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A WEIGHTED MULTIPLIER THEOREM FOR THE MODIFIED HANKEL TRANSFORM

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Abstract. We prove a weighted version of Hörmander multiplier theorem for the modified Hankel transform. To do this, we connect Riesz function technique with weighted estimates associated with such a transform.

1. Introduction

The purpose of this paper is to derive a weighted version of Hörmander multiplier theorem for the modified Hankel transform, which extends earlier results in the subject [4], [5], [7]. In more details, the main result of [5] says that a bounded function on R_+ satysfying a Hörmander type condition for λ the least even integer $> \alpha+1$ is a multiplier for the modified Hankel transform on every $L^p(R_+, x^{2\alpha+1})$, $1<p<\infty$, $\alpha > -\frac{1}{2}$. Then the result of [5] was improved by author in [7] allowing λ be fractional $> \alpha+1$. It is an usual practice to extend Hörmander multiplier theorem into weighted spaces by introducing the additional weight x^a. The paper concludes that a similar condition (λ is fractional $> \alpha+1$) allows to get multiplier on every $L^p(R_+, x^a x^{2\alpha+1})$, for $1<p<\infty$, $\alpha > -\frac{1}{2}$ and $a>0$ [Theorem 1].

Let $C_c^\infty(0,\infty)$ be the space of compactly supported C^∞ functions on $(0,\infty)$. For a suitable function f on $(0,\infty)$ (for example for $f \in C_c^\infty(0,\infty)$) the modified Hankel transform, of order α, of f is defined by the formula

$$H_\alpha f(y) = \int_0^\infty \frac{J_\alpha(xy)}{(xy)^\alpha} f(x) x^{2\alpha+1} dx,$$

where J_α is the Bessel function of the first kind (of order α). Note that the inversion theorem holds

$$H_\alpha(H_\alpha f)(x) = f(x), \qquad x>0,$$

see [5, p. 656].

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For $\alpha > -\frac{1}{2}$, $a \in R$, and $1 < p < \infty$, let $L^{p,a}_\alpha$ denote the set of measurable functions f on $(0,\infty)$ for which $\int_0^\infty |f(x)|^p x^a x^{2\alpha+1} dx < \infty$. Endowed with the norm

$$||f||_{L^{p,a}_\alpha} := \left(\int_0^\infty |f(x)|^p x^a x^{2\alpha+1} dx\right)^{1/p} < \infty$$

$L^{p,a}_\alpha$ is a Banach space.

A bounded, measurable function m on $(0,\infty)$ is said to be a multiplier on $L^{p,a}_\alpha$, for H_α, if the inequality

$$||H_\alpha(m \cdot H_\alpha f)||_{L^{p,a}_\alpha} \leq C||f||_{L^{p,a}_\alpha}$$

holds for all $f \in C_c^\infty(0,\infty)$, where C is a constant independent of f.

For suitable functions f and g on $(0,\infty)$, let $f * g$ denote the (generalized) convolution of f and g defined as

$$f * g(x) = \int_0^\infty f(y) T^y g(x) y^{2\alpha+1} dy,$$

where T^y is the generalized translation operator given by

$$T^y g(x) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \int_0^\pi g((x,y)_\theta) \sin^{2\alpha}(\theta) d\theta,$$

with $(x,y)_\theta = (x^2+y^2-2xy\cos\theta)^{\frac{1}{2}}$, if $x, y > 0$. Note that

$$H_\alpha(f * g) = H_\alpha(f) \cdot H_\alpha(g), \qquad f, g \in L^{1,0}_\alpha,$$

see [6].

In this paper, we make use of the following properties of T^y :

(1) $$\int_0^\infty T^y f(x) g(x) x^{2\alpha+1} dx = \int_0^\infty f(x) T^y g(x) x^{2\alpha+1} dx, \qquad y > 0,$$

(2) $$|T^y f(x)|^p \leq T^y(|f|^p)(x), \qquad (1 \leq p < \infty).$$

If $f = 0$ on the interval $(|x-y|, x+y)$ then

(3) $$T^y f(x) = 0,$$

see [5], [10].

Let $\lambda > 0$ and let $WBV_{2,\lambda}$ denote the set of bounded, continous functions m on $(0,\infty)$ such that

$$||m||_{2,\lambda} := ||m||_\infty + \sup_{j\in Z} \left(\int_{2^{j-1}}^{2^j} |x^\lambda m^{(\lambda)}(x)|^2 \frac{dx}{x}\right)^{1/2},$$

where $m^{(\lambda)}$ is the fractional derivative of order λ, in the sense given in [3]. Then the expression $||m||_{2,\lambda}$ defines a norm on $WBV_{2,\lambda}$, for which $WBV_{2,\lambda}$ becomes a Banach space. As usual we decompose the function m from $WBV_{2,\lambda}$ into small dyadic pieces by using a fixed bump function. Let Ψ be a C^∞ function on $(0,\infty)$ with support in $(1/2,2)$ such that $\sum_{j=-\infty}^{\infty}\Psi(2^{-j}x)=1$, and set $m_j(x)=m(x)\Psi(2^{-j}x)$. The following characterization of m in $WBV_{2,\lambda}$ will be needed through the paper. If $\lambda \geq \frac{1}{2}$ and $\Psi_{1/t}(x)=\Psi(x/t)$ then

$$\sup_{t>0}(t)^{\lambda-1/2}\left(\int_{-\infty}^{\infty}|(\Psi_{1/t}m)^{(\lambda)}(x)|^2dx\right)^{1/2} \leq C||m||_{2,\lambda}, \tag{4}$$

where C is a suitable positive constant independent of m and $\Psi_{1/t}$ [1, Theorem 2]. Substitution $t=2^j$ in (4) leads to

$$\left(\int_0^{2^{j+1}}|m_j^{(\lambda)}(x)|^2dx\right)^{\frac{1}{2}} \leq C(2^j)^{\frac{1}{2}-\lambda}||m||_{2,\lambda}, \tag{5}$$

where C does not depend on j. Moreover, for the function m_j with compact support in $(2^{j-1},2^{j+1})$ we have $\operatorname{supp} m_j^{(\lambda)} \subset (0,2^{j+1})$ and we can write the reproducing formula

$$m_j(x)=\frac{1}{\Gamma(\lambda)}\int_0^{2^{j+1}} m_j^{(\lambda)}(u)(u-x)_+^{\lambda-1}du \tag{6}$$

[3, Section 3].

Throughout this paper C, with or without subscription, will represent a positive constant which is not necessarily the same in each occurrence.

2. The main result

THEOREM 1. *Let $\alpha > -\frac{1}{2}$, $1 < p < \infty$, $(p-2)(\alpha+1) < a < (\alpha+1)\min\{p,2p-2\}$. Assume that m belongs to $WBV_{2,\lambda}$ for some $\lambda > \alpha+1$. Then m is a multiplier on $L^{p,a}_{\alpha}$, for H_α.*

Let $j \in Z$. Define new family of functions: $h(x)=m(x^2)$, $h_j(x)=m_j(x^2)$.

LEMMA 2. *Let $m \in WBV_{2,\lambda}$ for some $\lambda > \alpha+1$. Then, for all $1<p<2$,*

$$\sum_{j=-N}^{N}\left(\int_r^{2r}|H_\alpha h_j(y)|^p y^{2\alpha+1}dy\right)^{1/p} \leq C||m||_{2,\lambda}\, r^{(2\alpha+2)((1/p)-1)}, \tag{7}$$

with C independent of $r>0$ and $N=1,2,\ldots$.

Proof. As in [7] we use the fact that the modified Hankel transform of the function

$$R(x) = (u - x^2)_+^{\lambda-1}, \qquad \lambda > 0$$

has the simple form

$$H_\alpha R(y) = C_{\alpha,\lambda}\left(\frac{\sqrt{u}}{y}\right)^{\alpha+\lambda} J_{\alpha+\lambda}(\sqrt{u}y). \tag{8}$$

[9, §4 Theorem 4.15].

Substitution $x := x^2$ in (6) and using (8) lead to

$$H_\alpha h_j(y) = C\int_0^{2^{j+1}} m_j^{(\lambda)}(u)\left(\frac{\sqrt{u}}{y}\right)^{\alpha+\lambda} J_{\alpha+\lambda}(\sqrt{u}y)du. \tag{9}$$

Write λ in the form $\lambda = \alpha + 1 + \varepsilon$. According to [3, Theorem 4b] we may assume that $0 < \varepsilon < 1$. In order to prove (7) it is sufficient to obtain two inequalities:

$$\int_r^{2r} |H_\alpha h_j(y)|^p y^{2\alpha+1} dy \le C\|m\|_{2,\lambda}^p (r\sqrt{2^j})^{-\varepsilon p} r^{(2\alpha+2)(1-p)}, \tag{10}$$

which will work for $\sqrt{2^j}r \ge 1$;

$$\int_r^{2r} |H_\alpha h_j(y)|^p y^{2\alpha+1} dy \le C\|m\|_\infty^p (r\sqrt{2^j})^{(2\alpha+2)p} r^{(2\alpha+2)(1-p)}, \tag{11}$$

which will be enough whenever $\sqrt{2^j}r < 1$.

To prove (10) observe that

$$H_\alpha h_j(y) = H_{2\alpha+1+\varepsilon} g_j(y),$$

where

$$g_j(y) = Cm_j^{(\lambda)}(y^2)\chi_{[0,\sqrt{2^{j+1}}]}(y).$$

Now Hölder's inequality, the Plancherel formula [5, p. 656] applied to g_j and (5) give

$$\begin{aligned}
&\int_r^{2r} |H_\alpha h_j(y)|^p y^{2\alpha+1} dy \\
&\le Cr^{-\varepsilon p} r^{(2\alpha+2)(1-p)}\left(\int_r^{2r} |H_\alpha h_j(y)|^2 (y^{2\alpha+1+\frac{1}{2}+\varepsilon})^2 dy\right)^{\frac{p}{2}} \\
&\le Cr^{-\varepsilon p} r^{(2\alpha+2)(1-p)}\left(\int_0^{2^{j+1}} |m_j^{(\lambda)}(x)|^2 x^{2\alpha+1+\varepsilon} dx\right)^{\frac{p}{2}} \\
&\le C\|m\|_{2,\lambda}^p (r\sqrt{2^j})^{-\varepsilon p} r^{(2\alpha+2)(1-p)}.
\end{aligned}$$

To prove (11) we use asymptotic properties of Bessel functions: $J_\alpha(x) = O(x^\alpha)$, $x \to 0^+$, $J_\alpha(x) = O(x^{-1/2})$, $x \to \infty$. First we get

$$|H_\alpha h_j(y)| \le ||m||_\infty \int_0^{\sqrt{2^{j+1}}} \frac{|J_\alpha(xy)|}{(xy)^\alpha} x^{2\alpha+1} dx \le C||m||_\infty (\sqrt{2^j})^{2\alpha+2}$$

and therefore

$$\int_r^{2r} |H_\alpha h_j(y)|^p y^{2\alpha+1} dy \le C||m||_\infty^p (r\sqrt{2^j})^{(2\alpha+2)p} r^{(2\alpha+2)(1-p)}.$$

This completes the proof of (10) and consequently the proof of the lemma. □

Lemma 3. *Let* $Tf(y) = \int_0^\infty K(y,z) f(z) z^{2\alpha+1} dz$, *where* K *is a bounded, measurable function on* R_+^2 *and* $f \in C_c^\infty(0,\infty)$. *Assume that the operator* T *satisfies the inequality*

$$||Tf||_{L_\alpha^{p,0}} \le A||f||_{L_\alpha^{p,0}}, \tag{12}$$

with A *independent of* f. *Then* $||Tf||^p_{L_\alpha^{p,a}}$ *is bounded by the sum of*

$$C\int_0^\infty \left(\int_{z<\frac{r}{4}} \left[\int_{r/2}^{2r} |K(y,z)|^p y^{2\alpha+1} dy \right]^{\frac{1}{p}} |f(z)| z^{2\alpha+1} dz \right)^p r^{a-1} dr, \tag{13}$$

$$CA||f||^p_{L_\alpha^{p,a}}, \tag{14}$$

$$C\int_0^\infty \left(\int_{z>4r} \left[\int_{r/2}^{2r} |K(y,z)|^p y^{2\alpha+1} dy \right]^{\frac{1}{p}} |f(z)| z^{2\alpha+1} dz \right)^p r^{a-1} dr, \tag{15}$$

where C *is a constant depending only on* a *and* p.

Proof. We closely follow the original proof of [8, Lemma 4.3]. Firstly, observe that

$$||Tf||^p_{L_\alpha^{p,a}} = \sum_{n=-\infty}^{\infty} \int_{2^n}^{2^{n+1}} \left| \int_0^\infty K(y,z) f(z) z^{2\alpha+1} dz \right|^p y^a y^{2\alpha+1} dy.$$

Fix n and write $f(z) = \sum_{j=1}^3 f_j(z)$, where $f_j \in C_c^\infty(0,\infty)$, $|f_j(z)| \le |f(z)|$ for $1 \le j \le 3$; $f_1(z) = 0$ for $z > 2^{n-2}$, $f_2(z) = 0$ for $z < 2^{n-3}$ and $z > 2^{n+4}$, $f_3(z) = 0$ for $z < 2^{n+3}$. Then $||Tf||^p_{L_\alpha^{p,a}}$ is bounded by 3^p times the sum of

$$\sum_{n=-\infty}^{\infty} \int_{2^n}^{2^{n+1}} \left| \int_0^\infty K(y,z) f_j(z) z^{2\alpha+1} dz \right|^p y^a y^{2\alpha+1} dy \tag{16}$$

for $1 \le j \le 3$.

To estimate (16) with $j = 1$, use Minkowski's integral inequality to get the bound

$$\sum_{n=-\infty}^{\infty}\left(\int_0^{2^{n-2}}\left[\int_{2^n}^{2^{n+1}}|K(y,z)f_1(z)|^p y^a y^{2\alpha+1}dy\right]^{1/p} z^{2\alpha+1}dz\right)^p.$$

This is bounded by $(\log 2)^{-1}$ times the sum of

$$\sum_{n=-\infty}^{\infty}\int_{2^n}^{2^{n+1}}\left(\int_{z<r/4}\left[\int_{r/2}^{2r}|K(y,z)f_1(z)|^p y^a y^{2\alpha+1}dy\right]^{\frac{1}{p}} z^{2\alpha+1}dz\right)^p \frac{dr}{r},$$

which is bounded by (13).

For (16) with $j = 2$ we have the bound

$$\sum_{n=-\infty}^{\infty} C2^{na}\int_{2^n}^{2^{n+1}}\left|\int_{2^{n-3}}^{2^{n+4}} K(y,z)f_2(z)z^{2\alpha+1}dz\right|^p y^{2\alpha+1}dy.$$

By (12) this is bounded by

$$\sum_{n=-\infty}^{\infty} C2^{na}A\int_{2^{n-3}}^{2^{n+4}}|f_2(z)|^p z^{2\alpha+1}dz,$$

whence we obtain (14).

Finally, (15) can be obtained from (16) using, for $j = 3$, an argument similar to the argument considered for $j = 1$. □

Lemma 4. *Let $m \in WBV_{2,\lambda}$ for some $\lambda > \alpha + 1$. Then for $1 < p < 2$, $(p-2)(\alpha+1) < a < (p-1)(2\alpha+2)$ and $f \in C_c^\infty(0,\infty)$,*

$$\left|\left|\sum_{j=-N}^{N}(H_\alpha h_j) * f\right|\right|_{L_\alpha^{p,a}} \le C||m||_{2,\lambda}||f||_{L_\alpha^{p,a}},$$

with C independent of f and $N = 1, 2, \ldots$.

Proof. To prove this we will apply Lemma 3 with $K(y,z) = \sum_{j=-N}^{N} T^z(H_\alpha h_j)(y)$. Note that if $(Tf)(y) := \int_0^\infty K(y,z)f(z)z^{2\alpha+1}dz$, $y > 0$, then $Tf = \sum_{j=-N}^{N} T_{h_j}$, where $T_{h_j}f = (H_\alpha h_j) * f$, for every $f \in C_c^\infty(0,\infty)$. Let us observe that the assumption of Lemma 3 holds. To be precise, if m belongs to $WBV_{2,\lambda}$ and $\lambda > \alpha + 1$ then

$$\int_t^\infty |H_\alpha h_j(y)|y^{2\alpha+1}dy \le C||m||_{2,\lambda}(\sqrt{2^j}t)^{\lambda-a-1},$$

and

$$\int_0^\infty |H_\alpha h_j(y)| y^{2\alpha+1} dy \leq C||m||_{2,\lambda}.$$

The above inequalities can be proven, from (5), along the same lines as in [7, pp. 284-285]. Then we obtain that

$$(17) \qquad || \sum_{j=-N}^{N} T_{h_j} f ||_{L^{p,0}_\alpha} \leq C||m||_{2,\lambda} ||f||_{L^{p,0}_\alpha},$$

where C is independent of $N = 1, 2, \ldots$, as it is done in [7] for integer λ. So we apply Lemma 3. To complete the proof, we will show that (13) and (15) are bounded by $C||m||^p_{2,\lambda} \cdot ||f||^p_{L^{p,a}_\alpha}$ (note that the constant A of (14) includes the factor $||m||_{2,\lambda}$, for K as above).

To estimate (13) observe that (3) implies

$$(18) \qquad T^z \chi_{[\frac{r}{2}, 2r]} \leq \chi_{[\frac{r}{4}, 2r+\frac{r}{4}]} \quad \text{for} \quad 0 < z < r/4.$$

Now (1), (2) with (18) lead to

$$\left(\int_{r/2}^{2r} | \sum_{j=-N}^{N} T^z (H_\alpha h_j)(y)|^p y^{2\alpha+1} dy \right)^{1/p}$$

$$\leq \sum_{j=-N}^{N} \left(\int_{r/2}^{2r} T^z (|H_\alpha h_j|^p)(y) y^{2\alpha+1} dy \right)^{1/p}$$

$$= \sum_{j=-N}^{N} \left(\int_0^\infty T^z \chi_{[r/2,2r]}(y) |H_\alpha h_j(y)|^p y^{2\alpha+1} dy \right)^{1/p}.$$

$$\leq \sum_{j=-N}^{N} \left(\int_{r/4}^{2r+r/4} |H_\alpha h_j(y)|^p y^{2\alpha+1} dy \right)^{1/p}.$$

By Lemma 2, (13) is bounded by

$$4^p C ||m||^p_{2,\lambda} \int_0^\infty \left(\int_{z<r/4} |f(z)| z^{2\alpha+1} dz \right)^p r^{(2\alpha+2)(1-p)} r^{a-1} dr.$$

Since $(2\alpha+2)(1-p) + a - 1 < -1$, Hardy's inequality [9, Lemma 3.14 p. 196] shows that (13) is bounded by $C||m||^p_{2,\lambda} \cdot ||f||^p_{L^{p,a}_\alpha}$ as desired.

To estimate (15), note that the assumption $(p-2)(\alpha+1) < a$ implies that $p(2\alpha+2)/(2\alpha+2+a) < 2$. We can, therefore, choose any q satisfying

$$\max\left(p, \frac{p(2\alpha+2)}{2\alpha+2+a}\right) < q < 2,$$

and use Hölder's inequality to obtain the bound

$$\left(\int_{r/2}^{2r}\Big|\sum_{j=-N}^{N} T^z(H_\alpha h_j)(y)\Big|^p y^{2\alpha+1}dy\right)^{1/p}$$
$$\leq \left(\int_{r/2}^{2r}\Big|\sum_{j=-N}^{N} T^z(H_\alpha h_j)(y)\Big|^q y^{2\alpha+1}dy\right)^{1/q} r^{(2\alpha+2)(1/p-1/q)}.$$

Observe that (3) implies

$$T^z\chi_{[\frac{r}{2},2r]} \leq \chi_{[\frac{z}{4},\frac{3}{2}z]} \quad \text{for} \quad z > 4r. \tag{19}$$

Now (1), (2) with (19) lead to

$$\left(\int_{r/2}^{2r}\Big|\sum_{j=-N}^{N} T^z(H_\alpha h_j)(y)\Big|^q y^{2\alpha+1}dy\right)^{1/q}$$
$$\leq \sum_{j=-N}^{N}\left(\int_{r/2}^{2r} T^z(|H_\alpha h_j|^q)(y)y^{2\alpha+1}dy\right)^{1/q}$$
$$= \sum_{j=-N}^{N}\left(\int_{0}^{\infty} T^z\chi_{[\frac{r}{2},2r]}|H_\alpha h_j(y)|^q y^{2\alpha+1}dy\right)^{1/q}$$
$$\leq \sum_{j=-N}^{N}\left(\int_{\frac{z}{4}}^{\frac{3z}{2}} |H_\alpha h_j(y)|^q y^{2\alpha+1}dy\right)^{1/q}.$$

By Lemma 2, (15) is bounded by

$$3^pC||m||_{2,\lambda}^p\int_0^\infty\left(\int_{z>4r}|f(z)|z^{2\alpha+1}z^{(2\alpha+2)(1/q-1)}dz\right)^p\frac{r^{(2\alpha+2)(1-p/q)}}{r^{1-a}}dr.$$

Now since $\frac{p(2\alpha+2)}{2\alpha+2+a} < q$, we have $(2\alpha+2)(1-p/q)+a-1 > -1$ and Hardy's inequality [9, Lemma 3.14 p. 196] shows that (15) is bounded by $C||m||_{2,\lambda}^p \cdot ||f||_{L_\alpha^{p,a}}^p$. This completes the proof of the Lemma. □

Let $M_\alpha^{p,a}$ denotes the space of all $L_\alpha^{p,a}$ multipliers for the modified Hankel transform H_α. We use the identity

$$M_\alpha^{p,a} = M_\alpha^{\frac{p}{p-1},-\frac{a}{p-1}},$$

which is a consequence of Plancherel formula and Hölder inequality, and derive the dual version of Lemma 4.

LEMMA 5. *Let $m \in WBV_{2,\lambda}$ for some $\lambda > \alpha + 1$. Then for $2 < p < \infty$, $-(2\alpha+2) < a < (p-2)(\alpha+1)$ and $f \in C_c^\infty(0,\infty)$*

$$\Big|\Big| \sum_{j=-N}^{N} (H_\alpha h_j) * f \Big|\Big|_{L_\alpha^{p,a}} \leq C||m||_{2,\lambda}||f||_{L_\alpha^{p,a}},$$

with C independent of f and $N = 1, 2, \ldots$.

3. Proof of Theorem 1

Proof. Firstly we will obtain the theorem for h. Then we will show how to deduce the thesis for the function m from the thesis for the function h.

As usual we denote the multiplier operator T_h defined by $H_\alpha(T_h f) = hH_\alpha f$ in the form

$$T_h = \lim_{N\to\infty} \sum_{j=-N}^{N} T_{h_j} \quad a.e., \qquad \text{where} \qquad T_{h_j} f = (H_\alpha h_j) * f.$$

Fix p and a satisfying the hypothesis. It is possible to choose p_0 such that $1 < p_0 < \min\{2, p\}$ and

$$(p_0 - 2)(\alpha+1) < \frac{a}{p}p_0 < (p_0 - 1)(2\alpha+2).$$

By Lemma 4

$$\left(\int_0^\infty \left|x^{\frac{a}{p}} \sum_{j=-N}^{N} (H_\alpha h_j) * f(x)\right|^{p_0} x^{2\alpha+1} dx\right)^{\frac{1}{p_0}} \leq C||m||_{2,\lambda} \left(\int_0^\infty \left|x^{\frac{a}{p}} f(x)\right|^{p_0} x^{2\alpha+1} dx\right)^{\frac{1}{p_0}}.$$

It is also possible to choose p_1 such that $\max\{2, p\} < p_1 < \infty$ and

$$-(2\alpha+2) < \frac{a}{p}p_1 < (p_1 - 2)(\alpha+1).$$

By Lemma 5

$$\left(\int_0^\infty \left|x^{\frac{a}{p}} \sum_{j=-N}^{N} (H_\alpha h_j) * f(x)\right|^{p_1} x^{2\alpha+1} dx\right)^{\frac{1}{p_1}} \leq C||m||_{2,\lambda} \left(\int_0^\infty \left|x^{\frac{a}{p}} f(x)\right|^{p_1} x^{2\alpha+1} dx\right)^{\frac{1}{p_1}}.$$

Interpolation of operators [9, Theorem 1.3, p. 179] shows that

$$\Big|\Big| \sum_{j=-N}^{N} (H_\alpha h_j) * f \Big|\Big|_{L^{p,a}_\alpha} \leq C ||m||_{2,\lambda} ||f||_{L^{p,a}_\alpha}.$$

Finally $N \to \infty$ completes the proof for the function h. The result for the function m follows then from the lemma below. □

LEMMA 6. *For $\beta > 0$ and $\lambda > 1/2$ the transformation $x \to x^\beta$ of $(0,\infty)$ induces the continuous isomorphism $m(x) \to m(x^\beta)$ of the $WBV_{2,\lambda}$ space.*

Proof. According to [3, Lemma 10 p. 255] we can restrict into the function m with compact support. Then for m with compact support we can write the reproducing formula

$$m(x) = \frac{1}{\Gamma(\lambda)} \int_0^\infty m^{(\lambda)}(u)(u-x)_+^{\lambda-1} du \tag{20}$$

[1]. Let (20) be the definition of the fractional derivative of function m of order λ. To complete the proof of the lemma it is sufficient to repeat exactly the proof of [2, Proposition 3.9, p. 325] changing the norm $||.||_{(\mu),2,1}$ of $AC^{\mu}_{2,1}$ into the norm $||.||_{2,\lambda}$ of $WBV_{2,\lambda}$. □

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