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ON TORIC VARIETIES OF HIGH ARITHMETICAL RANK

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Abstract. We describe a class of toric varieties in the N-dimensional affine space which are minimally defined by no less than $N-2$ binomial equations.

Introduction

The *arithmetical rank* (ara) of an algebraic variety is the minimum number of equations that are needed to define it set-theoretically. For every affine variety $V\subset K^{N}$ we have that $\operatorname{codim}V\leq\operatorname{ara}V\leq N$. This general upper bound was found by Eisenbud and Evans [7]. In particular cases a better upper bound can be obtained by direct computations based on Hilbert's Nullstellensatz: this was done for certain toric varieties in [1], [2], [3], [4], [5]. In all the examples treated there the arithmetical rank was close to the trivial lower bound, i.e., $\operatorname{ara}V\leq\operatorname{codim}V+1$. In this paper we present a class of toric varieties whose arithmetical rank is close to the general upper bound, namely $\operatorname{ara}V\geq N-2$. For proving this result, of course, we need a more efficient lower bound: this is provided by étale cohomology. The same kind of tools was used in [3]. There they were applied for showing that the arithmetical rank of certain toric varieties of codimension 2 depends on the characteristic of the ground field, and that $\operatorname{ara}V=\operatorname{codim}V$ in exactly one positive characteristic. In the present paper, however, we study toric varieties of any codimension, and obtain the same lower bound in all characteristics. This lower bound turns out to be sharp if $N=3$ or $N=5$.

1. The main theorem

Let K be an algebraically closed field, and let $n\geq 2$ be an integer. Let $e_1,\ldots,e_n$ be the standard basis of $\mathbb{Z}^n$. Set $N=2n-1$ and consider the following

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subset of $\mathbb{N}^N$:

$$T = \{e_1, \ldots, e_{n-1}, de_n, a_1e_1 + e_n, \ldots, a_{n-1}e_{n-1} + e_n\},$$

where $d, a_1, \ldots, a_{n-1} \in \mathbb{N}^*$. We also suppose that d is divisible by two distinct primes p and q. With T we can associate the variety V admitting the following parametrization

$$V : \begin{cases} x_1 &= u_1 \\ &\vdots \\ x_{n-1} &= u_{n-1} \\ x_n &= u_n^d \\ y_1 &= u_1^{a_1} u_n \\ &\vdots \\ y_{n-1} &= u_{n-1}^{a_{n-1}} u_n \end{cases},$$

which is a toric variety of codimension $n-1$ in the affine space K^N. Our aim is to show the following

THEOREM 1. $\operatorname{ara} V \geq N - 2$.

This will be done in the next section.

2. The lower bound

We show that $\operatorname{ara} V \geq N - 2$ by means of the following criterion, which is based on étale cohomology and is cited from [6], Lemma 3′.

LEMMA 2. *Let $W \subset \tilde{W}$ be affine varieties. Let $d = \dim \tilde{W} \setminus W$. If there are s equations $F_1, \ldots, F_s$ such that $W = \tilde{W} \cap V(F_1, \ldots, F_s)$, then*

$$H_{\text{et}}^{d+i}(\tilde{W} \setminus W, \mathbb{Z}/r\mathbb{Z}) = 0 \qquad \textit{for all } i \geq s$$

and for all $r \in \mathbb{Z}$ which are prime to $\operatorname{char} K$.

Since p and q are distinct primes, we may assume that $\operatorname{char} K \neq p$. Hence our claim will follow once we have shown

PROPOSITION 3. *If* $\operatorname{char} K \neq p$, *then*

$$H_{\text{et}}^{2N-3}(K^N \setminus V, \mathbb{Z}/p\mathbb{Z}) \neq 0.$$

Proof. In the sequel H_{et} and H_c will denote étale cohomology and étale cohomology with compact support with respect to the coefficient group $\mathbb{Z}/p\mathbb{Z}$: we shall omit the latter for the sake of simplicity. By Poincaré Duality (see [8], Theorem 14.7, p. 83) we have

$$\mathrm{Hom}_{\mathbb{Z}}(H^{2N-3}_{\mathrm{et}}(K^N \setminus V), \mathbb{Z}/p\mathbb{Z}) \simeq H^3_{\mathrm{c}}(K^N \setminus V). \tag{1}$$

Moreover, we have a long exact sequence of cohomology with compact support

$$\cdots \to H^2_{\mathrm{c}}(K^N) \to H^2_{\mathrm{c}}(V) \to H^3_{\mathrm{c}}(K^N \setminus V) \to H^3_{\mathrm{c}}(K^N) \to \cdots,$$

where $H^2_{\mathrm{c}}(K^N) = H^3_{\mathrm{c}}(K^N) = 0$, since $N \geq 3$ (see [8], Example 16.3, pp.98–99). Hence

$$H^3_{\mathrm{c}}(K^N \setminus V) \simeq H^2_{\mathrm{c}}(V). \tag{2}$$

By (1) and (2) it thus suffices to show that

$$H^2_{\mathrm{c}}(V) \neq 0. \tag{3}$$

On K^n fix the coordinates $u_1, \ldots, u_n$ and let X be the subvariety of K^n defined by $u_1 = u_2 = \cdots = u_{n-1} = 0$. Then X is a 1-dimensional affine space over K, on which we fix the coordinate u_n. Consider the surjective map

$$\begin{gathered} \phi : K^n \to V \\ (u_1, \ldots, u_n) \mapsto (u_1, \ldots, u_{n-1}, u_n^d, u_1^{a_1} u_n, \ldots, u_{n-1}^{a_{n-1}} u_n) \end{gathered}$$

and the restriction map

$$\tilde{\phi} : K^n \setminus X \to V \setminus \phi(X),$$

which is a bijective morphism of affine schemes. For all $i = 1, \ldots, n-1$ let $V_i = \{(\bar{x}_1, \ldots, \bar{x}_{n-1}, \bar{x}_n, \bar{y}_1, \ldots, \bar{y}_{n-1}) \in V \mid \bar{x}_i \neq 0\}$. These sets form an open cover of $V \setminus \phi(X)$, and $U_i = \phi^{-1}(V_i) = \{(u_1, \ldots, u_n) \in K^n \mid u_i \neq 0\}$. Moreover, for all $i = 1 \ldots, n-1$, the morphism

$$\begin{gathered} \psi_i : V_i \to U_i \\ (\bar{x}_1, \ldots, \bar{x}_{n-1}, \bar{x}_n, \bar{y}_1, \ldots, \bar{y}_{n-1}) \mapsto (\bar{x}_1, \ldots, \bar{x}_{n-1}, \frac{\bar{y}_i}{\bar{x}_i^{a_i}}) \end{gathered}$$

is the inverse map of the restriction of $\tilde{\phi}$ to U_i. Hence $\tilde{\phi}$ is an isomorphism of affine schemes, so that it induces an isomorphism of groups

$$\tilde{\phi}_i^* : H^i_{\mathrm{c}}(V \setminus \phi(X)) \simeq H^i_{\mathrm{c}}(K^n \setminus X)$$

for all indices i. The restriction of ϕ to X

$$\bar{\phi} : X \to \phi(X)$$

maps u_n to $(0,\dots,0,u_n^d,\dots,0)$. Hence $\phi(X)$ is a 1-dimensional affine space and $\bar{\phi}$ induces multiplication by d on the second cohomology group with compact support (see [8], Remark 24.2 (f), p. 135). Now, as it is well-known (see [8], Example 16.3, pp. 98–99),

$$H_c^i(\phi(X)) \simeq H_c^i(X) \simeq \begin{cases} \mathbb{Z}/p\mathbb{Z} & \text{for } i=2 \\ 0 & \text{else} \end{cases}.$$

Since p divides d, it follows that the induced maps

$$\bar{\phi}_i^* : H_c^i(\phi(X)) \longrightarrow H_c^i(X)$$

are all equal to the zero map. Thus ϕ gives rise to the following morphism of acyclic complexes:

$$\begin{array}{ccccc} & & \mathbb{Z}/p\mathbb{Z} & & \\ & & \wr\| & & \\ H_c^2(V) & \longrightarrow & H_c^2(\phi(X)) & \xrightarrow{f} & H_c^3(V\setminus\phi(X)) \\ \downarrow & & \bar{\phi}_2^*\downarrow 0 & & \wr\|\downarrow\bar{\phi}_3^* \\ H_c^2(K^n) & \longrightarrow & H_c^2(X) & \xrightarrow[g]{} & H_c^3(K^n\setminus X) \\ & & \wr\| & & \\ & & \mathbb{Z}/p\mathbb{Z} & & \end{array}.$$

Note that $\bar{\phi}_3^* f = g0 = 0$, so that $\bar{\phi}_3^* f$ is not injective. Since $\bar{\phi}_3^*$ is an isomorphism, it follows that f is not injective. This implies (3) and completes the proof. ■

3. On the defining equations

Finally we show that the lower bound established in Theorem 1 is sharp, since it is attained by the arithmetical rank when $n=2$ or $n=3$.

PROPOSITION 4. *If $n=2$, V is defined set-theoretically by $F=y_1^d-x_1^{a_1 d}x_2$. If $n=3$, then V is set-theoretically defined by the following three binomials:*

$$F_1=y_1^d-x_1^{a_1d}x_3,\quad F_2=y_2^d-x_2^{a_2d}x_3,\quad G=y_1^{d-1}y_2-x_1^{a_1(d-1)}x_2^{a_2}x_3.$$

Proof. If $n=2$, the claim is clear, since the defining ideal of V is the principal ideal generated by F. So suppose that $n=3$. It is straightforward to check that for all $\boldsymbol{w}\in V$, $F_1(\boldsymbol{w})=F_2(\boldsymbol{w})=G(\boldsymbol{w})=0$. Conversely, we have to prove that for every $\boldsymbol{w}\in K^5$ such that

(4) $$F_1(\boldsymbol{w})=F_2(\boldsymbol{w})=G(\boldsymbol{w})=0,$$

we have that $\boldsymbol{w} \in V$. Let $\boldsymbol{w} = (\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{y}_1, \bar{y}_2) \in K^5$ be a point fulfilling (4). Set $u_i = \bar{x}_i$ for $i = 1, 2$. We show that, for a suitable choice of parameter u_3, we can write $\boldsymbol{w} = (u_1, u_2, u_3^d, u_1^{a_1}u_3, u_2^{a_2}u_3)$. This is certainly true if $\bar{x}_3 = 0$: in this case (4) implies that $\bar{y}_1 = \bar{y}_2 = 0$, and $u_3 = 0$ is the required parameter. Suppose that $\bar{x}_3 \neq 0$. Let u_3 be a d-th root of $\bar{x}_3$. By (4) we have that, for $i = 1, 2$,

$$\bar{y}_i^d = u_i^{a_i d} u_3^d,$$

which implies that

$$\bar{y}_i = u_i^{a_i} u_3 \omega_i \tag{5}$$

for some d-th root ω_i of 1. On the other hand, from (4) we also deduce that

$$\bar{y}_1^{d-1}\bar{y}_2 = u_1^{a_1(d-1)} u_2^{a_2} u_3^d. \tag{6}$$

Note that, by (4), since $\bar{x}_3 \neq 0$, for $i = 1, 2$, we have $\bar{x}_i = 0$ if and only if $\bar{y}_i = 0$. If $\bar{x}_1 = 0$, set $u_3' = u_3\omega_2$. Then $\bar{x}_3 = u_3'^d$, and, in view of (5), $\bar{y}_i = u_i^{a_i}u_3'$ for $i = 1, 2$. Hence u_3' is the required parameter. Similarly one can reason if $\bar{x}_2 = 0$. So assume that $\bar{x}_i \neq 0$ for $i = 1, 2$. Replacing (5) on the left-hand side of (6) gives

$$u_1^{a_1(d-1)} u_3^{d-1} \omega_1^{d-1} u_2^{a_2} u_3 \omega_2 = u_1^{a_1(d-1)} u_2^{a_2} u_3^d. \tag{7}$$

Since u_1, u_2, u_3 are non zero, from this we deduce that

$$\omega_1^{d-1}\omega_2 = 1, \tag{8}$$

which implies that

$$\omega_1 = \omega_2,$$

i.e., ω_1 and ω_2 are both equal to the same d-th root ω of 1. Thus $u_3' = u_3\omega$ is the required parameter. ∎

The computation of the arithmetical rank for $n \geq 4$ remains an open question.

References

[1] M. Barile, Almost set-theoretic complete intersections in characteristic zero. Preprint (2005). **arXiv:math.AG/0504052**. To appear in *Collect. Math.*

[2] M. Barile, A note on Veronese varieties, *Rend. Circ. Mat. Palermo.*, **54** (2005), 359–366.

[3] M. Barile, G. Lyubeznik, Set-Theoretic Complete Intersections in Characteristic p, *Proc. Amer. Math. Soc.*, **133** (2005), 3199–3209.

[4] M. Barile, M. Morales, A. Thoma, On Simplicial Toric Varieties Which Are Set-Theoretic Complete Intersections, *J. Algebra*, **226** (2000), 880–892.

[5] M. Barile, M. Morales, A. Thoma, Set-Theoretic Complete Intersections on Binomials, *Proc. Amer. Math. Soc.*, **130** (2002), 1893–1903.

[6] W. Bruns, R. Schwänzl, The number of equations defining a determinantal variety, *Bull. London Math. Soc.*, **22** (1990), 439–445.

[7] D. Eisenbud, E.G. Evans, jr. Every Algebraic Set in n-Space is the Intersection of n Hypersurfaces, *Inv. Math.*, **19** (1973), 107–112.

[8] J.S. Milne, "Lectures on étale cohomology". Available at `http://www.jmilne.org`.

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