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ON EXPONENTIALLY BOUNDED α-TIMES INTEGRATED C-COSINE FUNCTIONS

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Abstract. In this paper we apply some basic properties concerning α-times integrated C-cosine functions to deduce a characterization of an exponentially bounded α-times integrated C-cosine function in terms of its Laplace transform,and then use it to show that for each $x\in(\lambda^{2}-A)^{-1}CX$ the second order abstract Cauchy problem:
$u''(t)=Au(t)+\frac{t^{\alpha-1}}{\Gamma(\alpha)}x$ for $t>0, u(0)=u'(0)=0$ has a unique solution $u(\cdot)$ which satisfies $\|u(t)\|,\|u''(t)\|\in O(e^{\omega t})$ as $t\rightarrow\infty$ when the closed linear operator A : $D(A)\subset X\rightarrow X$ which generates an exponentially bounded α-times integrated C-cosine function $C(\cdot)$ on a Banach space X with $\|C(t)\|\leq Me^{\omega t}$ for all $t\geq 0$ and for some fixed $M, \omega\geq 0$.Moreover , we show that a closed linear operator in X generates an exponentially bounded α-times integrated C-cosine function on X also generates an exponentially bounded $\frac{\alpha}{2}$-times integrated C-semigroup on X.

1. Introduction

Let X be a Banach space with norm $\|\cdot\|$, and let $B(X)$ denote the set of all bounded linear operators from X into itself. For each $\alpha>0$, and $C\in B(X)$, a family $T(\cdot)(=\{T(t)|t\geq 0\})\subset B(X)$ is called an α-times integrated C-semigroup on X, if

(i) $T(\cdot)$ is strongly continuous. That is, for each $x\in X$, $T(\cdot)x:[0,\infty)\rightarrow X$ is continuous;

(ii) $T(\cdot)C=CT(\cdot)$. That is, $T(t)C=CT(t)$ on X for each $t\geq 0$;

(iii) $T(t)T(s)x=\frac{1}{\Gamma(\alpha)}[\int_{0}^{t+s}-\int_{0}^{t}-\int_{0}^{s}](t+s-r)^{\alpha-1}T(r)Cxdr$ for $x\in X$ and $t, s\geq 0$.

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Key words and phrases: α-times integrated C-cosine function,generator,abstract Cauchy problem (see [1, 5, 7, 8, 16, 17, 20, 21, 23]) Moreover, we say that $T(\cdot)$ is nondegenerate, if $x = 0$ whenever $T(t)x = 0$ for all $t \geq 0$. In this case, the closed linear operator $A : D(A) \subset X \to X$ defined by $D(A) = \{x | x \in X$ and there exists a $y_x \in X$ such that $T(t)x - \frac{t^\alpha}{\Gamma(\alpha+1)}Cx = \int_0^t T(s)y_x ds$ for all $t \geq 0\}$ and $Ax = y_x$ for all $x \in D(A)$, is called the (integral) generator of $T(\cdot)$. In general, we say that $T(\cdot)$ is exponentially bounded, if there exist $M, \omega \geq 0$ such that $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$. In this paper we consider the following two abstract Cauchy problems:

$$ACP_1(f,x) \begin{cases} u'(t) = Au(t) + f(t) \text{ for } t > 0, \\ u(0) = x, \end{cases}$$

and

$$ACP_2(f,x,y) \begin{cases} u''(t) = Au(t) + f(t) \text{ for } t > 0, \\ u(0) = x, \ u'(0) = y, \end{cases}$$

where x,$y \in X$ are given, $A : D(A) \subset X \longrightarrow X$ is a closed linear operator and f is an X-valued function defined on a subset of $\mathbb{R}$ containing $(0, \infty)$. The concept of (exponentially bounded) α-times integrated C-semigroups has been extensively applied to discuss the existence of (strong or weak) solutions to $ACP_1(f,x)$ (see [1–5, 7, 8, 16, 17, 20, 23]). Some equivalence conditions between the existence of an α-times integrated C-semigroup (or a C-semigroup) and the unique existence of (strong or weak) solutions of $ACP_1(f,x)$ are also deduced as in [9, 10]. Recently many authors have to study the relation between the existence of a C-cosine function (see [1,5-7,11,14,15,19]) or an α-times integrated C-cosine function for $\alpha \in \mathbb{N}$ (see [15,22,23]) and the existence of (strong or weak) solutions of $ACP_2(f,x,y)$. When $\alpha > 0$ is arbitrarily given , the formation of an α-times integrated C-cosine function has been constructed as in [12] which is presented as below : A family $C(\cdot)(= \{C(t)|t \geq 0\}) \subset B(X)$ is called an α-times integrated C-cosine function on X, if it is strongly continuous, $C(\cdot)C = CC(\cdot)$, and satisfies $2C(t)C(s)x = \frac{1}{\Gamma(\alpha)}\{[\int_0^{t+s} - \int_0^t - \int_0^s](t+s-r)^{\alpha-1}C(r)Cxdr + \int_{|t-s|}^t (s-t+r)^{\alpha-1}C(r)Cxdr + \int_{|t-s|}^s (t-s+r)^{\alpha-1}C(r)Cxdr + \int_0^{|t-s|}(|t-s|+r)^{\alpha-1}C(r)Cxdr\}$ for $x \in X$ and t,$s \geq 0$. In this case, its (integral) generator $A : D(A) \subset X \longrightarrow X$ is a closed linear operator in X defined by $D(A) = \{x|x \in X$ and there exists a $y_x \in X$ such that $C(t)x - \frac{t^\alpha}{\Gamma(\alpha+1)}Cx = \int_0^t \int_0^s C(r)y_x drds$ for all $t \geq 0\}$ and$Ax = y_x$ for all $x \in D(A)$ when $C(\cdot)$ is nondegenerate. Some results concerning $ACP_2(f,x,y)$ are also deduced in there and in [12,13],and examples of exponentially bounded α-times integrated C-semigroup and C-cosine function generated by partial differential operators given as in [8] and [25], respectively.

As in [9–11] for cases of C-cosine function andα-times integrated C-semigroup, we shall first prove a characterization of an exponentially bounded α-times integrated C-cosine function in terms of its Laplace transform, and then use it to show that if $C(\cdot)$ is a nondegenerate α-times integrated C-cosine function on X with generator A, and satisfies $\|C(t)\| \le Me^{\omega t}$ for all $t \ge 0$ and for some M, $\omega \ge 0$, then for each $\lambda > \omega$ and $x \in (\lambda^2 - A)^{-1}CX$, $ACP_2(j_{\alpha-1}(\cdot)x, 0, 0)$ has a unique (strong) solution $u(\cdot)$ which satisfies $\|u(t)\|,\|u''(t)\| \in O(e^{\omega t})$ as t approaches ∞, where $j_{\alpha-1}(t) = \dfrac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$. Moreover, we can also show that a closed linear operator A which generates an exponentially bounded α-times integrated C-cosine function on X also generates an exponentially bounded $\dfrac{\alpha}{2}$-times integrated C-semigroup on X.

2. Exponentially bounded α-times integrated C-cosine functions

In this section, we always assume that α is a positive number and $C \in B(X)$ is an injection, and first note some basic properties concerning α-times integrated C-cosine functions which have been deduced in [12] and frequently applied in this paper.

PROPOSITION 2.1 (see [12]). *Let A be the generator of a nondegenerate α-times integrated C-cosine function $C(\cdot)$ on X. Then*

$$C(t)x \in D(A),\ AC(t)x = C(t)Ax \text{ for all } x \in D(A) \text{ and } t \ge 0; \tag{2.1}$$

$$\int_0^t S(r)xdr \in D(A),\ A\int_0^t S(r)xdr = C(t)x - j_\alpha(t)Cx \tag{2.2}$$

$$\text{for all } t \ge 0 \text{ and } x \in X, \text{ where } S(r)x = \int_0^r C(s)xds;$$

$$C^{-1}AC = A; \tag{2.3}$$

$$C(\cdot) \text{ is uniquely determined.} \tag{2.4}$$

DEFINITION 2.2. A function $u : [0,\infty) \longrightarrow X$ is called a (strong) solution of $ACP_2(f,x,y)$, if $u \in C^2((0,\infty);X) \cap C^1([0,\infty);X) \cap C((0,\infty);[D(A)])$, and satisfies $ACP_2(x,y,f)$, where $[D(A)]$ denotes the Banach space $D(A)$ with the graph norm $|x|_A = \|x\| + \|Ax\|$.

PROPOSITION 2.3 (see [12]). *Let A be the generator of a nondegenerate α-times integrated C-cosine function $C(\cdot)$ on X and $C^1 = \{x \in X | C(\cdot)x$ is continuously differentiable on $(0,\infty)\}$. Then*

$$S(t)C^1 \subset D(A) \text{ for each } t \ge 0; \tag{2.5}$$

$$S(\cdot)x \text{ is the unique solution of } ACP_2(j_{\alpha-1}(\cdot)Cx, 0, 0) \text{ for each } x \in C^1; \tag{2.6}$$

$$S(\cdot)x \text{ is the unique solution of } ACP_2(j_{\alpha-1}(\cdot)Cx, 0, 0) \text{ in } C^2((0,\infty);X) \cap C^1([0,\infty);[D(A)]) \text{ for all } x \in D(A). \tag{2.7}$$

PROPOSITION 2.4 (see [12]). *Let A be the generator of a nondegenerate α-times integrated C-cosine function $C(\cdot)$ on X and $x \in X$. Assume that $C(t)x \in R(C)$ for all $t \geq 0$ and $C^{-1}C(\cdot)x$ is continuously differentiable on $(0, \infty)$. Then $C^{-1}S(t)x \in D(A)$ for all $t \geq 0$ and $C^{-1}S(\cdot)x$ is the unique solution of $ACP_2(j_{\alpha-1}(\cdot)x, 0, 0)$.*

Next we shall prove an important lemma which can be used to obtain the main result of this paper.

LEMMA 2.5. *Let A be the generator of a nondegenerate α-times integrated C-cosine function $C(\cdot)$ on X and $\widetilde{S}(t)x = \int_0^t S(r)xdr$ for all $x \in X$ and $t \geq 0$. For given $\lambda > 0$,we set D_λ to denote the set of all those $x \in X$ for which $L_\lambda x = \int_0^\infty e^{-\lambda t}C(t)xdt$ exists and both $\int_0^\infty e^{-\lambda t}\|S(t)x\|dt$ and $\int_0^\infty e^{-\lambda t}\|\widetilde{S}(t)x\|dt$ are finite. Then $L_\lambda D_\lambda \subset D(A)$ and $(\lambda^2 - A)\lambda^\alpha L_\lambda x = \lambda Cx$ for all $x \in D_\lambda$.*

Proof. Indeed, if $x \in D_\lambda$ is given,then from integration by parts, we have

$$\begin{aligned}\int_0^\tau e^{-\lambda t}C(t)xdt &= e^{-\lambda\tau}\int_0^\tau C(s)xds + \lambda e^{-\lambda\tau}\int_0^\tau\int_0^t C(s)xdsdt \\ &\quad + \lambda^2\int_0^\tau e^{-\lambda t}\int_0^t\int_0^s C(r)xdrdsdt \\ &= e^{-\lambda\tau}S(\tau)x + \lambda e^{-\lambda\tau}\widetilde{S}(\tau)x + \lambda^2\int_0^\tau e^{-\lambda t}\widetilde{S}(t)xdt,\end{aligned}$$

which converges to $\lambda^2\int_0^\infty e^{-\lambda t}\widetilde{S}(t)xdt$ as $\tau \to \infty$. Therefore, $L_\lambda x = \lambda^2\int_0^\infty e^{-\lambda t}\widetilde{S}(t)xdt$. It follows from (2.2) and the closedness of A that we have $\lambda^{\alpha+2}\int_0^\tau e^{-\lambda t}\widetilde{S}(t)xdt \in D(A)$ and

$$\begin{aligned}\lambda^{\alpha+2}A\int_0^\tau e^{-\lambda t}\widetilde{S}(t)xdt &= \lambda^{\alpha+2}\int_0^\tau e^{-\lambda t}[C(t) - \frac{t^\alpha}{\Gamma(\alpha+1)}Cx]dt \\ &\to \lambda^{\alpha+2}L_\lambda x - \lambda Cx \text{ as } \tau \to \infty.\end{aligned}$$

Again, from the closedness of A, we have $L_\lambda x \in D(A)$ and $\lambda^\alpha AL_\lambda x = \lambda^{\alpha+2}L_\lambda x - \lambda Cx$, or equivalently, $(\lambda^2 - A)\lambda^\alpha L_\lambda x = \lambda Cx$.

THEOREM 2.6. *Let $\{C(t)|t \geq 0\}$ be a strongly continuous family of bounded linear operators on X which satisfies $\|C(t)\| \leq Me^{\omega t}$ for all $t \geq 0$ and for some M, $\omega \geq 0$. For $\lambda > \omega$, we define $R(\lambda)x = \lambda^\alpha L_\lambda x = \lambda^\alpha\int_0^\infty e^{-\lambda t}C(t)xdt$ for $x \in X$. Then $C(\cdot)$ is an α-times integrated C-cosine function on X if and only if $(\lambda^2 - \mu^2)R(\mu)R(\lambda) = [\lambda R(\mu) - \mu R(\lambda)]C$ for all λ, $\mu > \omega$.*

Proof. As in the proof of [9, Proposition 2.2], we have

$$\frac{1}{\Gamma(\alpha)}\int_0^\infty\int_0^\infty e^{-\mu s}e^{-\lambda t}[\int_0^{t+s}-\int_0^t-\int_0^s](t+s-r)^{\alpha-1}C(r)Cxdrd(s,t)$$
$$=\lambda^{-\alpha}\mu^{-\alpha}(\lambda-\mu)^{-1}[R(\mu)Cx-R(\lambda)Cx] \text{ for all } x\in X \text{ and } t,s\geq 0.$$

Next we shall show that

$$\frac{1}{\Gamma(\alpha)}\iint_{t\geq s\geq 0} e^{-\mu s}e^{-\lambda t}[\int_{|t-s|}^t (s-t+r)^{\alpha-1}C(r)Cxdr$$
$$+\int_{|t-s|}^s (t-s+r)^{\alpha-1}C(r)Cxdr+\int_0^{|t-s|}(|t-s|+r)^{\alpha-1}C(r)Cxdr]d(s,t)$$
$$=\lambda^{-\alpha}\mu^{-\alpha}(\lambda+\mu)^{-1}R(\lambda)Cx+\lambda^{-\alpha}\mu^{-\alpha}(\lambda+\mu)^{-1}R(\mu)Cx$$
$$-\frac{1}{\Gamma(\alpha)}\int_0^\infty\int_r^\infty(\lambda+\mu)^{-1}t^{\alpha-1}e^{-\mu t}e^{-\lambda r}C(r)Cxdtdr$$
$$-\frac{1}{\Gamma(\alpha)}\int_0^\infty\int_0^r(\lambda+\mu)^{-1}t^{\alpha-1}e^{-\lambda t}e^{-\mu r}C(r)Cxdtdr.$$

Indeed, applying Fubini's theorem and change of variables for double integrals, we have

$$\int_0^\infty\int_0^\infty e^{-\mu s}\int_r^{r+s}e^{-\lambda t}(s-t+r)^{\alpha-1}C(r)Cxdtdrds$$
$$=\int_0^\infty\int_0^\infty e^{-\mu s}\int_r^{r+s}e^{-\lambda t}(s-t+r)^{\alpha-1}C(r)Cxdtdsdr$$
$$=\int_0^\infty\int_r^\infty e^{-\lambda t}\int_{t-r}^\infty e^{-\mu s}(s-t+r)^{\alpha-1}C(r)Cxdsdtdr$$
$$=\int_0^\infty\int_r^\infty e^{-\lambda t}\int_0^\infty e^{-\mu(s'-r+t)}s'^{\alpha-1}C(r)Cxds'dtdr$$
$$=\int_0^\infty\int_r^\infty e^{-\lambda t}e^{-\mu(-r+t)}\int_0^\infty e^{-\mu s'}s'^{\alpha-1}ds'C(r)Cxdtdr$$
$$=\int_0^\infty e^{\mu r}\int_r^\infty e^{-(\lambda+\mu)t}\mu^{-\alpha}\Gamma(\alpha)C(r)Cxdtdr$$
$$=\int_0^\infty \mu^{-\alpha}\Gamma(\alpha)e^{\mu r}\int_r^\infty e^{-(\lambda+\mu)t}dtC(r)Cxdr$$
$$=\int_0^\infty \mu^{-\alpha}\Gamma(\alpha)e^{\mu r}(\lambda+\mu)^{-1}e^{-(\lambda+\mu)r}C(r)Cxdr$$
$$=\mu^{-\alpha}(\lambda+\mu)^{-1}\Gamma(\alpha)\int_0^\infty e^{-\lambda r}C(r)Cxdr$$
$$=\Gamma(\alpha)\lambda^{-\alpha}\mu^{-\alpha}(\lambda+\mu)^{-1}R(\lambda)Cx$$

and

$$\int_0^\infty \int_0^s \int_r^s e^{-\mu s} e^{-\lambda t} (s-t+r)^{\alpha-1} C(r) Cx dt dr ds$$
$$= \int_0^\infty \int_r^\infty \int_r^s e^{-\mu s} e^{-\lambda t} (s-t+r)^{\alpha-1} C(r) Cx dt ds dr$$
$$= \int_0^\infty \int_r^\infty \int_t^\infty e^{-\mu s} e^{-\lambda t} (s-t+r)^{\alpha-1} C(r) Cx ds dt dr$$
$$= \int_0^\infty \int_r^\infty \int_0^\infty e^{-\mu(s'+t)} e^{-\lambda t} (s'+r)^{\alpha-1} C(r) Cx ds' dt dr$$
$$= \int_0^\infty \int_r^\infty \int_0^\infty e^{-(\lambda+\mu)t} e^{-\mu s'} (s'+r)^{\alpha-1} C(r) Cx ds' dt dr$$
$$= \int_0^\infty \int_0^\infty \int_r^\infty e^{-(\lambda+\mu)t} e^{-\mu s'} (s'+r)^{\alpha-1} C(r) Cx dt ds' dr$$
$$= \int_0^\infty \int_0^\infty \int_r^\infty e^{-(\lambda+\mu)t} dt e^{-\mu s'} (s'+r)^{\alpha-1} C(r) Cx ds' dr$$
$$= \int_0^\infty \int_0^\infty (\lambda+\mu)^{-1} e^{-(\lambda+\mu)r} e^{-\mu s'} (s'+r)^{\alpha-1} C(r) Cx ds' dr$$
$$= \int_0^\infty \int_r^\infty (\lambda+\mu)^{-1} e^{-(\lambda+\mu)r} e^{-\mu(s''-r)} s''^{\alpha-1} C(r) Cx ds'' dr$$
$$= \int_0^\infty \int_r^\infty (\lambda+\mu)^{-1} e^{-\lambda r} e^{-\mu s''} s''^{\alpha-1} C(r) Cx ds'' dr$$
$$= \int_0^\infty \int_r^\infty (\lambda+\mu)^{-1} t^{\alpha-1} e^{-\mu t} e^{-\lambda r} C(r) Cx dt dr.$$

Combining these equalities we obtain:

$$\frac{1}{\Gamma(\alpha)} \iint_{t\geq s\geq 0} e^{-\mu s} e^{-\lambda t} \int_{|t-s|}^t (s-t+r)^{\alpha-1} C(r) Cx dr d(s,t)$$
$$= \frac{1}{\Gamma(\alpha)} \iint_{t\geq s\geq 0} e^{-\mu s} e^{-\lambda t} \int_{t-s}^t (s-t+r)^{\alpha-1} C(r) Cx dr d(s,t)$$
$$= \frac{1}{\Gamma(\alpha)} \int_0^\infty \int_s^\infty \int_{t-s}^t e^{-\mu s} e^{-\lambda t} (s-t+r)^{\alpha-1} C(r) Cx dr dt ds$$
$$= \frac{1}{\Gamma(\alpha)} \int_0^\infty \int_s^\infty \int_r^{r+s} e^{-\mu s} e^{-\lambda t} (s-t+r)^{\alpha-1} C(r) Cx dt dr ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_0^\infty \int_0^s \int_s^{r+s} e^{-\mu s} e^{-\lambda t} (s-t+r)^{\alpha-1} C(r) Cx dt dr ds$$
$$= \frac{1}{\Gamma(\alpha)} \int_0^\infty \int_0^\infty \int_r^{r+s} e^{-\mu s} e^{-\lambda t} (s-t+r)^{\alpha-1} C(r) Cx dt dr ds$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha)}\int_0^\infty\int_0^s\int_s^r e^{-\mu s}e^{-\lambda t}(s-t+r)^{\alpha-1}C(r)Cxdtdrds\\
&=\lambda^{-\alpha}\mu^{-\alpha}(\lambda+\mu)^{-1}R(\lambda)Cx\\
&-\frac{1}{\Gamma(\alpha)}\int_0^\infty\int_r^\infty(\lambda+\mu)^{-1}t^{\alpha-1}e^{-\mu t}e^{-\lambda r}C(r)Cxdtdr,
\end{aligned}
$$

which together with the fact

$$
\begin{aligned}
&\iint_{t\geq s\geq 0} e^{-\mu s}e^{-\lambda t}[\int_{|t-s|}^s(t-s+r)^{\alpha-1}C(r)Cxdr\\
&\quad+\int_0^{|t-s|}(|t-s|+r)^{\alpha-1}C(r)Cxdr]d(s,t)\\
&=\iint_{t\geq s\geq 0} e^{-\mu s}e^{-\lambda t}\int_0^s(t-s+r)^{\alpha-1}C(r)Cxdrd(s,t)\\
&=\int_0^\infty\int_s^\infty\int_0^s e^{-\mu s}e^{-\lambda t}(t-s+r)^{\alpha-1}C(r)Cxdrdtds\\
&=\int_0^\infty\int_0^s\int_s^\infty e^{-\mu s}e^{-\lambda t}(t-s+r)^{\alpha-1}C(r)Cxdtdrds\\
&=\int_0^\infty\int_0^s\int_0^\infty(t'+r)^{\alpha-1}e^{-\lambda(t'+s)}e^{-\mu s}C(r)Cxdt'drds\\
&=\int_0^\infty\int_0^\infty\int_0^s(t'+r)^{\alpha-1}e^{-(\lambda+\mu)s}e^{-\lambda t'}C(r)Cxdrdt'ds\\
&=\int_0^\infty\int_0^\infty\int_0^s(t'+r)^{\alpha-1}e^{-(\lambda+\mu)s}e^{-\lambda t'}C(r)Cxdrdsdt'\\
&=\int_0^\infty\int_0^\infty\int_r^\infty(t'+r)^{\alpha-1}e^{-(\lambda+\mu)s}e^{-\lambda t'}C(r)Cxdsdrdt'\\
&=\int_0^\infty\int_0^\infty\int_r^\infty(t'+r)^{\alpha-1}e^{-(\lambda+\mu)s}e^{-\lambda t'}C(r)Cxdsdt'dr\\
&=\int_0^\infty\int_0^\infty\int_r^\infty e^{-(\lambda+\mu)s}ds(t'+r)^{\alpha-1}e^{-\lambda t'}C(r)Cxdt'dr\\
&=\int_0^\infty\int_0^\infty(\lambda+\mu)^{-1}e^{-(\lambda+\mu)r}(t'+r)^{\alpha-1}e^{-\lambda t'}C(r)Cxdt'dr\\
&=\int_0^\infty\int_r^\infty(\lambda+\mu)^{-1}e^{-(\lambda+\mu)r}e^{-\lambda(t-r)}t^{\alpha-1}C(r)Cxdtdr\\
&=(\lambda+\mu)^{-1}\int_0^\infty\int_r^\infty e^{-\mu r}e^{-\lambda t}t^{\alpha-1}C(r)Cxdtdr\\
&=(\lambda+\mu)^{-1}[\int_0^\infty\int_0^\infty e^{-\mu r}e^{-\lambda t}t^{\alpha-1}C(r)Cxdtdr
\end{aligned}
$$

$$- \int_0^\infty \int_0^r e^{-\mu r} e^{-\lambda t} t^{\alpha-1} C(r)Cxdtdr]$$
$$= (\lambda+\mu)^{-1}\lambda^{-\alpha}\mu^{-\alpha}\Gamma(\alpha)R(\mu)Cx$$
$$- (\lambda+\mu)^{-1} \int_0^\infty \int_0^r e^{-\mu r} e^{-\lambda t} t^{\alpha-1} C(r)Cxdtdr$$

implies that

$$\frac{1}{\Gamma(\alpha)} \iint\limits_{t\geq s\geq 0} e^{-\mu s} e^{-\lambda t} [\int_{|t-s|}^t (s-t+r)^{\alpha-1} C(r)Cxdr$$
$$+ \int_{|t-s|}^s (t-s+r)^{\alpha-1} C(r)Cxdr + \int_0^{|t-s|} (|t-s|+r)^{\alpha-1} C(r)Cxdr] d(s,t)$$
$$= \lambda^{-\alpha}\mu^{-\alpha}(\lambda+\mu)^{-1}R(\lambda)Cx + \lambda^{-\alpha}\mu^{-\alpha}(\lambda+\mu)^{-1}R(\mu)Cx$$
$$- \frac{1}{\Gamma(\alpha)} \int_0^\infty \int_r^\infty (\lambda+\mu)^{-1} t^{\alpha-1} e^{-\mu t} e^{-\lambda r} C(r)Cxdtdr$$
$$- \frac{1}{\Gamma(\alpha)} \int_0^\infty \int_0^r (\lambda+\mu)^{-1} t^{\alpha-1} e^{-\lambda t} e^{-\mu r} C(r)Cxdtdr.$$

Similarly, we can show that

$$\frac{1}{\Gamma(\alpha)} \iint\limits_{s\geq t\geq 0} e^{-\mu s} e^{-\lambda t} [\int_{|t-s|}^t (s-t+r)^{\alpha-1} C(r)Cxdr$$
$$+ \int_{|t-s|}^s (t-s+r)^{\alpha-1} C(r)Cxdr + \int_0^{|t-s|} (|t-s|+r)^{\alpha-1} C(r)Cxdr] d(s,t)$$
$$= \lambda^{-\alpha}\mu^{-\alpha}(\lambda+\mu)^{-1}R(\lambda)Cx + \lambda^{-\alpha}\mu^{-\alpha}(\lambda+\mu)^{-1}R(\mu)Cx$$
$$- \frac{1}{\Gamma(\alpha)} \int_0^\infty \int_r^\infty (\lambda+\mu)^{-1} t^{\alpha-1} e^{-\lambda t} e^{-\mu r} C(r)Cxdtdr$$
$$- \frac{1}{\Gamma(\alpha)} \int_0^\infty \int_0^r (\lambda+\mu)^{-1} t^{\alpha-1} e^{-\mu t} e^{-\lambda r} C(r)Cxdtdr.$$

Consequently, $C(\cdot)$ is an α-times integrated C-cosine function on X if and only if $(\lambda^2-\mu^2)R(\mu)R(\lambda) = [\lambda R(\mu) - \mu R(\lambda)]C$ for all λ, $\mu > \omega$.

As an application of Theorem 2.6, we can deduce the following characterization of an exponentially bounded α-times integrated C-cosine function in terms of its Laplace transform, which has been deduced by [15,22] when $\alpha = n$.

THEOREM 2.7. *A strongly continuous family $C(\cdot)$ with $\|C(t)\| \leq Me^{\omega t}$ for all $t \geq 0$ and for some M, $\omega \geq 0$,is a nondegenerate α-times integrated C-cosine* *function on X with generator A if and only if $CC(\cdot) = C(\cdot)C$, $C^{-1}AC = A$, and $\lambda^2 - A$ is injective, $R(C) \subset R(\lambda^2 - A)$, and for each $\lambda > \omega$*

$$\lambda^\alpha L_\lambda(\lambda^2 - A) \subset \lambda^\alpha(\lambda^2 - A)L_\lambda = \lambda C. \tag{2.8}$$

Proof. Indeed, if $C(\cdot)$ is a nondegenerate α-times integrated C-cosine function on X with the generator A, then for large λ,the set D_λ as defined in Lemma 2.5 is clearly equal to X,which together with (2.1) yields that for each $\lambda > \omega$, $\lambda^2 - A$ is injective, $L_\lambda \in B(X)$, $R(L_\lambda) \subset D(A)$, $R(C) \subset R(\lambda^2 - A)$, and (2.8) holds.

Conversely, suppose that $CC(\cdot) = C(\cdot)C$, $C^{-1}AC = A$, and for each $\lambda > \omega$, $\lambda^2 - A$ is injective, $R(C) \subset R(\lambda^2 - A)$ and (2.8) holds. Then $R(\lambda)(\lambda^2 - A) \subset (\lambda^2 - A)R(\lambda) = \lambda C$. Since

$$\begin{aligned}\lambda R(\mu)C - \mu R(\lambda)C &= R(\mu)\lambda C - \mu C R(\lambda) \\ &= R(\mu)(\lambda^2 - A)R(\lambda) - R(\mu)(\mu^2 - A)R(\lambda) \\ &= (\lambda^2 - \mu^2)R(\mu)R(\lambda),\end{aligned}$$

Theorem 2.6 implies that $C(\cdot)$ is an α-times integrated C-cosine function on X. Since $\lambda^2 - A$ and C are injective, we conclude from (2.8) that $C(\cdot)$ is nondegenerate. Now let B denote its generator. Then the " only if " part of this theorem asserts that $C^{-1}BC = B$, and for each $\lambda > \omega$, $\lambda^2 - B$ is injective, $R(C) \subset R(\lambda^2 - B)$, and $R(\lambda)(\lambda^2 - B) \subset (\lambda^2 - B)R(\lambda) = \lambda C$. Next if $x \in D(A)$ is given, then $\lambda Cx = R(\lambda)(\lambda^2 - A)x \in D(B)$ and $\lambda(\lambda^2 - B)Cx = (\lambda^2 - B)R(\lambda)(\lambda^2 - A)x = \lambda C(\lambda^2 - A)x$, so that $x \in D(C^{-1}BC) = D(B)$ and $Ax = C^{-1}BCx = Bx$. Hence $A \subset B$. By symmetry, we also have $B \subset A$. This completes the proof of this theorem.

Combining Theorem 2.7 with Proposition 2.4, we can deduce the following theorem.

THEOREM 2.8. *Let A be the generator of an exponentially bounded α-times integrated C-cosine function $C(\cdot)$ on X with $\|C(t)\| \le Me^{\omega t}$ for all $t \ge 0$ and for some M, $\omega \ge 0$, and let $\lambda > \omega$. Then for each $x \in (\lambda^2 - A)^{-1}CX = \lambda^{\alpha-1}L_\lambda X$, $u(\cdot) = C^{-1}S(\cdot)x$ is the unique solution of $ACP_2(j_{\alpha-1}(\cdot)x, 0, 0)$. Moreover, $\|u(t)\|, \|u''(t)\| \in O(e^{\omega t})$ as $t \to \infty$.*

Proof. From Proposition2.4, it suffices to show that $C(t)x \in R(C)$ and $C^{-1}C(\cdot)x \in C^1([0,\infty); X)$ for each $x \in (\lambda^2 - A)^{-1}CX$. Indeed, if $x = (\lambda^2 - A)^{-1}Cy = \lambda^{\alpha-1}L_\lambda y$ for some $y \in X$, then

$$\begin{aligned}C(t)x &= C(t)\lambda^{\alpha-1}L_\lambda y \\ &= \lambda^{\alpha-1}L_\lambda C(t)y\end{aligned}$$

$$
\begin{aligned}
&= \lambda^{\alpha-1}[\int_0^\infty e^{-\lambda s} C(s)C(t)yds] \\
&= \lambda^{\alpha-1} \int_0^\infty e^{-\lambda s} \frac{1}{2\Gamma(\alpha)} \{[\int_0^{t+s} - \int_0^t - \int_0^s](t+s-r)^{\alpha-1} C(r)Cydr \\
&\quad + \int_{|t-s|}^t (s-t+r)^{\alpha-1} C(r)Cydr + \int_{|t-s|}^s (t-s+r)^{\alpha-1} C(r)Cydr \\
&\quad + \int_0^{|t-s|} (|t-s|+r)^{\alpha-1} C(r)Cydr\} ds \\
&= \lambda^{\alpha-1} C \int_0^\infty e^{-\lambda s} \frac{1}{2\Gamma(\alpha)} \{[\int_0^{t+s} - \int_0^t - \int_0^s](t+s-r)^{\alpha-1} C(r)ydr \\
&\quad + \int_{|t-s|}^t (s-t+r)^{\alpha-1} C(r)ydr + \int_{|t-s|}^s (t-s+r)^{\alpha-1} C(r)ydr \\
&\quad + \int_0^{|t-s|} (|t-s|+r)^{\alpha-1} C(r)ydr\} ds \\
&\in R(C),
\end{aligned}
$$

so that
$C(t)x - j_\alpha(t)Cx = A\int_0^t S(r)xdr = AC\int_0^t C^{-1}S(r)xdr = CA\int_0^t C^{-1}S(r)xdr.$
Hence $C^{-1}C(t)x - j_\alpha(t)x = A\int_0^t C^{-1}S(r)xdr$. Since $(\lambda^2 - A)x = Cy$, we have $Ax = \lambda^2 x - Cy$ and $C(t)x - j_\alpha(t)Cx = \int_0^t S(r)Axdr = \int_0^t S(r)(\lambda^2 x - Cy)dr = \lambda^2 \int_0^t S(r)xdr - C\int_0^t S(r)ydr$, which implies that $C^{-1}C(t)x - j_\alpha(t)x = \lambda^2 \int_0^t C^{-1}S(r)xdr - \int_0^t S(r)ydr$. Consequently, $C^{-1}C(\cdot)x = j_\alpha(\cdot)x + \lambda^2 \int_0^\cdot C^{-1} S(r)xdr - \int_0^\cdot S(r)ydr \in C^1([0,\infty);X)$.

3. Connection between integrated C-semigroups and integrated C-cosine functions

In the following we shall deduce that the generator of a 2α-times integrated C-cosine function on Banach space X also generates an α-times integrated C-semigroup on X which has been known when $\alpha \in \mathbb{N}$ (see [1, 15]).

Definition 3.1. A function $u : [0,\infty) \longrightarrow X$ is called a (strong) solution of $ACP_1(f,x)$, if $u \in C^1((0,\infty);X) \cap C([0,\infty);X) \cap C((0,\infty);[D(A)])$ and satisfies

$ACP_1(f, x)$.

THEOREM 3.2. *Let A be the generator of an exponentially bounded α-times integrated C-cosine function $C(\cdot)$ on X with $\|C(t)\| \leq Me^{\omega t}$ for all $t \geq 0$ and for some M, $\omega \geq 0$. Assume that $K_\alpha = \dfrac{2^{1-\alpha}\Gamma(\alpha+1)}{\Gamma(\frac{\alpha+1}{2})\Gamma(\frac{\alpha}{2}+1)}$ and $T(t)x = K_\alpha \int_0^\infty e^{-u^2} C(2t^{\frac{1}{2}}u)xdu$ for all $t \geq 0$ and $x \in X$. Then $T(\cdot)$ is an exponentially bounded $\frac{\alpha}{2}$-times integrated C-semigroup on X with generator A , and satisfies $\|T(t)\| \leq Ke^{\omega^2 t}$ for all $t \geq 0$ and for some $K \geq 0$.*

Proof. We shall first show that $T(\cdot)$ is exponentially bounded. Indeed, for each $x \in X$ and $t \geq 0$, we have

$$\begin{aligned}\|T(t)x\| &\leq K_\alpha \int_0^\infty e^{-u^2}\|C(2t^{\frac{1}{2}}u)\|xdu \leq K_\alpha \int_0^\infty e^{-u^2} Me^{2t^{\frac{1}{2}}\omega u}du\|x\| \\ &\leq K_\alpha M e^{\omega^2 t}\int_0^\infty e^{-(u-t^{\frac{1}{2}}\omega)^2}du\|x\| \leq 2K_\alpha M e^{\omega^2 t}\int_0^\infty e^{-u^2}du\|x\| \\ &= K_\alpha M \pi^{\frac{1}{2}} e^{\omega^2 t}\|x\| = Ke^{\omega^2 t}\|x\|.\end{aligned}$$

It is easy to see from the exponential boundedness of $C(\cdot)$ and the dominated convergence theorem that $T(\cdot)$ is strongly continuous. Combining the closedness of A with (2.1) , we have for $x \in D(A)$ and $t \geq 0$, $T(t)x \in D(A)$ and

$$\begin{aligned}T(t)Ax &= K_\alpha \int_0^\infty e^{-u^2}C(2t^{\frac{1}{2}}u)Axdu \\ &= K_\alpha \int_0^\infty e^{-u^2}AC(2t^{\frac{1}{2}}u)xdu \\ &= A(K_\alpha \int_0^\infty e^{-u^2}C(2t^{\frac{1}{2}}u)xdu) \\ &= AT(t)x.\end{aligned}$$

Next we shall show that $\int_0^t T(s)xds \in D(A)$ and $A\int_0^t T(s)xds = T(t)x - \dfrac{t^{\frac{\alpha}{2}}}{\Gamma(\frac{\alpha}{2}+1)}Cx$ for all $x \in X$ and $t \geq 0$. Indeed, if $x \in X$ is given, then

$$\begin{aligned}&\int_0^t T(s)xds \\ &= K_\alpha \int_0^\infty e^{-u^2}\int_0^t C(2s^{\frac{1}{2}}u)xdsdu\end{aligned}$$

$$= K_\alpha \int_0^\infty e^{-u^2}(2u^2)^{-1}\int_0^{2t^{\frac{1}{2}}u} rC(r)xdrdu$$

$$= -\frac{K_\alpha}{2}\int_0^\infty e^{-u^2}u^{-2}\int_0^{2t^{\frac{1}{2}}u}(2t^{\frac{1}{2}}u - r)C(r)xdrdu$$

$$+ K_\alpha t^{\frac{1}{2}}\int_0^\infty e^{-u^2}u^{-1}\int_0^{2t^{\frac{1}{2}}u} C(r)xdrdu$$

$$= -\frac{K_\alpha}{2}\int_0^\infty e^{-u^2}u^{-2}\int_0^{2t^{\frac{1}{2}}u}\int_0^s C(r)xdrdsdu$$

$$+ K_\alpha t^{\frac{1}{2}}\int_0^\infty e^{\frac{-u'^2}{4t}}u'^{-1}\int_0^{u'} C(r)xdrdu'$$

$$= -\frac{K_\alpha}{2}\int_0^\infty e^{-u^2}u^{-2}\int_0^{2t^{\frac{1}{2}}u}\int_0^s C(r)xdrdsdu$$

$$+ K_\alpha t^{\frac{1}{2}}e^{\frac{-u'^2}{4t}}u'^{-1}\int_0^{u'}\int_0^s C(r)xdrds\,|_{u'=0}^{u'=\infty}$$

$$+ K_\alpha t^{\frac{1}{2}}\int_0^\infty e^{\frac{-u'^2}{4t}}[u'^{-2} + (2t)^{-1}]\int_0^{u'}\int_0^s C(r)xdrdsdu'$$

$$= -\frac{K_\alpha}{2}\int_0^\infty e^{-u^2}u^{-2}\int_0^{2t^{\frac{1}{2}}u}\int_0^s C(r)xdrdsdu$$

$$+ K_\alpha(2t)\int_0^\infty e^{-u^2}[(4t)^{-1}u^{-2} + (2t)^{-1}]\int_0^{2t^{\frac{1}{2}}u}\int_0^s C(r)xdrdsdu$$

$$= K_\alpha\int_0^\infty e^{-u^2}\int_0^{2t^{\frac{1}{2}}u}\int_0^s C(r)xdrdsdu,$$

which together with the closedness of A implies that $\int_0^t T(s)xds \in D(A)$ and

$$A\int_0^t T(s)xds = K_\alpha\int_0^\infty e^{-u^2}A\int_0^{2t^{\frac{1}{2}}u}\int_0^s C(r)xdrdsdu$$

$$= K_\alpha\int_0^\infty e^{-u^2}[C(2t^{\frac{1}{2}}u)x - \frac{(2t^{\frac{1}{2}}u)^\alpha}{\Gamma(\alpha+1)}Cx]du$$

$$= T(t)x - K_\alpha\int_0^\infty e^{-u^2}2^\alpha t^{\frac{\alpha}{2}}\frac{u^\alpha}{\Gamma(\alpha+1)}Cxdu$$

$$= T(t)x - K_\alpha 2^{\alpha-1}t^{\frac{\alpha}{2}}\int_0^\infty e^{-u'}\frac{u'^{\frac{\alpha-1}{2}}}{\Gamma(\alpha+1)}Cxdu'$$

$$= T(t)x - K_\alpha 2^{\alpha-1}t^{\frac{\alpha}{2}}\frac{\Gamma(\frac{\alpha+1}{2})}{\Gamma(\alpha+1)}Cx$$

$$= T(t)x - \frac{t^{\frac{\alpha}{2}}}{\Gamma(\frac{\alpha}{2}+1)}Cx$$

for all $t \geq 0$. It follows from the uniqueness of solutions of $ACP_2(0,0,0)$ that $\int_0^{\cdot} T(s)xds$ is the unique solution of $ACP_1(j_{\frac{\alpha}{2}}(\cdot)Cx, 0)$ in $C^1([0,\infty);X) \cap C([0,\infty);[D(A)])$. We conclude from [10,Theorem 2.3] that $T(\cdot)$ is a nondegenerate $\frac{\alpha}{2}$-times integrated C-semigroup on X with generator $A = C^{-1}AC$.

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