup_\beta \|x_{\alpha_\beta} - T(t)x_{\alpha_\beta}\| \\ &= \limsup_t \|x_0 - T(t)x_0\|.\end{aligned}$$

Therefore, for any $s \in S$, we obtain

$$\begin{aligned}\|x_0 - T(s)x_0\| &\le \limsup_t \|x_0 - T(t)x_0\| + \limsup_t \|T(t)x_0 - T(s)x_0\| \\ &= \limsup_t \|T(t+s)x_0 - T(s)x_0\| \\ &\le \limsup_t k(s)\|T(t)x_0 - x_0\| \\ &= 0\end{aligned}$$

and hence $x_0 \in F(\mathcal{S})$.

We can prove the following lemma from Lemma 3.2.

LEMMA 3.6. *Let C be a nonempty compact convex subset of a strictly convex Banach space E, let $\mathcal{S} = \{T(t) : t \in S\}$ be an asymptotically nonexpansive semigroup on C with Lipschitz constants $\{k(t) : t \in S\}$ and let $x \in C$. Let $\{\mu_\alpha : \alpha \in I\}$ and $\{\lambda_\beta : \beta \in J\}$ be nets of finite means on S such that*

$$\lim_\alpha \|\mu_\alpha - r_t{}^*\mu_\alpha\| = 0 \text{ and } \lim_\beta \|\lambda_\beta - \mathrm{r}_t{}^*\lambda_\beta\| = 0$$

for every $t \in S$. Then, there exist nets $\{p_\alpha : \alpha \in I\}$ and $\{q_\beta : \beta \in J\}$ in S such that for any $z \in F(\mathcal{S})$,

$$\lim_\alpha \left\| \int T(p_\alpha + t)x d\mu_\alpha(t) - z \right\| = \lim_\beta \left\| \int T(q_\beta + t)x d\lambda_\beta(t) - z \right\|.$$

Proof. Let $\varepsilon > 0$. From Lemma 3.2 , for $\alpha \in I$ and $\beta \in J$, there exist $h_0, p_\alpha, q_\beta \in S$ with $p_\alpha, q_\beta \ge h_0$ such that

$$\sup_{h\in S} \left\| \int T(h)T(w + p_\alpha + t)xd\mu_\alpha(t) - T(h)\int (T(w + p_\alpha + t)xd\mu_\alpha(t)) \right\| < \varepsilon$$

and

$$\sup_{h\in S} \left\| \int T(h)T(w + q_\beta + s)xd\lambda_\beta(s) - T(h)\int (T(w + q_\beta + s)xd\lambda_\beta(s)) \right\| < \varepsilon$$

for every $h \geq h_0$ and $w \in S$. Fix $z \in F(\mathcal{S})$ and consider

$$L = \left\| \int T(p_\alpha + t)x d\mu_\alpha(t) - z \right\|,$$

$$I_1 = \left\| \int T(p_\alpha + t)x d\mu_\alpha(t) - \iint T(p_\alpha + t + q_\beta + s)x d\lambda_\beta(s) d\mu_\alpha(t) \right\|,$$

$$I_2 = \left\| \iint T(p_\alpha + t + q_\beta + s)x d\lambda_\beta(s) d\mu_\alpha(t) - z \right\|,$$

$$J_1^{(2)} = \left\| \iint T(p_\alpha + t + q_\beta + s)x d\lambda_\beta(s) d\mu_\alpha(t) - \int T(p_\alpha + t) \left(\int T(q_\beta + s)x d\lambda_\beta(s) \right) d\mu_\alpha(t) \right\|$$

and

$$J_2^{(2)} = \left\| \int T(p_\alpha + t) \left(\int T(q_\beta + s)x d\lambda_\beta(s) \right) d\mu_\alpha(t) - z \right\|.$$

Then, we have $L \leq I_1 + I_2$ and $I_2 \leq {J_1}^{(2)} + {J_2}^{(1)}$. Suppose

$$\mu_\alpha = \sum_{i=1}^n a_i \delta_{t_i}, \quad (a_i \geq 0, \sum_{i=1}^n a_i = 1) \quad \text{and } \lambda_\beta = \sum_{j=1}^m b_{\delta_{s_j}} \quad (b_j \geq 0, \sum_{j=1}^m b_j = 1).$$

Then, we have

$$\begin{aligned} J_1^{(2)} &= \sum_{i=1}^n a_i \left\| \int T(p_\alpha + t_i) T(q_\beta + s)x d\lambda_\beta(s) - T(p_\alpha + t_i) \left(\int T(q_\beta + s)x d\lambda_\beta(s) \right) \right\| \\ &\leq \sup_{h \geq h_0} \left\| \int T(h) T(q_\beta + s)x d\lambda_\beta(s) - T(h) \left(\int T(q_\beta + s)x d\lambda_\beta(s) \right) \right\| < \varepsilon. \end{aligned}$$

Since $z \in F(\mathcal{S})$, we obtain

$$\begin{aligned} J_2^{(2)} &\leq \sum_{i=1}^n a_i \left\| \int T(p_\alpha + t_i) \left(\int T(q_\beta + s)x d\lambda_\beta(s) \right) d\mu_\alpha(t) - z \right\| \\ &\leq \left\| \int T(q_\beta + s)x d\lambda_\beta(s) - z \right\|. \end{aligned}$$

Then, we have

$$I_2 \leq J_1^{(2)} + J_2^{(2)} < \varepsilon + \left\| \int T(q_\beta + s)x d\lambda_\beta(s) - z \right\|.$$

On the other hand, we obtain

$$\begin{aligned}
I_1 &= \left\| \int T(p_\alpha + t)x d\mu_\alpha(t) - \sum_{j=1}^{m} b_j \int T(p_\alpha + t + q_\beta + s_j)x d\mu_\alpha(t) \right\| \\
&\le \sum_{j=1}^{m} b_j \left\| \int T(p_\alpha + t)x d\mu_\alpha(t) - \int T(p_\alpha + t + q_\beta + s_j)x d\mu_\alpha(t) \right\| \\
&\le \sum_{j=1}^{m} b_j \left\| \int T(p_\alpha + t)x d\mu_\alpha(t) - \int T(p_\alpha + t)x d(r^*_{q_\beta + s_j}\mu_\alpha)(t) \right\| \\
&\le \sum_{i=1}^{m} b_j \sup_{g\in S} \|T(g)x\| \; \|\mu_\alpha - r^*_{q_\beta+s_j}\mu_\alpha\|.
\end{aligned}$$

Therefore, from $\lim_\alpha I_1 = 0$, we have

$$\begin{aligned}
\limsup_\alpha \left\| \int T(p_\alpha + t)x d\mu_\alpha(t) - z \right\| &= \limsup_\alpha L \le \limsup_\alpha (I_1 + I_2) \\
&\le \varepsilon + \left\| \int T(q_\beta + s)x d\lambda_\beta(s) - z \right\|.
\end{aligned}$$

Then, we have

$$\limsup_\alpha \left\| \int T(p_\alpha + t)x d\mu_\alpha(t) - z \right\| \le \varepsilon + \liminf_\beta \left\| \int T(q_\beta + s)x d\lambda_\beta(s) - z \right\|.$$

Since $\varepsilon > 0$ is arbitrary, we obtain

$$\limsup_\alpha \left\| \int T(p_\alpha + t)x d\mu_\alpha(t) - z \right\| \le \liminf_\beta \left\| \int T(q_\beta + s)x d\lambda_\beta(s) - z \right\|.$$

Similarly, we have

$$\limsup_\beta \left\| \int T(q_\beta + s)x d\lambda_\beta(s) - z \right\| \le \liminf_\alpha \left\| \int T(p_\alpha + t)x d\mu_\alpha(t) - z \right\|.$$

Therefore, we have

$$\lim_\alpha \left\| \int T(p_\alpha + t)x d\mu_\alpha(t) - z \right\| = \lim_\beta \left\| \int T(q_\beta + s)x d\lambda_\beta(s) - z \right\|.$$

□

Remark. In Lemma 3.6, take nets $\{p_\alpha{}'\}$ and $\{q_\beta{}'\}$ in S such that $p_\alpha{}' \ge p_\alpha$ and $q'_\beta \ge q_\beta$. Then, repeating the above argument, we have the following:

$$\lim_\alpha \left\| \int T(p_\alpha{}' + t)x d\mu_\alpha(t) - z \right\| = \lim_\beta \left\| \int T(q_\beta{}' + s)x d\lambda_\beta(s) - z \right\|$$

for every $z \in F(\mathcal{S})$.

4. Main Result

Before proving our main theorem, we need one more lemma.

LEMMA 4.1. *Let C be a nonempty compact convex subset of a strictly convex Banach space E and let $\mathcal{S} = \{T(t) : t \in S\}$ be an asymptotically nonexpansive semigroup on C with Lipschitz constants $\{k(t) : t \in S\}$. Let D be a subspace of $B(S)$ such that $1 \in D$, D is r_s-invariant for each $s \in S$ and the function $t \mapsto \langle T(t)x, x^* \rangle$ is an element of D for each $x \in C$ and $x^* \in E^*$. Let $\{\mu_\alpha\}$ be a net of finite means on S such that*

$$\lim_\alpha \|\mu_\alpha - r_t{}^* \mu_\alpha\| = 0 \text{ for every } t \in S.$$

Then, for any $x \in C$, $\int T(p+t)x d\mu_\alpha(t)$ converges strongly to a common fixed point y_0 of $\mathcal{S}$ uniformly in $p \in S$. Furthermore, y_0 is independent of $\{\mu_\alpha : \alpha \in I\}$ and for any invariant mean μ on D, $y_0 = T_\mu x = \int T(t)x d\mu(t)$.

Proof. Let $\{\mu_\alpha : \alpha \in I\}$ and $\{\lambda_\beta : \beta \in J\}$ be nets of finite means on S such that

$$\lim_\alpha \|\mu_\alpha - r_t^* \mu_\alpha\| = 0 \text{ and } \lim_\beta \|\lambda_\beta - r_t^* \lambda_\beta\| = 0$$

for each $t \in S$. From Lemma 3.5 , we can take a net $\{p_\alpha\}$ in S such that for any $z \in F(\mathcal{S})$,

$$\lim_\alpha \left\| \int T(p_\alpha + t)x d\mu_\alpha(t) - z \right\|$$

exists. Let $\{\Phi_\alpha\} = \{\int T(p_\alpha + t)x d\mu_\alpha(t) : \alpha \in I\}$. As in the proof of Remark 3, we can take a subnet $\{\Phi_{\alpha_\gamma}\}$ of $\{\Phi_\alpha\}$ which converges strongly to a common fixed point of $\mathcal{S}$. Therefore, we have

$$\lim_\alpha \|\Phi_\alpha - y_0\| = \lim_\gamma \|\Phi_{\alpha_\gamma} - y_0\| = 0.$$

This implies that $\Phi_\alpha \to y_0$. Next we prove that $\int T(h+t)x d\mu_\alpha(t)$ converges strongly to $y_0 \in F(\mathcal{S})$ uniformly in h. In the above argument, take a net $\{p'_\alpha : \alpha \in I\}$ in S such that $p'_\alpha \geq p_\alpha$ for each $\alpha \in I$. Then, repeating the above argument, we see that $\Phi'_\alpha = \int T(p'_\alpha + t)x d\mu_\alpha(t)$ converges strongly to a common fixed point y_1 of $T(t), t \in S$. We show $y_0 = y_1$. From Lemma 3.6 and Remark 3 , we know

$$\lim_\alpha \left\| \int T(p'_\alpha + t)x d\mu_\alpha(t) - z \right\| = \lim_\alpha \left\| \int T(p_\alpha + t)x d\mu_\alpha(t) - z \right\|$$

for every $z \in F(\mathcal{S})$. Suppose $y_0 \neq y_1$. Then Φ_α does not converge strongly to y_1. Since y_0 and y_1 are in $F(\mathcal{S})$, we have

$$0 \leq \lim_\alpha \|\Phi_\alpha - y_1\| = \lim_\alpha \|\Phi'_\alpha - y_1\| = 0$$

and hence $\Phi'_\alpha \to y_1$. This is a contradiction. So, we have $y_0 = y_1 \in F(\mathcal{S})$. Since $\{p'_\alpha\}$ is an arbitrary net in S such that $p'_\alpha \geq p_\alpha$ for each $\alpha \in I$, we have that $\int T(h+p_\alpha+t)x d\mu_\alpha(t)$ converges strongly to y_0 uniformly in $h \in S$. Let $\varepsilon > 0$. Then, there exists $\alpha_0 \in I$ such that

$$\left\| \int T(h+p_\alpha+s)x d\mu_\alpha(s) - y_0 \right\| < \frac{\varepsilon}{2}$$

for every $\alpha \geq \alpha_0$ and $h \in S$. Suppose

$$\mu_{\alpha_0} = \sum_{k=1}^m b_k \delta_{s_k} (b_k \geq 0, \sum_{k=1}^m b_k = 1).$$

Put $\mu_0 = \mu_{\alpha_0}$ and $p_0 = p_{\alpha_0}$. Then we have

$$\begin{aligned}
&\left\| \iint T(h+t+p_0+s)x d\mu_0(s) d\lambda_\beta(t) - y_0 \right\| \\
&= \left\| \iint T(h+t+p_0+s)x d\mu_0(s) d\lambda_\beta(t) - \int y_0 d\lambda_\beta(t) \right\| \\
&\leq \sup_{t,h\in S} \left\| \int T(h+t+p_0+s)x d\mu_0(s) - y_0 \right\| < \frac{\varepsilon}{2}
\end{aligned}$$

for every $h \in S$ and $\beta \in J$. Since $\{\lambda_\beta\}$ satisfies $\lim_\beta \|\lambda_\beta - r_t^* \lambda_\beta\| = 0$, there exists β_1 such that

$$\|\lambda_\beta - r^*_{p_0+s_k}\lambda_\beta\| < \frac{\varepsilon}{2\max\{1, M\}}$$

for every $k \in \{1, 2, \ldots, m\}$ and $\beta \geq \beta_1$, where $M = \sup_{g\in S} \|T(g)x\|$. Then, we have

$$\begin{aligned}
&\left\| \int T(h+t)x d\lambda_\beta(t) - \iint T(h+t+p_0+s)x d\mu_0(s) d\lambda_\beta(t) \right\| \\
&= \left\| \int T(h+t)x d\lambda_\beta(t) - \sum_{k=1}^m b_k \int T(h+t+p_0+s_k)x d\lambda_\beta(t) \right\| \\
&\leq \sum_{k=1}^m b_k \left\| \int T(h+t)x d\lambda_\beta(t) - \int T(h+t+p_0+s_k)x d\lambda_\beta(t) \right\| \\
&\leq \sum_{k=1}^m b_k \left\| \int T(h+t)x d\lambda_\beta(t) - \int T(h+t)x d(r^*_{p_0+s_k}\lambda_\beta)(t) \right\| \\
&\leq \sum_{k=1}^m b_k M \|\lambda_\beta - r^*_{p_0+s_k}\lambda_\beta\| < \frac{\varepsilon}{2}
\end{aligned}$$

for every $\beta \geq \beta_1$ and $h \in S$. Therefore, we have

$$\begin{aligned}
&\left\| \int T(h+t)x d\lambda_\beta(t) - y_0 \right\| \\
&\leq \left\| \int T(h+t)x d\lambda_\beta(t) - \iint T(h+t+p_0+s)x d\mu_0(s) d\lambda_\beta(t) \right\| \\
&+ \left\| \iint T(h+t+p_0+s)x d\mu_0(s) d\lambda_\beta(t) - y_0 \right\| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

for every $\beta \geq \beta_1$ and $h \in S$. Hence, $\int T(h+t)x d\lambda_\beta(t)$ converges strongly to y_0 uniformly in $h \in S$. Since nets $\{\mu_\alpha : \alpha \in I\}$ and $\{\lambda_\beta : \beta \in J\}$ of finite means such that

$$\lim_\alpha \|\mu_\alpha - r_t^* \mu_\alpha\| = 0 \text{ and } \lim_\beta \|\lambda_\beta - r_t^* \lambda_\beta\| = 0$$

are arbitrary, we can obtain that $y_0 \in F(\mathcal{S})$ is independent of such nets of finite means. Finally, we show that, for any invariant mean μ, $y_0 = T_\mu x$. As in the proof of [5], for any invariant mean μ, there exists a net $\{\mu_\alpha\}$ of finite means such that $\lim_\alpha \|\mu_\alpha - r_t^* \mu_\alpha\| = 0$ and $\{\mu_\alpha\}$ converges to μ in weak* topology. Therefore, we can obtain

$$\begin{aligned}
\lim_\alpha \left\langle \int T(s)x d\mu_\alpha(s), y^* \right\rangle &= \lim_\alpha \int \langle T(s)x, y* \rangle d\mu_\alpha(s) \\
&= \int \langle T(s)x, y^* \rangle d\mu(s) \\
&= \left\langle \int T(s)x d\mu(s), y* \right\rangle \\
&= \langle T_\mu x, y* \rangle
\end{aligned}$$

for every $y^* \in E^*$. On the other hand, we have that $\int T(s)x d\mu_\alpha(s)$ converges strongly to y_0. Since C is compact, we obtain $y_0 = T_\mu x$. □

Now, we can prove a nonlinear ergodic theorem for an asymptotically nonexpansive semigroup with compact domain in a Banach space.

THEOREM 4.1. *Let C be a nonempty compact convex subset of a strictly convex Banach space E and let $\mathcal{S} = \{T(t) : t \in S\}$ be an asymptotically nonexpansive semigroup on C with Lipschitz constants $\{k(t) : t \in S\}$. Let D be a subspace of $B(S)$ such that $1 \in D$, D is r_s-invariant for each $s \in S$ and the function $t \mapsto \langle T(t)x, x^* \rangle$ is an element of D for each $x \in C$ and $x^* \in E^*$. Let $\{\lambda_\alpha\}$ be a* *strongly regular net of continuous linear functionals on D and let $x \in C$. Then $\int T(h+t)d\lambda_\alpha(t)$ converges strongly to a common fixed point y_0 of $\mathcal{S}$ uniformly in $p \in S$. Further, y_0 is independent of $\{\lambda_\alpha\}$ and for any invariant mean μ on D, $y_0 = T_\mu x = \int T(t)xd\mu(t)$. In this case, putting $Qx = \lim_\alpha \int T(t)xd\lambda_\alpha(t)$ for each $x \in C$, Q is a nonexpansive mapping of C onto $F(\mathcal{S})$ such that $QT(t) = T(t)Q = Q$ for every $t \in S$ and $Qx \in \overline{co}\{T(s)x : s \in S\}$ for every $x \in C$.*

Proof. Let $\{\lambda_\alpha : \alpha \in A\}$ be a strongly regular net of continuous linear functionals on D and let $\{\mu_\beta\}$ be a net of finite means on S such that

$$\lim_\beta \|\mu_\beta - r_t^* \mu_\beta\| = 0$$

for every $t \in S$. From Lemma 4.1 , we have that $\int T(h+t)xd\mu_\beta(t)$ converges strongly to a common fixed point y_0 of $\mathcal{S}$ uniformly in $h \in S$. Let $\varepsilon > 0$ and let μ be an invariant mean on D. From Lemma 4.1 , we also know $y_0 = T_\mu x$. Further, there exists β_1 such that

$$\left\| \int T(h+t)xd\mu_\beta(t) - T_\mu x \right\| < \frac{\varepsilon}{\sup_\alpha \|\lambda_\alpha\|}$$

for all $\beta \geq \beta_1$ and $h \in S$. Suppose

$$\mu_{\beta_1} = \sum_{i=1}^n b_i \delta_{t_i} \quad (b_i \geq 0, \sum_{i=1}^n b_i = 1)$$

and put $\mu_1 = \mu_{\beta_1}$. Then, we have

$$\left\| \int T(h+t)xd\mu_1(t) - T_(\mu)x \right\| < \frac{\varepsilon}{\sup_\alpha \|\lambda_\alpha\|}$$

for every $h \in S$. Since $\{\lambda_\alpha\}$ is strongly regular, there exists α_0 such that

$$|1 - \lambda_\alpha(1)| < \frac{\varepsilon}{\max\{1, \|T_\mu x\|\}}$$

and

$$\|\lambda_\alpha - r_{t_i}^* \lambda_\alpha\| < \frac{\varepsilon}{\max\{1, M\}}$$

for every $i \in \{1, 2, \dots, n\}$ and $\alpha \geq \alpha_0$, where $M = \sup_{h \in S} \|T(h)x\|$. Then, we have

$$\begin{aligned} \left\| T_\mu x - \int T_\mu x d\lambda_\alpha(s) \right\| &= \sup_{x^* \in S_1(E^*)} \left| \langle T_\mu x, x^* \rangle - \int \langle T_\mu x, x^* \rangle d\lambda_\alpha(s) \right| \\ &\leq \sup_{x^* \in S_1(E^*)} |\langle T_\mu x, x^* \rangle| \cdot |1 - \lambda_\alpha(s)| < \varepsilon \end{aligned}$$

for every $\alpha \geq \alpha_0$ and

$$\begin{aligned}&\left\|\iint T(h+s+t)x d\mu_1(t) d\lambda_\alpha(s) - \int T_\mu x\right\| \\ &\leq \|\lambda_\alpha\| \cdot \sup_{s,h\in S}\left\|\int T(h+s+t)x d\mu_1(t) - T_\mu x\right\| < \varepsilon\end{aligned}$$

for every $h \in S$ and $\alpha \in A$. Thus, we obtain

$$\left\|\iint T(h+s+t)x d\mu_1(t) d\lambda_\alpha(s) - T_\mu x\right\| < \varepsilon + \varepsilon = 2\varepsilon$$

for every $h \in S$ and $\alpha \geq \alpha_0$. On the other hand, we have

$$\begin{aligned}&\left\|\int T(h+s)x d\lambda_\alpha(s) - \iint T(h+s+t)x d\mu_1(t) d\lambda_\alpha(s)\right\| \\ &= \left\|\int T(h+s)x d\lambda_\alpha(s) - \sum_{i=1}^n b_i \int T(h+s+t_i)x d\lambda_\alpha(s)\right\| \\ &\leq \sum_{i=1}^n b_i \left\|\int T(h+s)x d\lambda_\alpha(s) - \int T(h+s+t_i)x d\lambda_\alpha(s)\right\| \\ &= \sum_{i=1}^n b_i \left\|\int T(h+s)x d(\lambda_\alpha - r_{t_i}^* \lambda_\alpha)(s)\right\| \\ &\leq \sum_{i=1}^n b_i \|\lambda_\alpha - r_{t_i}^* \lambda_\alpha\| \cdot M < \varepsilon\end{aligned}$$

for every $h \in S$ and $\alpha \geq \alpha_0$. Therefore, we obtain

$$\begin{aligned}&\left\|\int T(h+s)x d\lambda_\alpha(s) - T_\mu x\right\| \\ &\leq \left\|\int T(h+s)x d\lambda_\alpha(s) - \iint T(h+s+t)x d\mu_1(t) d\lambda_\alpha(s)\right\| \\ &+ \left\|\iint T(h+s+t)x d\mu_1(t) d\lambda_\alpha(s) - T_\mu x\right\| \\ &< \varepsilon + 2\varepsilon = 3\varepsilon\end{aligned}$$

for every $h \in S$ and $\alpha \geq \alpha_0$. Then, $\int T(h+t)x d\lambda_\alpha(t)$ converges strongly to a common fixed point y_0 of $\mathcal{S}$ uniformly in h. Further, such an element y_0 is independent of $\{\lambda_\alpha\}$ and $y_0 = T_\mu x$ for any invariant mean μ on D. If $Qx = \lim_\alpha \int T(t)x d\lambda_\alpha(t)$ for each $x \in S$, then Q is a nonexpansive mapping of C onto $F(\mathcal{S})$ such that $QT(t) = T(t)Q = Q$ for every $t \in S$ and $Qx \in \overline{\text{co}}\{T(s)x : s \in S\}$ for every $x \in C$. $\square$

5. Yoshimoto's theorems

Using Theorem 4.1, we can prove the following two theorems which are obtained by Yoshimoto [15].

THEOREM 5.1. *Let C be a nonempty compact convex subset of a strictly convex Banach space E and let $\{T(t) : t \geq 0\}$ be a one-parameter asymptotically nonexpansive semigroup on C. Let $x_1 \in C$ and $r > 0$. Then $r\int_0^\infty e^{-rt}T(t+s)xdt$ converges strongly as $r \to 0+$ to a common fixed point of $T(t), t \geq 0$ uniformly in $s \geq 0$.*

Proof. Let $S = \mathbb{R}^+$, $\mathcal{S} = \{T(t) : t \in S\}$, $D = B(S)$ and $\lambda_r(f) = r\int_0^\infty e^{-rt}f(t)dt$ for $r > 0$ and $f \in B(S)$. Then, we prove that $\{\lambda_r\}$ is a strongly regular net of means. In fact, for $f \in B(S)$,

$$\begin{aligned} |\lambda_r(f)| &= \left| r\int_0^\infty e^{-rt}f(t)dt \right| \\ &\leq r\int_0^\infty e^{-rt}\|f\|dt = \|f\| \end{aligned}$$

and

$$\lambda_r(1) = r\int_0^\infty e^{-rt}\cdot 1dt = 1.$$

Then, we have $\|\lambda_r\| = \lambda_r(1) = 1$. Next, for $h \in \mathbb{R}^+$and $f \in B(S)$, we have

$$\begin{aligned} |\lambda_r(f) - \lambda_r(r_hf)| &= \left| r\int_0^\infty e^{-rt}f(t)dt - r\int_0^\infty e^{-rt}f(t+h)dt \right| \\ &= \left| r\int_0^\infty e^{-rt}f(t)dt - e^{rh}r\int_h^\infty e^{-rt}f(t)dt \right| \\ &\leq \left| r(1-e^{rh})\int_0^\infty e^{-rt}f(t)dt \right| + \left| re^{rh}\int_0^h e^{-rt}f(t)dt \right| \\ &\leq |1-e^{rh}|\|f\| + e^{rh}|1-e^{-rh}|\|f\| \\ &= 2|1-e^{rh}|\|f\| \to 0 \end{aligned}$$

as $r \to 0+$. Therefore $\{\lambda_r\}$ is strongly regular. From Theorem 4.1, we obtain the conclusion. □

THEOREM 5.2. *Let C be a nonempty compact convex subset of a strictly convex Banach space E and let $\{T(t) : t \geq 0\}$ be a one-parameter asymptotically nonexpansive semigroup on C. Let $x_1 \in C$ and $r > 0$. Then $\frac{\alpha}{r^\alpha}\int_0^r (r-t)^{\alpha-1}T(t+s)xdt$ converges strongly as $r \to \infty$ to a common fixed point of $T(t), t \geq 0$ uniformly in $s \geq 0$.*

Proof. Let $S = \mathbb{R}^+$, $\mathcal{S} = \{T(t) : t \in S\}$, $D = B(S)$ and $\lambda_r(f) = \frac{\alpha}{r^\alpha}\int_0^r (r - t)^{\alpha-1} f(t)dt$ for $r > 0$ and $f \in B(S)$. Then, we prove that $\{\lambda_r\}$ is a strongly regular net of means. In fact, for $f \in B(S)$, we have

$$\begin{aligned} |\lambda_r(f)| &= \left| \frac{\alpha}{r^\alpha} \int_0^r (r-t)^{\alpha-1} f(t)dt \right| \\ &\le \frac{\alpha}{r^\alpha} \int_0^r (r-t)^{\alpha-1} \|f\| dt = \|f\| \end{aligned}$$

and

$$\lambda_r(1) = \frac{\alpha}{r^\alpha} \int_0^r (r-t)^{\alpha-1} \cdot 1dt = 1.$$

Then, we have $\|\lambda_s\| = \lambda_r(1) = 1$. Next, for $h \in \mathbb{R}^+$and $f \in B(S)$, we get

$$\begin{aligned} |\lambda_r(f) - \lambda_r(r_h f)| &= \frac{\alpha}{r^\alpha} \left| \int_0^r (r-t)^{\alpha-1} f(t)dt - \int_0^r (r-t)^{\alpha-1} f(t+h)dt \right| \\ &= \frac{\alpha}{r^\alpha} \left| \int_0^r (r-t)^{\alpha-1} f(t)dt - \int_h^{r+h} (r-t+h)^{\alpha-1} f(t)dt \right| \\ &\le \frac{\alpha}{r^\alpha} \left| \int_0^h (r-t)^{\alpha-1} f(t)dt \right| + \left| \int_r^{r+h} (r-t+h)^{\alpha-1} f(t)dt \right| \\ &\quad + \left| \int_h^r \{(r-t)^{\alpha-1} - (r-t+h)^{\alpha-1}\} f(t)dt \right| \\ &\le \frac{\alpha}{r^\alpha} \int_0^h (r-t)^{\alpha-1} \|f\| dt + \int_r^{r+h} (r-t+h)^{\alpha-1} \|f\| dt \\ &\quad + \int_h^r \left| \{(r-t)^{\alpha-1} - (r-t+h)^{\alpha-1}\} \right| \|f\| dt. \end{aligned}$$

For $\alpha \in (0, 1)$, we have

$$\begin{aligned} |\lambda_r(f) - \lambda_r(r_h f)| &\le \|f\| \left\{ \frac{\alpha}{r^\alpha} \int_0^h (r-t)^{\alpha-1} dt + \int_r^{r+h} (r-t+h)^{\alpha-1} dt \right. \\ &\qquad \left. + \int_h^r (r-t+h)^{\alpha-1} - (r-t)^{\alpha-1} dt \right\} \\ &= 2 \left(\frac{s}{t} \right)^\alpha \|f\|. \end{aligned}$$

For $\alpha \in (1, \infty)$, we also have

$$\begin{aligned}|\lambda_r(f) - \lambda_r(r_h f)| \leq \|f\| \Bigg\{ \frac{\alpha}{r^\alpha} \int_0^h (r-t)^{\alpha-1} dt + \int_r^{r+h} (r-t+h)^{\alpha-1} dt \\ + \int_h^r (r-t)^{\alpha-1} - (r-t+h)^{\alpha-1} dt \Bigg\} \\ = 2\left\{1 - \left(1 - \frac{s}{t}\right)^\alpha\right\} \|f\|.\end{aligned}$$

Therefore, for each $\alpha \in \mathbb{R}^+$, $\{\lambda_r\}$ is strongly regular. From Theorem 4.1, we obtain the conclution. □

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