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AN APPLICATION OF A THREE CRITICAL POINTS THEOREM TO A SYSTEM OF DIFFERENTIAL EQUATIONS

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Abstract. In this paper we have applied a variant of Ricceri's three critical points theorem provided by Averna and Bonanno to establish an existence and multiplicity result for periodic solutions of a system of differential equations involving a real parameter.

1. Introduction

In this paper we consider the existence and multiplicity of periodic solutions of the system

$$\begin{gathered} \ddot{u} - A(t)u = \lambda b(t)V'(u) \quad t \in [0,T] \\ \dot{u}(T) - \dot{u}(0) = u(T) - u(0) = 0 \end{gathered} \tag{P}$$

where λ is a real parameter, $A \in L^{\infty}(\mathbb{R}, \mathbb{R}^{N\times N})$ is positive definite, $V : \mathbb{R}^N \to \mathbb{R}$, and $b : \mathbb{R} \to \mathbb{R}$.

Recently, Ricceri developed a critical points theorem [4] which we have used to prove results in [7]. For the reader's benefit, we state here the three critical points theorem of Ricceri's in [4]:

THEOREM 1.1. *Let X be a separable and reflexive real Banach space; $\phi : X \to \mathbb{R}$ a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*; $\psi : X \to \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact; $I \subseteq \mathbb{R}$ is an interval. Assume that*

i) $\displaystyle\lim_{\|x\|\to\infty} (\phi(x) + \lambda\psi(x)) = +\infty$ *for all* $\lambda \in I$.

ii) *And there exists a continuous concave function $h : I \to \mathbb{R}$ such that*

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$$\sup_{\lambda\in I}\inf_{x\in\mathbf{X}}(\phi(x)+h(\lambda)+\lambda\psi(x)) < \inf_{x\in\mathbf{X}}\sup_{\lambda\in I}(\phi(x)+h(\lambda)+\lambda\psi(x)).$$

Then there exists an open interval $J \subseteq I$ and a positive number ρ such that for each $\lambda \in J$ the equation

$$\phi'(x)+\lambda\psi'(x)=0$$

admits at least three distinct solutions whose norms are less than ρ.

The above Ricceri critical points theorem establishes a uniform upper bound for the norms of all eigen vectors corresponding to each admissible eigen value λ for which they are solutions of problem (P). However, this theorem, though a valuable tool, contains an inequality (ii) which is usually difficult to verify. The theorem has since been improved upon by authors like G. Bonanno [5], B. Ricceri [1], D. Averna and G. Bonanno [2].
We recall theorem B in [2] which we use to prove our main result:

THEOREM 1.2. *Let X be a reflexive Banach space, $\phi : X \to \mathbb{R}$ be continuously Gâteaux differentiable and sequentially weakly lower semi continuous, whose Gâteaux derivative has a continuous inverse on $X^*, \psi : X \to \mathbb{R}$ be continuously Gâteaux differentiable with a compact Gâteaux derivative.*
For $r \in \mathbb{R}$ let us define,

$$\varphi_1(r) = \inf_{x\in\phi^{-1}(]-\infty,r[)} \frac{\psi(x) - \inf_{\overline{\phi^{-1}(]-\infty,r[)}^{w}}\psi}{r-\phi(x)} \tag{1.1}$$

and

$$\varphi_2(r) = \inf_{x\in\phi^{-1}(]-\infty,r[)}\ \sup_{y\in\phi^{-1}([r,+\infty[)} \frac{\psi(x)-\psi(y)}{\phi(y)-\phi(x)} \tag{1.2}$$

where $\overline{\phi^{-1}(]-\infty,r[)}^{w}$ denotes the closure of $\phi^{-1}(]-\infty,r[)$ in the weak topology.
Assume that

i) $\lim_{\|x\|\to+\infty}(\phi(x)+\lambda\psi(x)) = +\infty$ *for all $\lambda \geq 0$;*
ii) *there exists $r \in \mathbb{R}, r > \inf_X \phi$, and $\varphi_1(r) < \varphi_2(r)$.*

Then for each $\lambda \in]\frac{1}{\varphi_2(r)}, \frac{1}{\varphi_1(r)}[$, the equation

$$\phi'(x)+\lambda\psi'(x)=0$$

has at least three solutions in X.

Remark. If $\varphi_1(r)=0$, $\frac{1}{0}$ is denoted as $+\infty$.

It is very interesting to note that in theorem 1.2, the circumvention of inequality ii) in theorem 1.1 sacrifices the uniform upper estimate (apriori bound) for norms of solutions of problem (P) for each admissible parameter λ. As we apply the variant theorem 1.2 here, a fascinating open problem is to prove an investigative tool which does not include inequality ii) of theorem 1.1 but preserves the apriori bound given in theorem 1.1. The difference between results obtained in [7] and in this paper is mainly in the apriori bounds for solutions for each admissible λ; however, the results in this paper are obtained under comparatively less stringent conditions on the nonlinearity.

We shall investigate solutions of (P) in the Sobolev space $W^{1,2}(0,T;\mathbb{R}^N) = H_T^1$ with the standard norm

$$\|u\| = \left(\int_0^T (|\dot{u}|^2 + |u|^2)\right)^{\frac{1}{2}} \quad \text{for all} \quad u \in H_T^1.$$

Since A is positive definite, we have an equivalent norm on H_T^1 given by

$$\|u\|_1 = \left(\int_0^T (|\dot{u}|^2 + (Au,u))\right)^{\frac{1}{2}} \quad \text{for all} \quad u \in H_T^1.$$

DEFINITION 1.3. We define the functions ϕ and ψ, respectively as follows:

$$\phi(u) = \frac{\|u\|_1^2}{2} \quad \text{and} \quad \psi(u) = \int_0^T b(t)V(u)dt \tag{1.3}$$

2. Main result

THEOREM 2.1. *Assume that*

i) $A = (a_{ij})$ *is a positive definite symmetric N-order matrix such that* $a_{ij} \in L^\infty(0,T), a_{ij}(T) = a_{ij}(0), i,j = 1,...,N.$
ii) $V \in C^1(\mathbb{R}^N,\mathbb{R})$, *with* $V(0) = 0$, *and* $\lim\limits_{|x|\to+\infty} V(x) = +\infty.$
iii) $b \in C^0(\mathbb{R},\mathbb{R})$ *and* $b(t) \leq 0$ *for all* $t \in [0,T], b \neq 0.$
iv) *There exist* $a > 0, s < 2 : V(x) \leq a(1+|x|^s)$ *for all* $x \in \mathbb{R}^N.$
v) *There exists* $\gamma > 0$ *so that for any* $x \in \mathbb{R}^N$, *where* $|x| \leq \gamma$, *we have* $V(x) \leq V(0) = 0.$

Then there is an unbounded interval I such that for every $\lambda \in I$, problem (P) admits at least three solutions.

Remark. With regards to condition ii), our result also holds if there exists $\delta > 0 : V(x) > 0 \quad a.e \quad |x| \geq \delta.$

It is obvious that the critical points of the functional

$$I_\lambda(x) = \phi(x) + \lambda\psi(x) \quad \text{for each real} \quad \lambda$$

correspond to solutions of (P).

LEMMA 2.2. *If* $b \in C[0,T]$ *is as in* (iii) *of theorem* 2.1, $V \in C(\mathbb{R}^N, \mathbb{R})$, *and condition* iv) *in Theorem* 2.1 *holds, then for each* $r > 0$, *there exists* $M_r > 0$ *such that*

$$\psi(x) \geq -M_r \quad \textit{for all} \quad x \in \phi^{-1}([0,r]).$$

Proof. From the definition of ψ in Definition 1.2 above, we have

$$\psi(x) = \int_0^T b(t)V(x).$$

So, for $|x| \leq 1$, it follows from condition iv) of theorem 2.1 that the proposition clearly holds, since then ψ is bounded below by a constant $2amT$, where $m = \min_{t\in[0,T]} b(t)$. So we assume that $|x| > 1$. Then from (iv),we have

$$\begin{aligned}\psi(x) \quad &\geq am\int_0^T (1+|x|^s) \\ &\geq 2amT\|x\|_\infty^s \\ &\geq 2^{\frac{s+2}{2}} amTc_\infty^s[\phi(x)]^{\frac{s}{2}},\end{aligned}$$

(It is well known that there exists $c_\infty > 0 : \|u\|_\infty \leq c_\infty \|u\|_1$, for all $u \in H_T^1$) and this completes the proof. □

LEMMA 2.3. *Given conditions ii), iii), and v) in Theorem 2.1, then there exists* $r > 0$ *such that* $\varphi_1(r) = 0$ *and* $\varphi_2(r) > 0$.

Proof. Since A is positive definite, there exists $\beta > 0$ such that $A(t)u.u \geq \beta|u|^2$ for all $t \in [0,T], u \in \mathbb{R}^N$. So, letting $\beta^* = \min\{1,\beta\}$ we can choose $c_\infty = \frac{1+T}{(T\beta^*)^{1/2}}$, and then given condition v) in theorem 2.1, we choose $r > 0 :$ $2c_\infty^2 r \leq \gamma^2$.

We shall show that there is a minimum point $x_r^* \in \phi^{-1}([0,r])$ such that $\phi(x_r^*) \neq r$; in other words, $\phi(x_r^*) < r$:
Since

$$\|x\|_\infty \leq \frac{1+T}{(T\beta^*)^{1/2}}\|x\|_1 \quad \text{for all} \quad x \in H_T^1,$$

it follows from condition (v) and the non-positiveness of $b(t)$ on [0,T] that for all vector functions (constant or non-constant) $x \in \phi^{-1}([0,r]), \psi(x) \geq \psi(0) = 0$. And so, $x_r^* = 0$ is a minimum point of ψ. Since by definition of ϕ, $\phi(x) \to +\infty$ as $\|x\|_1 \to +\infty$, we can fix $x^* \in H_T^1, \|x^*\|_1$ sufficiently large such that

$$\phi(0) < r < \phi(x^*).$$

The existence of such a point follows from condition (v) as well. Besides,

$$\inf_{\phi^{-1}([0,r])} \psi = \psi(0) = 0 > \psi(x^*).$$

Therefore, from the definition in (1.1), $\varphi_1(r) = 0$.
Furthermore,

$$\varphi_2(r) \geq \frac{\psi(x_r) - \psi(x^*)}{\phi(x^*)} > 0.$$

Thus, $\varphi_1(r) < \varphi_2(r)$.

This completes the proof. □

Remark.

a) The set $\phi^{-1}([0,r])$ is convex. So, if the functional ψ is strictly convex, then ψ has at most one minimum point on $\phi^{-1}([0,r])$ ([8], Theorem 38.C). Since we have not imposed conditions on the potential V which could make ψ strictly convex, we cannot claim unique existence of minimum point of ψ in $\phi^{-1}([0,r])$.

b) $\overline{\phi^{-1}(]0,r[)}^{w} = \phi^{-1}([0,r])$; and since ϕ is continuous and convex (and so weak sequentially continuous), the set $\phi^{-1}([0,r])$ is closed (or weak sequentially closed). Reflexivity of H_T^1 implies that every bounded sequence in it has a weakly convergent subsequence ([8], Proposition 38.2 (2)). Furthermore, ψ' is strongly continuous, and so ψ is weak sequentially continuous ([8], Corollary 41.9). It then follows from [8] (Corollary 38.9) that there is a minimum point $x_r \in \phi^{-1}([0,r])$ such that

$$\inf_{\phi^{-1}([0,r])} \psi = \psi(x_r).$$

Proof of theorem 2.1. To prove the assertion, it is sufficient to show that the assumptions of Theorem 1.2 are satisfied by ϕ and ψ . Clearly, ϕ and ψ as defined in equation (1.3) both satisfy the smoothness conditions in Theorem 1.2. For each real parameter $\lambda \geq 0$,

$$\begin{aligned}
\phi(x) + \lambda\psi(x) &= \frac{\|x\|_1^2}{2} + \lambda \int_0^T b(t)V(x)dt \\
&\geq \frac{\|x\|_1^2}{2} + a\lambda mT\|x\|_\infty^s + a\lambda mT \\
&\geq \frac{\|x\|_1^2}{2} - c\|x\|_1^{\frac{s}{2}} - d, \quad 0 < c,d < \infty.
\end{aligned}$$

Thus,

$$\lim_{\|x\|\to+\infty}(\phi(x)+\lambda\psi(x))=+\infty \quad \text{for all} \quad \lambda\geq 0.$$

$\phi(x)\geq 0$ for all $x\in H^1_T$; $\phi(0)=0$. Therefore, $\inf_{H^1_T}\phi=0$. Hence, for all $r>0$, $r>\inf_{H^1_T}\phi$.

Furthermore, we have shown in Lemma 2.3 that there exists $r>0$ such that

$$\varphi_2(r)=\inf_{x\in\phi^{-1}(]-\infty,r[)}\ \sup_{y\in\phi^{-1}([r,+\infty[)}\frac{\psi(x)-\psi(y)}{\phi(y)-\phi(x)}>\varphi_1(r)$$
$$=\inf_{x\in\phi^{-1}(]-\infty,r[)}\frac{\psi(x)-\inf_{\overline{\phi^{-1}(]-\infty,r[)}^{w}}\psi}{r-\phi(x)}=0.$$

And so

$$\varphi_2(r)>\varphi_1(r).$$

Therefore, for every $\lambda\in]\frac{1}{\varphi_2(r)},+\infty[$, the problem (P) admits at least three solutions. □

Example 1. Let us consider the system

$$\ddot{u}_i(t)-\sum_{k=1}^{N}a_{ik}(t)u_i(t)$$
$$=\lambda(t-1)\sin\pi t\left[\frac{4\left(1-\cos\left(\sum_{k=1}^{N}u_k^2(t)\right)\right)}{2\left(\sum_{k=1}^{N}u_k^2(t)\right)+1}+2\sin\left(\sum_{k=1}^{N}u_k^2(t)\right)\log_e(M(t))\right]u_i(t),$$

where $M(t)=\frac{1}{2}+\sum_{k=1}^{N}u_k^2(t)$.

$\dot{u}_i(1)-\dot{u}_i(0)=u_i(1)-u_i(0)=0$, $i=1,2,\ldots,N$. $(a_{ij}(t))^T=(a_{ij}(t))$ is positive definite, $i,j=1,\ldots,N$ for each $t\in[0,1]$ and $a_{ij}(0)=a_{ij}(1)$, while $b(t)=(t-1)\sin\pi t\leq 0$ on $[0,1]$.

Here,

$$V(u)=(1-\cos|u|^2)\log_e\left(\frac{1}{2}+|u|^2\right)\to+\infty \quad \text{as} \quad |u|\to+\infty,$$
$$(V(u)=0 \quad \text{if} \quad |u|^2=2k\pi, k\in\mathbb{N}).$$

and

$$V(u)\leq 2(1+\log_e(1+2|u|^2)).$$

Clearly, fixing $s : 1 < s < 2$, we choose $a = 4$ and thus have

$$V(u) < 4(1 + |u|^s) \quad \text{for all} \quad u \in \mathbb{R}^N.$$

Furthermore,

$$V(u) \leq V(0) = 0 \quad \text{for every} \quad u \in \mathbb{R}^N, |u| \leq 1/\sqrt{2}.$$

So, we choose $\gamma = 1/\sqrt{2}$.

3. conclusion

In a 3-D representation, the function V in the system (P) produces a prototype of a *juicer*; we refer to the real parameter λ as a *zooming control* since it regulates size. And from the scaling location of the *zooming control*, the parameter takes a minimum value $1/\varphi_2(r)$, while there is no maximum enlargement. So, in a way, we can say that the volume of the *juicer* can be maximized as much as desired while it cannot go below a certain value.

Fig. 1 below illustrates the concept.

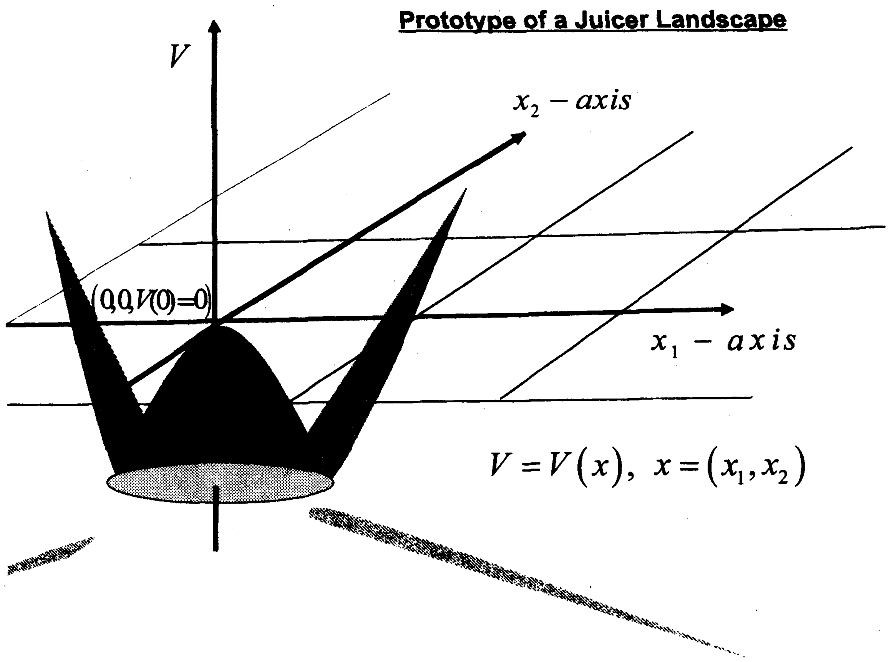

Figure 1 Juicer Landscape

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