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A CHARACTERIZATION OF THE RIEMANNIAN SYMMETRIC SPACE $Sp(n)/U(n)$

By

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Abstract. We characterize the symmetric space $M = Sp(n)/U(n)$ by using the shape operator of small geodesic spheres in M, and a certain tensor field that satisfies various algebraic properties. We also give a partial generalization to any isotropy irreducible symmetric space.

1. Introduction

This work is a contribution to the problem of characterizing the isotropy irreducible symmetric spaces of classical type and their non-compact duals by small geodesic spheres. Historically, the problem was motivated by L. Vanhecke and T. J. Wilmore in [12] who characterized spaces of constant curvature and spaces of constant holomorphic curvature. The real oriented Grassmann manifolds $SO(p+q)/SO(p)\times SO(q)$ were considered later on by D. E. Blair and A. J. Ledger in [1], and B. J. Papantoniou in [9]. The complex Grassmann manifolds $SU(p+q)/S(U(p)\times U(q))$ were studied by A. J. Ledger in [5], who later on gave a unified treatment of all Grassmann manifolds including the quaternionic case $Sp(p+q)/Sp(p)\times Sp(q)$ ([6]). The symmetric space $SO(2n)/U(n)$ was characterized by A. J. Ledger and A. M. Shahin in [7], and in the sequel B. J. Papantoniou characterized the symmetric space $SU(n)/SO(n)$ in [11]. The cases left to be characterized are the symmetric spaces $Sp(n)/U(n)$, $SU(2n)/Sp(n)$, and the ones determined by various exceptional Lie groups.

The aim of this work is firstly, to give a characterization of the symmetric space $Sp(n)/U(n)$, and secondly to highlight a few key points which can be generalized for any symmetric space.

All the characterizations mentioned before used a property of geodesic spheres in Riemannian locally symmetric spaces. More specifically, let M be a Riemannian manifold of dimension at least three, S_r be a geodesic sphere with center a

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point $p \in M$ and radius r contained in a normal neighborhood U of p, and let N be a unit vector field on $U \setminus \{p\}$ tangent to a geodesic γ from p. Then for any vector field X on $U \setminus \{p\}$, we have that on γ the shape operator A_N of the geodesic sphere S_r and the curvature tensor R of M are related by

$$R(N,X)N = A_N^2 X - (\nabla_N A_N)X.$$

The left-hand side in the above equation is known as the curvature endomorphism $R_N : T_pM \to T_pM$ given by $R_N(X) = R(N,X)N$. This is a self-adjoint map and its restriction to the hyperplane orthogonal to N is referred to as tidal force operator (cf. [8, p. 219]) with special significance in general relativity. Now, a fundamental consequence of the previous relation is that if M is a Riemannian locally symmetric space the following well known result holds (e.g. [12], [6]):

PROPOSITION 1. *Let p be a point in a Riemannian locally symmetric space M of dimension at least 3. Then p has a normal neighborhood U such that for each unit vector $N \in T_pM$ and corresponding geodesic γ through p, the parallel translation of an eigenspace of the linear map R_N along γ is contained in an eigenspace of the shape operator A_N, for each geodesic sphere in U about p.*

Furthermore, these characterizations used certain properties of a parallel tensor field T of type $(1,3)$, and additionally in some cases of another parallel tensor field S of type $(1,2)$, defined as an appropriate portion of the curvature tensor R of M. The tensor field T plays a significant role in the geometry of Grassmann manifolds, somewhat analogous to the underlying almost complex structure on a Kähler manifold (cf. Proposition 2 (P3)).

We will begin by presenting various properties of the symmetric space $M = Sp(n)/U(n)$, and then we will express the curvature tensor of M in terms of the $(1,3)$ tensor field T satisfying various properties. Then we will select vectors N as in Proposition 1 that satisfy an extra geometrical condition, to give an expression of the shape operator A_N of geodesic spheres in M. It turns out that these properties characterize the symmetric space $M = Sp(n)/U(n)$.

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2. Properties of the symmetric space $Sp(n)/U(n)$

Le $M = G/K$ be the symmetric space $Sp(n)/U(n)$. The imbedding of $U(n)$ into $Sp(n)$ is given by $A+iB \mapsto \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$, where A, B are $n \times n$ real matrices. Let $\mathfrak{g} = \mathfrak{sp}(n)$ and $\mathfrak{k} = \mathfrak{u}(n)$ be the Lie algebras of $Sp(n)$ and $U(n)$ respectively, and let g be the G-invariant metric on M determined by the $\mathrm{Ad}^{G/K}$-invariant inner product on $\mathfrak{g}$ given by

$$\langle X, Y\rangle = -\frac{1}{4}\operatorname{tr} XY \quad (X, Y \in \mathfrak{g}). \tag{1}$$

Here $\mathrm{Ad}^{G/K}$ denotes the isotropy representation of K in the tangent space T_pM ($p \in M$). Since M is an isotropy irreducible space, g is an Einstein metric, that is, the Ricci curvature of M is a multiple of g. Consider the reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}'$, with respect to this inner product. Then $\mathfrak{m}'$ consists of all matrices of the form

$$\{i\begin{pmatrix} X_1 & X_2 \\ X_2 & -X_1 \end{pmatrix} : \ X_1, X_2 \text{ real } n\times n \text{ symmetric matrices}\},$$

which from now on it will be identified with the set

$$\mathfrak{m} = \{X = \begin{pmatrix} X_1 & X_2 \\ X_2 & -X_1 \end{pmatrix} : \ X_1, X_2 \text{ real } n\times n \text{ symmetric matrices}\}.$$

The tangent space at a fixed point $o = eK$ can be identified with $\mathfrak{m}$, and its dimension is $n(n+1)$. A G-invariant complex structure is determined by the $\mathrm{Ad}^{G/K}$-invariant operator J on $\mathfrak{m}$ given by $JX = \begin{pmatrix} -X_2 & X_1 \\ X_1 & X_2 \end{pmatrix}$. Also, since $\langle JX, JY\rangle = \langle X, Y\rangle$, the metric g is Hermitian with respect to J, and furthermore it is a G-invariant Kähler metric on M. The curvature tensor at $o \in M$ is given by

$$R(X,Y)Z = -[[X,Y],Z] = (YXZ + ZXY) - (XYZ + ZYX).$$

We note that for the non-compact dual the curvature tensor is the negative of the above expression. Let T be the $(1,3)$ tensor at o defined by

$$T(X,Y,Z) = XYZ + ZYX \qquad (X,Y,Z \in \mathfrak{m}).$$

This is an $\mathrm{Ad}^{G/K}$-invariant tensor on a symmetric space, hence it is a parallel tensor field on M (cf. [8, p. 326]). Consequently, R can be expressed in terms of T as

$$R(X,Y)Z = -T(X,Y,Z) + T(Y,X,Z).$$

For each $X, Y, Z \in \mathfrak{m}$ we define the following endomorphisms on $\mathfrak{m}$:

$$T_{XY} : \mathfrak{m} \to \mathfrak{m}, \quad T_{XY}(Z) = T(X,Y,Z)$$
$$T_Y^X : \mathfrak{m} \to \mathfrak{m}, \quad T_Y^X(Z) = T(X,Z,Y).$$

PROPOSITION 2. *The tensor T defined above satisfies the following properties:*

$$T(X,Y,Z) = T(Z,Y,X) \tag{P1}$$

$$JT(X,Y,Z) = T(JX,Y,Z) = -T(X,JY,Z) \tag{P2}$$

$$(i)\ JT_{XX}g = 0, \quad (ii)\ JT_{XX}T = 0 \tag{P3}$$

$$(i)\ \operatorname{tr} T_{XX} = 4(n+1)g(X,X) \tag{P4}$$

$$(ii)\ \operatorname{tr}(T_X^X)^2 = 4g(T_X^X X, X) + 16(g(X,X))^2.$$

Proof. Properties (P1) and (P2) can be easily verified. Concerning properties (P3), the $(1,1)$ tensor JT_{XX} ($X \in \mathfrak{m}$) is defined by $JT_{XX}Y = J(T_{XX}Y)$, and is viewed as a derivation on the tensor algebra at o. Conditions (i) and (ii) are understood as generalizations of the properties $\nabla_X g = 0$ on a Riemannian manifold, and $\nabla_X J = 0$ on a Kähler manifold. Next we prove properties (P4).

We introduce an orthonormal basis for $\mathfrak{m}$. Let E_{ij} be the $n\times n$ matrix with 1 in the (i,j)-position and zeros elsewhere, and let $e_{ij}^* = E_{ij} + E_{ji}$ $(1 \le i < j \le n)$. Define matrices

$$e_{ij} = \begin{pmatrix} e_{ij}^* & 0 \\ 0 & -e_{ij}^* \end{pmatrix} \quad (1 \le i < j \le n)$$

and

$$f_{ii} = \begin{pmatrix} E_{ii} & 0 \\ 0 & -E_{ii} \end{pmatrix} \quad (1 \le i \le n).$$

Then the set $\{e_{ij}, Je_{ij}, \sqrt{2}f_{ii}, \sqrt{2}Jf_{ii}\}$ constitutes an orthonormal basis of $\mathfrak{m}$ with respect to the inner product $\langle X,Y\rangle = \frac{1}{4}\operatorname{tr} XY$. We use property (P2) and the relation $T_{XX}Z = T(X,X,Z) = X^2Z + ZX^2$, to compute:

$$\begin{aligned}
\operatorname{tr} T_{XX} &= \sum_{i<j}\{\langle T_{XX}e_{ij}, e_{ij}\rangle + \langle T_{XX}Je_{ij}, Je_{ij}\rangle\} \\
&\quad + 2\sum_i\{\langle T_{XX}f_{ii}, f_{ii}\rangle + \langle T_{XX}Jf_{ii}, Jf_{ii}\rangle\} \\
&= 2\sum_{i<j}\langle T_{XX}e_{ij}, e_{ij}\rangle + 4\sum_i\langle T_{XX}f_{ii}, f_{ii}\rangle \\
&= \frac{1}{2}\sum_{i<j}\operatorname{tr}(X^2e_{ij} + e_{ij}X^2)e_{ij} + \sum_i \operatorname{tr}(X^2f_{ii} + f_{ii}X^2)f_{ii} \\
&= \sum_{i<j}\operatorname{tr} X^2e_{ij}^2 + 2\sum_i \operatorname{tr} X^2 f_{ii}^2 \\
&= \operatorname{tr}(X^2\sum_{i<j}e_{ij}^2) + 2\operatorname{tr}(X^2\sum_i f_{ii}^2).
\end{aligned}$$

Since $e_{ij}^2 = \begin{pmatrix} (e_{ij}^*)^2 & 0 \\ 0 & (e_{ij}^*)^2 \end{pmatrix}$ and $(e_{ij}^*)^2 = (E_{ij}+E_{ji})^2 = E_{ii}+E_{jj}$, we obtain that

$$\sum_{i<j} e_{ij}^2 = \sum_{i<j} \begin{pmatrix} E_{ii}+E_{jj} & 0 \\ 0 & E_{ii}+E_{jj} \end{pmatrix} = (n-1)I_{2n}$$

and

$$\sum_i f_{ii}^2 = \sum_i \begin{pmatrix} E_{ii}^2 & 0 \\ 0 & E_{ii}^2 \end{pmatrix} = \sum_i \begin{pmatrix} E_{ii} & 0 \\ 0 & E_{ii} \end{pmatrix} = I_{2n}.$$

Thus

$$\operatorname{tr} T_{XX} = (n-1)\operatorname{tr} X^2 + 2\operatorname{tr} X^2 = (n+1)\operatorname{tr} X^2 = 4(n+1)g(X,X),$$

and property (P4) (i) has been proven. For (P4) (ii), we use (P2) and the relation $T_X^X Z = T(X,Z,X) = 2XZX$ to compute:

$$\begin{aligned}
\operatorname{tr}(T_X^X)^2 &= \sum_{i<j}\{\langle (T_X^X)^2 e_{ij}, e_{ij}\rangle + \langle (T_X^X)^2 J e_{ij}, J e_{ij}\rangle\} \\
&\quad + 2\sum_i \{\langle (T_X^X)^2 f_{ii}, f_{ii}\rangle + \langle (T_X^X)^2 J f_{ii}, J f_{ii}\rangle\} \\
&= 2\sum_{i<j}\langle (T_X^X)^2 e_{ij}, e_{ij}\rangle + 4\sum_i \langle (T_X^X)^2 f_{ii}, f_{ii}\rangle \\
&= 2\sum_{i<j}\operatorname{tr}(X^2 e_{ij})^2 + 4\sum_i \operatorname{tr}(X^2 f_{ii})^2.
\end{aligned}$$

Now if $X = \begin{pmatrix} X_1 & X_2 \\ X_2 & -X_1 \end{pmatrix} \in \mathfrak{m}$, then

$$X^2 = \begin{pmatrix} X_1^2 + X_2^2 & X_1X_2 - X_2X_1 \\ X_2X_1 - X_1X_2 & X_2^2 + X_1^2 \end{pmatrix} = \begin{pmatrix} A & B \\ -B & A \end{pmatrix},$$

where $A = (a_{ij})$ is a symmetric $(a_{ij} = a_{ji})$ matrix, and $B = (b_{ij})$ is a skew-symmetric $(b_{ii} = 0, b_{ij} = -b_{ji})$ matrix. Then $X^2 e_{ij} = \begin{pmatrix} C & D \\ D & -C \end{pmatrix}$, where C is the $n \times n$ matrix with i and j columns the vectors $(a_{1j}, \dots, a_{nj})^t$ and $(a_{1i}, \dots, a_{ni})^t$ respectively, and zeros elsewhere. The matrix D has i and j columns the vectors $(-b_{1j}, \dots, -b_{nj})^t$ and $(-b_{1i}, \dots, -b_{ni})^t$ respectively, and zeros elsewhere. We also find that $X^2 f_{ii} = \begin{pmatrix} E & F \\ F & -E \end{pmatrix}$, where E is the $n \times n$ matrix with i-column $(a_{1i}, \dots, a_{ni})^t$ and zeros elsewhere, and F is the $n \times n$

matrix with i column $(-b_{1i},\dots,-b_{ni})^t$ and zeros elsewhere. Thus, we obtain that

$$\begin{aligned}\operatorname{tr}(X^2e_{ij})^2 &= 4(a_{ij}^2 + a_{ii}a_{jj} + b_{ij}^2) \text{ and}\\ \operatorname{tr}(X^2f_{ii})^2 &= 2(a_{ii}^2 + b_{ii}^2) = 2a_{ii}^2.\end{aligned}$$

We also find that the following relations hold:

$$\begin{aligned}\operatorname{tr} X^2 &= 2\operatorname{tr} A = 2\sum_i a_{ii}\\ (\operatorname{tr} A)^2 &= (\sum_i a_{ii})^2 = \sum_i a_{ii}^2 + \sum_{i<j}(2a_{ii}a_{jj})\\ \operatorname{tr} A^2 &= \sum_i a_{ii}^2 + \sum_{i<j} 2a_{ij}^2, \quad \operatorname{tr} B^2 = -2\sum_{i<j} b_{ij}^2\\ \operatorname{tr} X^4 &= \operatorname{tr} X^2X^2 = 2\operatorname{tr}(A^2 - B^2).\end{aligned}$$

Consequently,

$$\begin{aligned}\sum_{i<j}\operatorname{tr}(X^2e_{ij})^2 &= 2\sum_{i<j}(2a_{ij}^2 + 2a_{ii}a_{jj} + 2b_{ij}^2) + \sum_i(a_{ii}^2 - a_{ii}^2 + a_{ii}^2 - a_{ii}^2)\\ &= 2(\operatorname{tr} A^2 - \operatorname{tr} B^2 + (\operatorname{tr} A)^2 - 2\sum_i a_{ii}^2)\\ &= 2\operatorname{tr}(A^2 - B^2) + 2(\operatorname{tr} A)^2 - 4\sum_i a_{ii}^2\\ &= \operatorname{tr} X^4 + \frac{1}{2}(\operatorname{tr} X^2)^2 - 2\sum_i \operatorname{tr}(X^2f_{ii})^2\end{aligned}$$

and finally,

$$\operatorname{tr}(T_X^X)^2 = 2\operatorname{tr} X^4 + (\operatorname{tr} X^2)^2 = 4g(T_X^X X, X) + 16(g(X,X))^2. \qquad \square$$

Next, we identify $\mathfrak{m} \cong \mathfrak{m}'$ with the vector space $\operatorname{Sym}_n \mathbb{C}$ of all $n \times n$ complex symmetric matrices by means of the identification

$$\mathfrak{m} \ni X = \begin{pmatrix} X_1 & X_2\\ X_2 & -X_1\end{pmatrix} \mapsto \tilde{X} = X_1 + iX_2 \in \operatorname{Sym}_n \mathbb{C}.$$

Under these correspondences the inner product $\langle X, Y\rangle = \frac{1}{4}\operatorname{tr} XY$ corresponds to the inner product $\langle \tilde{X}, \tilde{Y}\rangle' = \frac{1}{2}\operatorname{Re}\operatorname{tr}\tilde{X}\overline{\tilde{Y}}$ on $\operatorname{Sym}_n \mathbb{C}$, and the tensor T corresponds to the tensor $\tilde{T}$ defined by

$$\tilde{T}(\tilde{X}, \tilde{Y}, \tilde{Z}) = \tilde{X}\overline{\tilde{Y}}\tilde{Z} + \tilde{Z}\overline{\tilde{Y}}\tilde{X}$$

also denoted by T. The complex structure J corresponds to $\tilde{J}\tilde{X} = i\tilde{X} = -X_2 + iX_1$ also denoted by J. Then the curvature tensor is expressed on $\mathrm{Sym}_n\,\mathbb{C}$ by

$$\begin{aligned} R(\tilde{X},\tilde{Y})\tilde{Z} &= -(\tilde{X}\overline{\tilde{Y}}\tilde{Z} + \tilde{Z}\overline{\tilde{Y}}\tilde{X}) + (\tilde{Y}\overline{\tilde{X}}\tilde{Z} + \tilde{Z}\overline{\tilde{X}}\tilde{Y}) \\ &= -T(\tilde{X},\tilde{Y},\tilde{Z}) + T(\tilde{Y},\tilde{X},\tilde{Z}). \end{aligned}$$

We will now make use of unit vectors N in $\mathfrak{m}$ that satisfy a relation of the form $T(X,X,X) = kX$ $(k > 0)$. Geometrically, these vectors are realized as critical points of the function $|T(X,X,X)|^2$ (cf. [10]). Their existence is guaranteed by the following proposition.

PROPOSITION 3. *Let N be a unit vector in $\mathfrak{m} \cong \mathrm{Sym}_n\,\mathbb{C}$. Then $T(N,N,N) = 4N$ if and only if the rank of N is 1.*

Proof. Let $T(N,N,N) = 4N$. Then $2N\overline{N}N = 4N$, thus $N\overline{N}N = 2N$. We set $N\overline{N} = A$. Then $A^2 = N\overline{N}N\overline{N} = 2N\overline{N} = 2A$, and since N is symmetric,

$$\overline{A}^t = (\overline{N}N)^t = N^t\overline{N}^t = N\overline{N} = A,$$

i.e. A is a Hermitian matrix. Hence there exists an $n \times n$ unitary matrix P such that $PA\overline{P}^t = D$, where $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ is real diagonal with $D^2 = PA\overline{P}^tPA\overline{P}^t = PA^2\overline{P}^t = 2PA\overline{P}^t = 2D$, and $\mathrm{tr}\,D = \mathrm{tr}\,A = \mathrm{tr}\,N\overline{N} = \mathrm{Re}\,\mathrm{tr}\,N\overline{N} = 2$. Since $\mathrm{diag}(d_1^2, d_2^2, \cdots, d_n^2) = \mathrm{diag}(2d_1, 2d_2, \cdots, 2d_n)$, each of the d_i's must be 0 or 2, and as $\mathrm{tr}\,D = 2$ we finally obtain that $D = \mathrm{diag}(2, 0, \ldots, 0)$. We now set $B = PN$. Then

$$DB = PA\overline{P}^tPN = PAN = PN\overline{N}N = 2PN = 2B,$$

therefore the matrix B has all entries zeros except the first row, so its rank is 1. Therefore, $1 = \mathrm{rk}\,B = \mathrm{rk}\,PN = \mathrm{rk}\,N$ (as $P\overline{P}^t = I$).

For the converse, assume that $\mathrm{rk}\,N = 1$. Then there exists an $n \times n$ unitary matrix Q such that

$$QNQ^t = \mathrm{diag}(a, 0, \ldots, 0) \qquad (a \in \mathbb{C}) \tag{2}$$

(cf. [3]). Since N is a unit matrix we have that $|a|^2 = 2$.

We finally obtain that

$$\begin{aligned} &T(N,N,N) \\ &= 2N\overline{N}N = 2Q^{-1}\,\mathrm{diag}(a,0,\cdots,0)\,\mathrm{diag}(\bar{a},0,\cdots,0)\,\mathrm{diag}(a,0,\cdots,0)(Q^t)^{-1} \\ &= 2Q^{-1}\,\mathrm{diag}(a|a|^2,0,\cdots,0)(Q^t)^{-1} = 4Q^{-1}\,\mathrm{diag}(a,0,\cdots,0)(Q^t)^{-1} = 4N. \end{aligned}$$

□

We now choose such an $N \in \mathfrak{m}$ and recall the self-adjoint linear map $R_N : \mathfrak{m} \to \mathfrak{m}$ given by $R_N(X) = R(N, X)N$. Then

$$\begin{aligned} R(N, JN)N &= -T(N, JN, N) + T(JN, N, N) \\ &= JT(N, N, N) + JT(N, N, N) \\ &= 2JT(N, N, N) = 8JN, \end{aligned}$$

so we conclude that if $N \in \mathfrak{m}$ is such that $T(N, N, N) = 4N$, then JN is an eigenvector of R_N. Applying now Proposition 1 we obtain the following:

PROPOSITION 4. *Let $p \in Sp(n)/U(n)$ and choose a normal neighborhood U of p as in Proposition 1. Then for each geodesic sphere S in U with center at p, and each unit normal vector N to S such that $T(N, N, N) = 4N$, the shape operator A_N of S satisfies the property*

$$A_N JN = f(N)JN \tag{P5}$$

for some $f(N) \in \mathbb{R}$.

3. A characterization of the symmetric space $Sp(n)/U(n)$

We can now state the main theorem:

THEOREM 5. *Let (M, g) be a non-flat, complete, simply connected Kähler manifold of dimension $n(n+1)$. Let T be a parallel tensor field of type $(1, 3)$ on M satisfying properties (P1)–(P4). Suppose that each point $p \in M$ has a normal neighborhood U such that for each geodesic sphere S in U centered at p, and for each unit normal vector N to S with $T(N, N, N) = 4N$, the shape operator of S satisfies (P5). Then M is homothetic to the Riemannian symmetric space $Sp(n)/U(n)$ or its non-compact dual.*

For the proof of this theorem we need the following proposition whose proof is based on a series of linear algebra arguments, and is similar to the one given in [7] and [11]. However, it is useful to summarize its central points modified to our problem.

PROPOSITION 6. *Let V be a real vector space of dimension $n(n+1)$ with complex structure J and Hermitian inner product $\langle\ ,\ \rangle$. Let T be a tensor of type $(1, 3)$ on V satisfying (P1)-(P4) with $\langle\ ,\ \rangle$ in place of g. Then there exists a linear isomorphism ϕ of V onto the real vector space $\mathrm{Sym}_n\,\mathbb{C}$ of all complex symmetric $n \times n$ matrices, which preserves inner products as well as the complex structures*

J and i on V and $\mathrm{Sym}_n\,\mathbb{C}$ *respectively. Furthermore, under this identification,* $JX = iX$, $T(X,Y,Z) = X\overline{Y}Z + Z\overline{Y}X$, *and* $\langle X, X\rangle = \frac{1}{2}\,\mathrm{tr}\,X\overline{X}$.

Proof. (Sketch) The aim is to exhibit a vector space isomorphism

$$\phi : V \to \mathrm{Sym}_n\,\mathbb{C} \tag{3}$$

by determining this between corresponding orthonormal bases in these spaces. It can been shown that there exists an orthonormal basis $\mathcal{A} = A \cup JA$ on V, such that $A = \{e_{jk}\ (1 \leq j < k \leq n), f_{ii}\ (1 \leq i \leq n)\}$ is an orthonormal subset of A, and $JA = \{Je_{jk}, Jf_{ii} : e_{jk}, e_{ii} \in A\}$. The elements of the set A are gradually defined so that the action of the tensor T on these satisfies various orthogonality relations (cf. [7, p. 17]). Next we choose an orthonormal basis $\mathcal{B}$ for $\mathrm{Sym}_n\,\mathbb{C}$ with respect to the inner product $\langle X, X\rangle' = \frac{1}{2}\,\mathrm{tr}\,X\overline{X}$, consisting of the matrices $\mathcal{B} = \{e'_{jk} = E_{jk} + E_{kj}, ie'_{jk}(1 \leq j < k \leq n), f'_{ii} = E_{ii}, if'_{ii}\ (1 \leq i \leq n)\}$, and define the isomorphism ϕ by

$$\phi(e_{jk}) = e'_{jk},\ \phi(Je_{jk}) = ie'_{jk},\ \phi(f_{ii}) = f'_{ii},\ \phi(Jf_{ii}) = if'_{ii}.$$

This isomorphism preserves inner products, as well as the complex structutes J and i on V and $\mathrm{Sym}_n\,\mathbb{C}$ respectively. Also, if we define a tensor T' of type $(1,3)$ on $\mathrm{Sym}_n\,\mathbb{C}$ by $T'(X,Y,Z) = X\overline{Y}Z + Z\overline{Y}X$, then with respect to the basis $\mathcal{B}$, T' satisfies properies (P1)-(P4) as well as the orthogonality relations satisfied by T. Furthermore, $\phi(T(X,Y,Z)) = T'(\phi(X), \phi(Y), \phi(Z))$ for all $X, Y, Z \in \mathcal{A}$, and this completes the proof. □

The following lemmas are also needed for the proof of Theorem 5.
Let $D = \{X \in V : T(X,X,X) = 4\langle X, X\rangle X\}$.

Lemma 7. *Let S be any tensor of type $(1,3)$ on V which satisfies the symmetry properties of the Riemannian curvature tensor including the first Bianchi identity. Suppose that S satisfies the relation*

$$\langle S(JX, JY)Z, W\rangle = \langle S(X,Y)Z, W\rangle \qquad \text{for all } X, Y, Z, W \in V, \tag{4}$$

and that for each $X \in D$ and $Y \in V$ which is orthogonal to X, the relation $\langle S(X, JX)X, JY\rangle = 0$ holds. Then the "holomorhic sectional curvature" determined by S (i.e. $K(X) = \langle S(X, JX)X, JX\rangle$) is constant on D.

Proof. We will show that $K(X)$ is constant for all unit vectors $X \in D$, by considering four cases.

Case 1 Let $Y \in D$ be a unit vector orthogonal to X such that $X + Y \in D$. Such vectors do exist, as by Proposition 3 we can write $X = \mathrm{diag}(x+iy, 0, \ldots, 0)$

$(x^2 + y^2 = 2)$, and then take $Y = JX = \mathrm{diag}(-y + ix, 0, \ldots, 0)$. Then it is clear that $X - Y \in D$ and is orthogonal to $X + Y$, so by hypothesis we get

$$\langle S(X+Y, J(X+Y))(X+Y), J(X-Y)\rangle = 0. \tag{5}$$

By using condition (4) on S together with the symmetry properties we obtain that

$$\langle S(X,Y)JZ, W\rangle = -\langle S(Z, JW)X, Y\rangle = \langle JS(X,Y)Z, W\rangle,$$

which implies that

$$\begin{aligned}
\langle S(X,JX)Y,JX\rangle = \langle S(X,JX)JY,X\rangle &= \langle S(X,JX)X,JY\rangle = 0\\
\langle S(X,JY)X,JX\rangle &= \langle S(X,JX)X,JY\rangle = 0\\
\langle S(Y,JX)X,JX\rangle &= \langle S(X,JX)X,JY\rangle = 0\\
\langle S(X,JY)Y,JY\rangle &= \langle S(Y,JY)Y,JX\rangle = 0\\
\langle S(X,JY)Y,JX\rangle &= \langle S(Y,JX)X,JY\rangle\\
\langle S(Y,JX)Y,JX\rangle &= \langle S(JY,X)JY,X\rangle\\
\langle S(Y,JY)X,JX\rangle &= \langle S(X,JX)Y,JY\rangle.
\end{aligned}$$

By expanding (5) and using the above identities we obtain that $K(X) = K(Y)$, i.e. K is constant for such Y's.

Case 2 Let $Y \in D$ be any unit vector with $X + Y \in D$. Choose a unit vector $Z \in D$ orthogonal to X and Y so that $X + Z \in D$ and $Y + Z \in D$. Then from Case 1 we obtain that $K(X) = K(Z) = K(Y)$. For example, for X as before, take $Y = \mathrm{diag}(\alpha + i\beta, 0, \ldots, 0)$ (appropriatelly normalized), and a $Z = \mathrm{diag}(r + is, 0, \ldots, 0)$ is found by solving the system $xr + ys = 2 = \alpha r + \beta s$ for r, s.

Case 3 Let $Y \in D$ be any unit vector orthogonal to X. For example, for X as in case 1, we may write $Y = \mathrm{diag}(\alpha + i\beta, 0, \ldots, 0)$ with $x\alpha + y\beta = 0$. Choose a $Z = \mathrm{diag}(r + is, 0, \ldots, 0) \in D$ with $-yr + sx = 0$. Then $X + Z \in D$, and by Case 2 we get that $K(X) = K(Z)$. On the other hand, Z is orthogonal to Y and $Y + Z \in D$, so by Case 1 $K(Y) = K(Z)$.

Case 4 Let Y be any unit vector in D. By choosing a $Z \in D$ orthogonal to X and Y, then from Case 3 it follows that $K(X) = K(Y)$. □

Lemma 8. *Let S be a tensor of type $(1,3)$ on V which satisfies the symmetry properties of the Riemannian curvature tensor including the first Bianchi identity, as well as relation (4). Suppose that $S(X,JX)X = 0$ for all $X \in D$, and $S(X,Y)T = 0$ for all $X, Y \in V$. Then $S = 0$ on V.*

The proof of this lemma is presented in several of the references cited (e.g. [7], [6]). Finally we also need the following:

LEMMA 9 ([4, pp. 261–262]). *Let M_1, M_2 be Riemannian symmetric spaces, and p_1, p_2 be points in M_1 and M_2 respectively. If there is a linear isometry $\phi : T_{p_1}M_1 \to T_{p_2}M_2$ that preserves curvatures, i.e. $\phi \circ R^1_p = R^2_{\phi(p)} \circ \phi$ for all $p \in T_{p_1}M_1$, then M_1 and M_2 are locally isometric.*

Proof of Theorem 5. Let $v \in T_pM$ be a unit vector satisfying $T(v,v,v) = 4v$ and let N be the unit tangent vector field to a geodesic γ through p with initial vector v. Since T is parallel then $T(N,N,N) = 4N$ along γ, and from property (P5) $A_N JN = fJN$ along $\gamma \setminus \{p\}$. Now, if Y is a parallel vector field along γ normal to N, then $g(R(N,JN)N, JY) = 0$ on $\gamma \setminus \{p\}$, and hence at p by continuity. Indeed, we use property (P5), the relation $R(N,X)N = A_N^2 X - (\nabla_N A_N)X$, and the Kähler condition for M, to compute:

$$\begin{aligned}
&R(N, JN)N \\
&= A_N^2 JN - (\nabla_N A_N)JN = A_N(fJN) - (\nabla_N(A_N JN) - A_N \nabla_N JN) \\
&= f^2 JN - (\nabla_N fJN - A_N J\nabla_N N) = f^2 JN - f'JN - f\nabla_N JN \\
&= (f^2 - f')JN.
\end{aligned}$$

Therefore,

$$\begin{aligned}
g(R(N,JN)N, JY) &= g((f^2 - f')JN, JY) = (f^2 - f')g(JN, JY) \\
&= (f^2 - f')g(N, Y) = 0.
\end{aligned}$$

Next, we view the tangent space T_pM as the vector space V in Proposition 6. Then the tensor T satisfies (P1)-(P4) at p, and as shown before $\langle R(X,JX)X, JY\rangle = 0$ for all $X \in D$ and Y orthogonal to X. Since property (4) in Lemma 7 is satisfied for $JX = iX$, we conclude that the holomorphic sectional curvature is constant at p for each unit vector $X \in D$, i.e. $R(X,JX)X = cJX$. Next we define the $(1,3)$-tensor

$$S(X,Y)Z = R(X,Y)Z - \frac{c}{4}(-T(X,Y,Z) + T(Y,X,Z)),$$

where $R'(X,Y)Z = -T(X,Y,Z) + T(Y,X,Z)$ is viewed by Proposition 6 as the curvature tensor of $Sp(n)/U(n)$. We check that S satisfies the conditions of Lemma 8. Condition (4) is obviously satisfied. Also, for each $X \in D$

$$\begin{aligned}
S(X,JX)X &= R(X,JX)X - \frac{c}{4}(-T(X,JX,X) + T(JX,X,X)) \\
&= cJX - \frac{c}{4}(2JT(X,X,X)) = cJX - cJX = 0,
\end{aligned}$$

and

$$S(X,Y)T = R(X,Y)T - \frac{c}{4}R'(X,Y)T = 0$$

Both terms above are zero; the first because T is parallel on M, and the second by using the algebraic properties of T on $Sp(n)/U(n)$. Hence we conclude that

$$R(X,Y)Z = \frac{c}{4}R'(X,Y)Z \qquad \text{on} \quad T_pM.$$

Note that the left-hand side above is the curvature tensor of M, and the right-hand side is the curvature tensor of $Sp(n)/U(n)$. Since p is an arbitray point in M we obtain that

$$R = FR' \qquad \text{on} \quad M \tag{6}$$

for some function F.

Since $Sp(n)/U(n)$ is an Einstein manifold (6) implies that the Ricci curvature Ric of M is given by Ric $= fg$ for some function f. Hence M is also an Einstein manifold (cf. [8, p. 96]). Therefore we obtain that

$$R = \frac{c}{4}R' \qquad \text{on} \quad M,$$

and $\nabla R = \frac{c}{4}\nabla R' = 0$, so (M,g) is a Riemannian locally symmetric space.

Since M is non-flat we assume that $c > 0$. By Proposition 6 there exists a linear isomorphism between the tangent spaces at any two points of M and $Sp(n)/U(n)$ that preserves inner products and curvature tensors. Hence, by Lemma 9 M and $Sp(n)/U(n)$ are locally isometric. Since M is complete and simply connected, M is globally isometric to $Sp(n)/U(n)$. If $c < 0$ we have the corresponding result for the non-compact dual of $Sp(n)/U(n)$.

It remains to obtain the equation $R(X,Y)Z = -T(X,Y,Z) + T(Y,X,Z)$ for a metric $\overline{g}$ homothetic to g. Define $\overline{g} = |\frac{c}{4}|g$ and $\overline{T}(X,Y,Z) = |\frac{c}{4}|T(X,Y,Z)$ on M. Then (P1)-(P5) are satisfied by $\overline{g}$ and $\overline{T}$, so the conditions of Theorem 5 are satisfied by these. Since the curvature tensor of $\overline{g}$ is unchanged by homotheties, we have that $R(X,Y)Z = \frac{c}{|c|}(-\overline{T}(X,Y,Z) + \overline{T}(X,Y,Z))$ for all vector fields X,Y,Z on M, and the proof has been completed. □

4. Remarks about the shape operator of geodesic spheres in a symmetric space

As shown in Proposition 4 an important role in the characterization described before was played by the shape operator of geodesic spheres in the symmetric space $Sp(n)/U(n)$. This operator has been used in more general studies (cf. [13], [14]). We will first describe the eigenspaces of the map R_N. Let $M = G/K$ be a symmetric space with symmetry θ. For simplicity we assume that M is of non-compact type. Considering the eigenspaces of θ with respect to the eigenvalues

1 and -1 we obtain the direct sum $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where $\mathfrak{k}$ is the Lie algebra of the subgroup K, and $\mathfrak{m}$, as usual, is identified with the tangent space of M at a fixed point $o \in M$. We fix a maximal Abelian subspace $\mathfrak{h}$ in $\mathfrak{m}$, and let α be a linear form on $\mathfrak{h}$. Define

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{h}\}.$$

A vector $\alpha \neq 0$ in the dual space $\mathfrak{h}^*$ is called a restricted root with respect to $\mathfrak{h}$ if $\mathfrak{g}_\alpha \neq 0$. Let R be the set of all restricted roots. It is known that

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\alpha \in R} \mathfrak{g}_\alpha$$

is a decomposition of the real semisimple Lie algebra $\mathfrak{g}$, where $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{h}$, and $\mathfrak{k}_0 = \mathfrak{g}_0 \cap \mathfrak{k}$. Concerning the decomposition above, for any $\alpha, \beta \in R \cup \{0\}$ we have that $\theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$ and $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$. We fix an element $\alpha \in R$ and let $\dim \mathfrak{g}_\alpha = m_\alpha$. Take a basis $\{X_1^\alpha, \ldots, X_{m_\alpha}^\alpha\}$ in $\mathfrak{g}_\alpha$, and consider the subspaces

$$\mathfrak{k}_\alpha = \sum_{i=1}^{m_\alpha} \mathbb{R}(X_i^\alpha + \theta(X_i^\alpha)), \quad \mathfrak{m}_\alpha = \sum_{i=1}^{m_\alpha} \mathbb{R}(X_i^\alpha - \theta(X_i^\alpha)).$$

Obviously $\mathfrak{k}_\alpha = \mathfrak{k}_{-\alpha}$ and $\mathfrak{m}_\alpha = \mathfrak{m}_{-\alpha}$. Let R^+ be the set of positive roots with respect to an arbitrary lexicographic ordering in $\mathfrak{h}$. Using the above relations we obtain the following decompositions of $\mathfrak{k}$ and $\mathfrak{m}$ with respect to the Killing form of $\mathfrak{g}$:

$$\mathfrak{k} = \mathfrak{h} \oplus \sum_{\alpha \in R^+} \mathfrak{k}_\alpha, \quad \mathfrak{m} = \mathfrak{h} \oplus \sum_{\alpha \in R^+} \mathfrak{m}_\alpha.$$

Now take a unit vector N in $\mathfrak{h}$ such that $\alpha(N)^2$ are different for each $\alpha \in R^+ \cup \{0\}$. We have the following:

Proposition 10. *The eigenspaces of the self-adjoint map $R_N : \mathfrak{m} \to \mathfrak{m}$ given by $R_N(X) = R(N, X)N$ are $\mathfrak{m}_\alpha$, with corresponding eigenvalues $\alpha(N)^2$ ($\alpha \in R^+ \cup \{0\}$).*

Proof. Without loss of generality let $X = X^\alpha - \theta(X^\alpha) \in \mathfrak{m}_\alpha$. We compute:

$$\begin{aligned} R_N(X) &= -[[N, X], N] = -[[N, X^\alpha - \theta(X^\alpha)], N] \\ &= -[[N, X^\alpha], N] + [[N, \theta(X^\alpha)], N] \\ &= -[\alpha(N)X^\alpha, N] + [\alpha(N)\theta(X^\alpha), N] \\ &= \alpha(N)^2 X^\alpha - \alpha(N)^2 \theta(X^\alpha) \\ &= \alpha(N)^2 (X^\alpha - \theta(X^\alpha)) = \alpha(N)^2 X. \quad \square \end{aligned}$$

As a consequence of this proposition we obtain the following result which is a generalization of Proposition 4 to any symmetric space of non-compact type.

PROPOSITION 11. *Let $M = G/K$ be a symmetric space of non-compact type and let U be a normal neighborhood of a point $o \in M$ as in Proposition 1. Then for each geodesic sphere S in U with center at o, and each unit vector N in $\mathfrak{h}$ such that $\alpha(N)^2$ are different for each $\alpha \in R^+ \cup \{0\}$, the shape operator A_N of S satisfies the property*

$$A_N(\mathfrak{m}_\alpha) = f(N)\mathfrak{m}_\alpha$$

for some $f(N) \in \mathbb{R}$.

It is unclear at this point, and worth of further investigation, what would be an analogue of the condition $T(N, N, N) = kN$ $(k > 0)$, and the effect of this on the eigenspaces of the shape operator A_N.

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