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EVALUATION OF SOME CONVOLUTION SUMS INVOLVING THE SUM OF DIVISORS FUNCTIONS

By

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Abstract. The convolution sums $\sum_{m<n/2} \sigma_e(m)\sigma_f(n-2m)$ and $\sum_{m<n/4} \sigma_e(m)\sigma_f(n-4m)$ are evaluated explicitly for certain values of e and f and all positive integers n.

1. Introduction

For $e \in \mathbb{N}$ and $n \in \mathbb{N}$ we set

$$\sigma_e(n) = \sum_{d|n} d^e.$$

If $n \notin \mathbb{N}$ we set $\sigma_e(n) = 0$. We also write $\sigma(n)$ for $\sigma_1(n)$. We define the convolution sum $S_{e,f}(n)$ $(e, f, n \in \mathbb{N})$ by

$$S_{e,f}(n) := \sum_{m=1}^{n-1} \sigma_e(m)\sigma_f(n-m). \tag{1.1}$$

We note that

$$S_{e,f}(n) = S_{f,e}(n). \tag{1.2}$$

Ramanujan's tau function $\tau(n)$ $(n \in \mathbb{N})$ is defined by

$$\Delta(q) = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n, \quad q \in \mathbb{C}, \quad |q| < 1. \tag{1.3}$$

The first twenty values of $\tau(n)$ are given in the following table.

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n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
1	1	6	−6048	11	534612	16	987136
2	−24	7	−16744	12	−370944	17	−6905934
3	252	8	84480	13	−577738	18	2727432
4	−1472	9	−113643	14	401856	19	10661420
5	4830	10	−115920	15	1211760	20	−7109760

A table of the values of $\tau(n)$ can be found in [12]. Ramanujan [11] and Lahiri [6], [7] have shown that $S_{e,f}(n)$ can be expressed as a linear combination of

$$\sigma_j(n)\ \ (j=1,3,\ldots,e+f+1),\ \ n\sigma_{e+f-1},\ \ \tau(n) \tag{1.4}$$

with rational coefficients for those pairs $(e,f)\in\mathbb{N}^2$ satisfying

$$e+f\le 12,\quad e\equiv f\equiv 1 \pmod 2). \tag{1.5}$$

Specifically they proved

$$S_{1,1}(n)=\frac{5}{12}\sigma_3(n)+\frac{(1-6n)}{12}\sigma(n), \tag{1.6}$$

$$S_{1,3}(n)=\frac{7}{80}\sigma_5(n)+\frac{(1-3n)}{24}\sigma_3(n)-\frac{1}{240}\sigma(n), \tag{1.7}$$

$$S_{1,5}(n)=\frac{5}{126}\sigma_7(n)+\frac{(1-2n)}{24}\sigma_5(n)+\frac{1}{504}\sigma(n), \tag{1.8}$$

$$S_{3,3}(n)=\frac{1}{120}\sigma_7(n)-\frac{1}{120}\sigma_3(n), \tag{1.9}$$

$$S_{1,7}(n)=\frac{11}{480}\sigma_9(n)+\frac{(2-3n)}{48}\sigma_7(n)-\frac{1}{480}\sigma(n), \tag{1.10}$$

$$S_{3,5}(n)=\frac{11}{5040}\sigma_9(n)-\frac{1}{240}\sigma_5(n)+\frac{1}{504}\sigma_3(n), \tag{1.11}$$

$$S_{1,9}(n)=\frac{455}{30404}\sigma_{11}(n)+\frac{(5-6n)}{120}\sigma_9(n)+\frac{1}{264}\sigma(n)-\frac{36}{3455}\tau(n), \tag{1.12}$$

$$S_{3,7}(n)=\frac{91}{110560}\sigma_{11}(n)-\frac{1}{240}\sigma_7(n)-\frac{1}{480}\sigma_3(n)+\frac{15}{2764}\tau(n), \tag{1.13}$$

$$S_{5,5}(n)=\frac{65}{174132}\sigma_{11}(n)+\frac{1}{252}\sigma_5(n)-\frac{3}{691}\tau(n), \tag{1.14}$$

$$S_{1,11}(n)=\frac{691}{65520}\sigma_{13}(n)+\frac{(1-n)}{24}\sigma_{11}(n)-\frac{691}{65520}\sigma(n), \tag{1.15}$$

$$S_{3,9}(n)=\frac{1}{2640}\sigma_{13}(n)-\frac{1}{240}\sigma_9(n)+\frac{1}{264}\sigma_3(n), \tag{1.16}$$

$$S_{5,7}(n)=\frac{1}{10080}\sigma_{13}(n)+\frac{1}{504}\sigma_7(n)-\frac{1}{480}\sigma_5(n). \tag{1.17}$$

In 1997, Melfi [8], [9] considered among others the convolution sums

$$\sum_{m<n/2} \sigma(m)\sigma(n-2m), \quad \sum_{m<n/2} \sigma(m)\sigma_3(n-2m), \quad \text{and} \quad \sum_{m<n/2} \sigma_3(m)\sigma(n-2m),$$

when n is odd, and proved that

$$\sum_{m<n/2} \sigma(m)\sigma(n-2m) = \frac{1}{12}\sigma_3(n) + \frac{(1-3n)}{24}\sigma(n), \quad n \equiv 1 \pmod 2,$$

$$\sum_{m<n/2} \sigma(m)\sigma_3(n-2m) = \frac{1}{48}\sigma_5(n) + \frac{(2-3n)}{48}\sigma_3(n), \quad n \equiv 1 \pmod 2,$$

$$\sum_{m<n/2} \sigma_3(m)\sigma(n-2m) = \frac{1}{240}\sigma_5(n) - \frac{1}{240}\sigma(n), \quad n \equiv 1 \pmod 2.$$

For $e, f, n \in \mathbb{N}$ we set

$$T_{e,f}(n) = \sum_{m<n/2} \sigma_e(m)\sigma_f(n-2m) \tag{1.18}$$

and

$$U_{e,f}(n) = \sum_{m<n/4} \sigma_e(m)\sigma_f(n-4m). \tag{1.19}$$

In 2002 Huard, Ou, Spearman and Williams [4] evaluated $T_{e,f}(n)$ when $(e,f) = (1,1), (1,3), (3,1)$ and $U_{e,f}(n)$ when $(e,f) = (1,1)$ for all $n \in \mathbb{N}$, using only elementary methods. They proved

$$T_{1,1}(n) = \frac{1}{12}\sigma_3(n) + \frac{1}{3}\sigma_3(n/2) + \frac{(1-3n)}{24}\sigma(n) + \frac{(1-6n)}{24}\sigma(n/2),$$

$$T_{1,3}(n) = \frac{1}{48}\sigma_5(n) + \frac{1}{15}\sigma_5(n/2) + \frac{(2-3n)}{48}\sigma_3(n) - \frac{1}{240}\sigma(n/2),$$

$$T_{3,1}(n) = \frac{1}{240}\sigma_5(n) + \frac{1}{12}\sigma_5(n/2) + \frac{(1-3n)}{24}\sigma_3(n/2) - \frac{1}{240}\sigma(n),$$

$$U_{1,1}(n) = \frac{1}{48}\sigma_3(n) + \frac{1}{16}\sigma_3(n/2) + \frac{1}{3}\sigma_3(n/4) + \frac{(2-3n)}{48}\sigma(n) + \frac{(1-6n)}{24}\sigma(n/4).$$

In this paper we evaluate $T_{e,f}(n)$ and $U_{e,f}(n)$ for all $n \in \mathbb{N}$ for those pairs $(e,f) \in \mathbb{N}^2$ satisfying $e + f \leq 10$, $e \equiv f \equiv 1 \pmod 2$ (Sections 3-7). In Section 8 we use the values of $T_{3,1}(n)$ and $U_{3,1}(n)$ to reprove in a very simple way the formula of Ono, Robins and Wahl for the number of representations of $n \in \mathbb{N}$ as the sum of 12 triangular numbers [10, Theorem 7, p. 85].

2. Preliminary results

We begin by recalling Ramanujan's functions

$$(2.1)\qquad L(q) = 1 - 24\sum_{n=1}^{\infty}\sigma(n)q^n, \qquad |q| < 1,$$

$$(2.2)\qquad M(q) = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n, \qquad |q| < 1,$$

$$(2.3)\qquad N(q) = 1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n, \qquad |q| < 1.$$

Ramanujan [11] and Lahiri [6], [7] actually use P, Q, R in place of L, M, N but we follow the usage of Berndt [1, p. 318]. For $k \in \mathbb{N}$ with $k > 1$ the Eisenstein series

$$E_{2k}(\tau) := 1 + \frac{2}{\zeta(1-2k)}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)q^n,$$

where $\tau \in H = \{x + iy \in \mathbb{C} \mid y > 0\}$ and $q = e^{2\pi i\tau}$, is a modular form of weight $2k$ for the modular group $\Gamma = SL(2,\mathbb{Z})$. Here as usual $\zeta(s)$ denotes the Riemann zeta function. As $\zeta(-3) = 1/120$ and $\zeta(-5) = -1/252$ we have $E_4(\tau) = M(q)$ and $E_6(\tau) = N(q)$. The function $E_2(\tau) := 1 - 24\sum_{n=1}^{\infty}\sigma(n)q^n = L(q)$ is not a modular form but is transformed under the action of Γ by

$$E_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2E_2(\tau) + \frac{6}{\pi i}c(c\tau+d), \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma,$$

see [1, p. 318]. We need the following identities which can be found in Lahiri [6], [7].

$$(2.4)\qquad L^2(q) = 1 - 2^5\cdot 3^2\sum_{n=1}^{\infty}n\sigma(n)q^n + 2^4\cdot 3\cdot 5\sum_{n=1}^{\infty}\sigma_3(n)q^n,$$

$$(2.5)\qquad M^2(q) = 1 + 2^5\cdot 3\cdot 5\sum_{n=1}^{\infty}\sigma_7(n)q^n,$$

$$(2.6)\qquad M^3(q) = 1 + \frac{2^4\cdot 3^2\cdot 5\cdot 7\cdot 13}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^n + \frac{2^7\cdot 3^3\cdot 5^3}{691}\sum_{n=1}^{\infty}\tau(n)q^n,$$

$$(2.7)\qquad N^2(q) = 1 + \frac{2^4\cdot 3^2\cdot 5\cdot 7\cdot 13}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^n - \frac{2^6\cdot 3^5\cdot 7^2}{691}\sum_{n=1}^{\infty}\tau(n)q^n,$$

$$L(q)M(q) = 1 + 2^4 \cdot 3^2 \cdot 5 \sum_{n=1}^{\infty} n\sigma_3(n)q^n - 2^3 \cdot 3^2 \cdot 7 \sum_{n=1}^{\infty} \sigma_5(n)q^n, \tag{2.8}$$

$$L(q)M^2(q) = 1 + 2^4 \cdot 3^2 \cdot 5 \sum_{n=1}^{\infty} n\sigma_7(n)q^n - 2^3 \cdot 3 \cdot 11 \sum_{n=1}^{\infty} \sigma_9(n)q^n, \tag{2.9}$$

$$L(q)N(q) = 1 - 2^4 \cdot 3^2 \cdot 7 \sum_{n=1}^{\infty} n\sigma_5(n)q^n + 2^5 \cdot 3 \cdot 5 \sum_{n=1}^{\infty} \sigma_7(n)q^n, \tag{2.10}$$

$$M(q)N(q) = 1 - 2^3 \cdot 3 \cdot 11 \sum_{n=1}^{\infty} \sigma_9(n)q^n, \tag{2.11}$$

$$\begin{aligned} L(q)M(q)N(q) = 1 &- \frac{2^4 \cdot 3^2 \cdot 11}{5} \sum_{n=1}^{\infty} n\sigma_9(n)q^n \\ &+ \frac{2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13}{691} \sum_{n=1}^{\infty} \sigma_{11}(n)q^n - \frac{2^3 \cdot 3^4 \cdot 11}{5 \cdot 691} \sum_{n=1}^{\infty} \tau(n)q^n. \end{aligned} \tag{2.12}$$

From this point on we restrict q to be a real number satisfying $0 < q < 1$. Then $0 < -\log q < \infty$. As usual we denote the Gaussian hypergeometric function by ${}_2F_1(a, b; c; x)$. The derivative y' of the function

$$y = \frac{\pi {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-x\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right)}$$

is given by

$$y' = -\frac{x^{-1}(1-x)^{-1}}{\left({}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right)\right)^2},$$

see [1, p. 87]. For $0 < x < 1$, we have

$${}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; x\right) = \sum_{n=0}^{\infty} \frac{1}{2^{4n}} \binom{2n}{n}^2 x^n > 0$$

so that $y' < 0$ for $0 < x < 1$. Hence y is a decreasing function of x for $0 < x < 1$. As $y(0) = \infty$ and $y(1) = 0$, the function y decreases from ∞ to 0 as x increases from 0 to 1. Hence there is a unique value of x between 0 and 1 such that

$$y = \frac{\pi {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-x\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right)} = -\log q. \tag{2.13}$$

Thus

$$q = e^{-y} = e^{-\frac{\pi {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-x\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right)}}. \tag{2.14}$$

We also set

$$w = {}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; x\right). \tag{2.15}$$

The following formulae are proved in [2, pp. 126–129]

$$L(q) = (1-5x)w^2 + 12x(1-x)w\frac{dw}{dx}, \tag{2.16}$$

$$M(q) = (1+14x+x^2)w^4, \tag{2.17}$$

$$N(q) = (1+x)(1-34x+x^2)w^6, \tag{2.18}$$

$$L(q^2) = (1-2x)w^2 + 6x(1-x)w\frac{dw}{dx}, \tag{2.19}$$

$$M(q^2) = (1-x+x^2)w^4, \tag{2.20}$$

$$N(q^2) = (1+x)(1-\frac{1}{2}x)(1-2x)w^6, \tag{2.21}$$

$$L(q^4) = (1-\frac{5}{4}x)w^2 + 3x(1-x)w\frac{dw}{dx}, \tag{2.22}$$

$$M(q^4) = (1-x+\frac{1}{16}x^2)w^4, \tag{2.23}$$

$$N(q^4) = (1-\frac{1}{2}x)(1-x-\frac{1}{32}x^2)w^6. \tag{2.24}$$

Finally, from (2.6) and (2.7), we obtain

$$M^3(q) - N^2(q) = 2^6 \cdot 3^3 \sum_{n=1}^{\infty} \tau(n)q^n. \tag{2.25}$$

Appealing to (2.17), (2.18) and (2.25), we deduce

$$\Delta(q) = \sum_{n=1}^{\infty} \tau(n)q^n = \frac{x(1-x)^4 w^{12}}{2^4}. \tag{2.26}$$

Applying the principle of duplication [2, p. 125]

$$q \to q^2, \quad x \to \left(\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right)^2, \quad w \to w\left(\frac{1+\sqrt{1-x}}{2}\right)$$

to (2.26), we obtain

$$\Delta(q^2) = \sum_{n=1}^{\infty} \tau(n)q^{2n} = \frac{x^2(1-x)^2 w^{12}}{2^8}. \tag{2.27}$$

Applying the principle of duplication to (2.27), we obtain

$$\Delta(q^4) = \sum_{n=1}^{\infty} \tau(n)q^{4n} = \frac{x^4(1-x) w^{12}}{2^{16}}. \tag{2.28}$$

3. Evaluation of $T_{e,f}(n)$ and $U_{e,f}(n)$ for $e+f=2$

We begin by determining $L(q)L(q^2)$ and $L(q)L(q^4)$.

THEOREM 3.1.
(i) $L(q)L(q^2) = \frac{1}{4}L^2(q) + L^2(q^2) - \frac{1}{20}M(q) - \frac{1}{5}M(q^2)$,

(ii) $L(q)L(q^4) = \frac{1}{8}L(q)^2 + 2L(q^4)^2 - \frac{3}{40}M(q) + \frac{3}{20}M(q^2) - \frac{6}{5}M(q^4)$.

Proof. From (2.16) and (2.19) we obtain

$$-L(q) + 2L(q^2) = (1+x)w^2. \tag{3.1}$$

Squaring we deduce that

$$L^2(q) - 4L(q)L(q^2) + 4L^2(q^2) = (1+2x+x^2)w^4.$$

From (2.17) and (2.20) we have

$$M(q) + 4M(q^2) = (5+10x+5x^2)w^4.$$

Hence

$$L^2(q) - 4L(q)L(q^2) + 4L^2(q^2) = \frac{1}{5}M(q) + \frac{4}{5}M(q^2),$$

from which part (i) follows. Part (ii) follows similarly. □

From Theorem 3.1 we deduce the values of $T_{1,1}(n)$ and $U_{1,1}(n)$ mentioned at the end of Section 1.

THEOREM 3.2.
(i) $T_{1,1}(n) = \frac{1}{12}\sigma_3(n) + \frac{1}{3}\sigma_3(n/2) + \frac{(1-3n)}{24}\sigma(n) + \frac{(1-6n)}{24}\sigma(n/2)$,

(ii) $U_{1,1}(n) = \frac{1}{48}\sigma_3(n) + \frac{1}{16}\sigma_3(n/2) + \frac{1}{3}\sigma_3(n/4) + \frac{(2-3n)}{48}\sigma(n) + \frac{(1-6n)}{24}\sigma(n/4)$.

Proof. We have

$$\begin{aligned}\sum_{n=1}^{\infty} T_{1,1}(n)q^n &= \sum_{n=1}^{\infty}\left(\sum_{m<n/2}\sigma(m)\sigma(n-2m)\right)q^n \\ &= \sum_{l=1}^{\infty}\sigma(l)q^l\sum_{m=1}^{\infty}\sigma(m)q^{2m} = \left(\frac{1-L(q)}{24}\right)\left(\frac{1-L(q^2)}{24}\right)\end{aligned}$$

$$= \frac{1}{2^6 \cdot 3^2}(1 - L(q) - L(q^2) + L(q)L(q^2))$$
$$= \frac{1}{2^8 \cdot 3^2 \cdot 5}(20 - 20L(q) - 20L(q^2) + 5L^2(q) + 20L^2(q^2) - M(q) - 4M(q^2)),$$

by Theorem 3.1 (i). Appealing to (2. 1), (2. 2) and (2. 4), and equating the coefficients of q^n, we obtain the asserted formula for $T_{1,1}(n)$. The formula for $U_{1,1}(n)$ follows similarly from Theorem 3.1(ii), (2.1), (2.2) and (2.4). □

Theorem 3.2 (i) is [4, Theorem 2, p. 247] and Theorem 3.2 (ii) is [4, Theorem 4, p. 249].

4. Evaluation of $T_{e,f}(n)$ and $U_{e,f}(n)$ for $e + f = 4$

We first determine $L(q)M(q^2)$, $L(q^2)M(q)$, $L(q)M(q^4)$ and $L(q^4)M(q)$. In order to do this we introduce the constants $a(n)$ $(n = 1, 2, \ldots)$ defined by

$$A(q) = \sum_{n=1}^{\infty} a(n)q^n = \Delta(q^2)^{1/2} = \frac{x(1-x)w^6}{2^4}. \tag{4.1}$$

The first twenty values of $a(n)$ are given in the following table.

n	$a(n)$	n	$a(n)$	n	$a(n)$	n	$a(n)$
1	1	6	0	11	540	16	0
2	0	7	−88	12	0	17	594
3	−12	8	0	13	−418	18	0
4	0	9	−99	14	0	19	836
5	54	10	0	15	−648	20	0

Similarly to the proof of Theorem 3.1 we obtain

THEOREM 4.1.

(i) $L(q)M(q^2) = 2L(q^2)M(q^2) + \frac{1}{21}N(q) - \frac{22}{21}N(q^2)$,

(ii) $M(q)L(q^2) = \frac{1}{2}L(q)M(q) - \frac{11}{42}N(q) + \frac{16}{21}N(q^2)$,

(iii) $L(q)M(q^4) = 4L(q^4)M(q^4) + \frac{1}{336}N(q) + \frac{5}{112}N(q^2) - \frac{64}{21}N(q^4) - \frac{45}{2}A(q)$,

(iv) $M(q)L(q^4) = \frac{1}{4}L(q)M(q) - \frac{4}{21}N(q) + \frac{5}{28}N(q^2) + \frac{16}{21}N(q^4) + 90A(q)$.

Appealing to (2.1), (2.2), (2.3), (2.8), (4.1) and Theorem 4.1, we obtain

THEOREM 4.2.

(i) $T_{3,1}(n) = \frac{1}{240}\sigma_5(n) + \frac{1}{12}\sigma_5(n/2) + \frac{(1-3n)}{24}\sigma_3(n/2) - \frac{1}{240}\sigma(n),$

(ii) $T_{1,3}(n) = \frac{1}{48}\sigma_5(n) + \frac{1}{15}\sigma_5(n/2) + \frac{(2-3n)}{48}\sigma_3(n) - \frac{1}{240}\sigma(n/2),$

(iii) $U_{3,1}(n) = \frac{1}{3840}\sigma_5(n) + \frac{1}{256}\sigma_5(n/2) + \frac{1}{12}\sigma_5(n/4) + \frac{(1-3n)}{24}\sigma_3(n/4) - \frac{1}{240}\sigma(n) + \frac{1}{256}a(n),$

(iv) $U_{1,3}(n) = \frac{1}{192}\sigma_5(n) + \frac{1}{64}\sigma_5(n/2) + \frac{1}{15}\sigma_5(n/4) + \frac{(4-3n)}{96}\sigma_3(n) - \frac{1}{240}\sigma(n/4) - \frac{1}{64}a(n).$

The values of $T_{1,3}(n)$ and $T_{3,1}(n)$ were given at the end of Section 1. Theorem 4.2 (i), (ii) comprise [4, Theorem 6, p. 250].

5. Evaluation of $T_{e,f}(n)$ and $U_{e,f}(n)$ for $e+f=6$

We first determine $L(q)N(q^2)$, $M(q)M(q^2)$, $L(q^2)N(q)$, $L(q)N(q^4)$, $M(q)M(q^4)$ and $L(q^4)N(q)$. In order to do this we introduce the constants $b(n)$ ($n \in \mathbb{N}$) defined by

$$(5.1) \qquad B(q) = \sum_{n=1}^{\infty} b(n)q^n = (\Delta(q)\Delta(q^2))^{1/3} = \frac{x(1-x)^2w^8}{2^4}.$$

Hence

$$(5.2) \quad B(q^2) = \sum_{n=1}^{\infty} b(n)q^{2n} = \sum_{n=1}^{\infty} b(n/2)q^n = (\Delta(q^2)\Delta(q^4))^{1/3} = \frac{x^2(1-x)w^8}{2^8}.$$

The first twenty values of $b(n)$ are given in the following table.

n	$b(n)$	n	$b(n)$	n	$b(n)$	n	$b(n)$
1	1	6	−96	11	1092	16	4096
2	−8	7	1016	12	768	17	14706
3	12	8	−512	13	1382	18	16344
4	64	9	−2043	14	−8128	19	−39940
5	−210	10	1680	15	−2520	20	−13440

Similarly to the proof of Theorem 3.1, we obtain the following result.

THEOREM 5.1.

(i) $L(q)N(q^2) = 2L(q^2)N(q^2) + \frac{1}{85}M^2(q) - \frac{86}{85}M^2(q^2) - \frac{504}{17}B(q)$,

(ii) $M(q)M(q^2) = \frac{1}{17}M^2(q) + \frac{16}{17}M^2(q^2) + \frac{3600}{17}B(q)$,

(iii) $N(q)L(q^2) = \frac{1}{2}L(q)N(q) - \frac{43}{170}M^2(q) + \frac{64}{85}M^2(q^2) - \frac{2016}{17}B(q)$,

(iv) $L(q)N(q^4) = 4L(q^4)N(q^4) + \frac{1}{5440}M^2(q) + \frac{63}{5440}M^2(q^2) - \frac{256}{85}M^2(q^4)$
$$- \frac{819}{34}B(q) - \frac{4788}{17}B(q^2),$$

(v) $M(q)M(q^4) = \frac{1}{272}M^2(q) + \frac{15}{272}M^2(q^2) + \frac{16}{17}M^2(q^4) + \frac{4050}{17}B(q) + \frac{64800}{17}B(q^2)$,

(vi) $N(q)L(q^4) = \frac{1}{4}L(q)N(q) - \frac{16}{85}M^2(q) + \frac{63}{340}M^2(q^2) + \frac{64}{85}M^2(q^4)$
$$- \frac{4788}{17}B(q) - \frac{104832}{17}B(q^2).$$

Appealing to (2.1), (2.2), (2.3), (2.5), (2.10), (5.1), (5.2) and Theorem 5.1, we obtain the next result.

THEOREM 5.2.

(i) $T_{5,1}(n) = \frac{1}{2142}\sigma_7(n) + \frac{2}{51}\sigma_7(n/2) + \frac{(1-2n)}{24}\sigma_5(n/2) + \frac{1}{504}\sigma(n) - \frac{1}{408}b(n)$,

(ii) $T_{3,3}(n) = \frac{1}{2040}\sigma_7(n) + \frac{2}{255}\sigma_7(n/2) - \frac{1}{240}\sigma_3(n) - \frac{1}{240}\sigma_3(n/2) + \frac{1}{272}b(n)$,

(iii) $T_{1,5}(n) = \frac{1}{102}\sigma_7(n) + \frac{32}{1071}\sigma_7(n/2) + \frac{(1-n)}{24}\sigma_5(n) + \frac{1}{504}\sigma(n/2) - \frac{1}{102}b(n)$,

(iv) $U_{5,1}(n) = \frac{1}{137088}\sigma_7(n) + \frac{1}{2176}\sigma_7(n/2) + \frac{2}{51}\sigma_7(n/4) + \frac{(1-2n)}{24}\sigma_5(n/4)$
$$+ \frac{1}{504}\sigma(n) - \frac{13}{6528}b(n) - \frac{19}{816}b(n/2),$$

(v) $U_{3,3}(n) = \frac{1}{32640}\sigma_7(n) + \frac{1}{2176}\sigma_7(n/2) + \frac{2}{255}\sigma_7(n/4) - \frac{1}{240}\sigma_3(n) - \frac{1}{240}\sigma_3(n/4)$
$$+ \frac{9}{2176}b(n) + \frac{9}{136}b(n/2),$$

(vi) $U_{1,5}(n) = \frac{1}{408}\sigma_7(n) + \frac{1}{136}\sigma_7(n/2) + \frac{32}{1071}\sigma_7(n/4) + \frac{(2-n)}{48}\sigma_5(n) + \frac{1}{504}\sigma(n/4)$
$$- \frac{19}{816}b(n) - \frac{26}{51}b(n/2).$$

The values of $3T_{1,5}(n)+8T_{3,3}(n)$ and $2T_{3,3}(n)+3T_{5,1}(n)$ have been derived in an elementary manner in [4, Theorem 15, p. 271]. We also note that [5, Theorem 8, p. 162] follows from Theorem 5.2.

In [5, Theorem 6, p. 157] it is shown that the number of representations of $n(\in \mathbb{N})$ as the sum of sixteen squares is given by

$$\begin{aligned} r_{16}(n) = &\frac{32}{15}(-1)^n\sigma_7(n) + \frac{8192}{15}\sigma_7(n/2) + \frac{512}{15}(-1)^{n-1}\sigma_3(n) \\ &- \frac{512}{15}\sigma_3(n/2) + 8192(-1)^{n-1}T_{3,3}(n), \end{aligned}$$

and, appealing to Theorem 5.2 (ii) for the value of $T_{3,3}(n)$, we obtain

$$r_{16}(n) = \frac{32}{17}(-1)^{n-1}(\sigma_7(n) - 256\sigma_7(n/2) + 16b(n)),$$

which is a result due to Glaisher [3, p. 480].

6. Evaluation of $T_{e,f}(n)$ and $U_{e,f}(n)$ for $e+f=8$

We first determine $L(q)M^2(q^2)$, $M(q)N(q^2)$, $N(q)M(q^2)$, $M^2(q)L(q^2)$, $L(q)M^2(q^4)$, $M(q)N(q^4)$, $N(q)M(q^4)$ and $M^2(q)L(q^4)$. In order to do this we introduce the constants $c(n)$, $d(n)$ and $e(n)$ $(n \in \mathbb{N})$ defined by

$$(6.1) \quad C(q) = \sum_{n=1}^{\infty} c(n)q^n = (\Delta(q)^4\Delta(q^2))^{1/6} = \frac{x(1-x)^3w^{10}}{2^4},$$

$$(6.2) \quad D(q) = \sum_{n=1}^{\infty} d(n)q^n = (\Delta(q)^2\Delta(q^2)\Delta(q^4)^2)^{1/6} = \frac{x^2(1-x)^2w^{10}}{2^8},$$

$$(6.3) \quad E(q) = \sum_{n=1}^{\infty} e(n)q^n = (\Delta(q^2)\Delta(q^4)^4)^{1/6} = \frac{x^3(1-x)w^{10}}{2^{12}}.$$

We observe that

$$(6.4) \quad C(q)E(q) = D(q)^2$$

so that

$$(6.5) \quad \sum_{m=1}^{n-1} c(m)e(n-m) = \sum_{m=1}^{n-1} d(m)d(n-m).$$

The first twenty values of $c(n)$, $d(n)$ and $e(n)$ are given in the following tables.

n	$c(n)$	n	$c(n)$	n	$c(n)$	n	$c(n)$
1	1	6	2496	11	−38996	16	−65536
2	−16	7	−4536	12	39936	17	311442
3	100	8	−4096	13	37806	18	−74448
4	−256	9	23085	14	15232	19	128244
5	−154	10	−13920	15	−146472	20	−222720

n	$d(n)$	n	$d(n)$	n	$d(n)$	n	$d(n)$
1	0	6	−156	11	−536	16	4096
2	1	7	112	12	−2496	17	−17472
3	−8	8	256	13	4384	18	4653
4	16	9	−576	14	−952	19	5848
5	32	10	870	15	336	20	13920

n	$e(n)$	n	$e(n)$	n	$e(n)$	n	$e(n)$
1	0	6	0	11	67	16	0
2	0	7	−14	12	0	17	2184
3	1	8	0	13	−548	18	0
4	0	9	72	14	0	19	−731
5	−4	10	0	15	−42	20	0

Similarly to the proof of Theorem 3.1, we obtain the following result.

THEOREM 6.1.

(i) $$L(q)M^2(q^2) = 2L(q^2)M^2(q^2) + \frac{1}{341}M(q)N(q) - \frac{342}{341}M(q^2)N(q^2) - \frac{720}{31}C(q) - \frac{23040}{31}D(q),$$

(ii) $$M(q)N(q^2) = \frac{5}{341}M(q)N(q) + \frac{336}{341}M(q^2)N(q^2) + \frac{7560}{31}C(q) + \frac{241920}{31}D(q),$$

(iii) $$N(q)M(q^2) = \frac{21}{341}M(q)N(q) + \frac{320}{341}M(q^2)N(q^2) - \frac{15120}{31}C(q) - \frac{483840}{31}D(q),$$

(iv) $$M^2(q)L(q^2) = \frac{1}{2}L(q)M^2(q) - \frac{171}{682}M(q)N(q) + \frac{256}{341}M(q^2)N(q^2) + \frac{5760}{31}C(q) + \frac{184320}{31}D(q),$$

(v) $$L(q)M^2(q^4) = 4L(q^4)M^2(q^4) + \frac{1}{87296}M(q)N(q) + \frac{255}{87296}M(q^2)N(q^2) - \frac{1024}{341}M(q^4)N(q^4) - \frac{23805}{992}C(q) - \frac{28125}{62}D(q) - \frac{39240}{31}E(q),$$

(vi) $$M(q)N(q^4) = \frac{5}{21824}M(q)N(q) + \frac{315}{21824}M(q^2)N(q^2) + \frac{336}{341}M(q^4)N(q^4)$$

$$+\frac{59535}{248}C(q)+\frac{187110}{31}D(q)+\frac{997920}{31}E(q),$$

(vii) $$N(q)M(q^4)=\frac{21}{5456}M(q)N(q)+\frac{315}{5456}M(q^2)N(q^2)+\frac{320}{341}M(q^4)N(q^4)$$
$$-\frac{31185}{62}C(q)-\frac{748440}{31}D(q)-\frac{7620480}{31}E(q),$$

(viii) $$M^2(q)L(q^4)=\frac{1}{4}L(q)M^2(q)-\frac{64}{341}M(q)N(q)+\frac{255}{1364}M(q^2)N(q^2)$$
$$+\frac{256}{341}M(q^4)N(q^4)+\frac{9810}{31}C(q)+\frac{900000}{31}D(q)+\frac{12188160}{31}E(q).$$

Appealing to (2.1), (2.2), (2.3), (2.5), (2.9), (2.11), (6.1), (6.2), (6.3) and Theorem 6.1, we obtain the next result.

THEOREM 6.2.

(i) $$T_{7,1}(n)=\frac{1}{14880}\sigma_9(n)+\frac{17}{744}\sigma_9(n/2)+\frac{(2-3n)}{48}\sigma_7(n/2)-\frac{1}{480}\sigma(n)$$
$$+\frac{1}{496}c(n)+\frac{2}{31}d(n),$$

(ii) $$T_{5,3}(n)=\frac{1}{31248}\sigma_9(n)+\frac{1}{465}\sigma_9(n/2)-\frac{1}{240}\sigma_5(n/2)+\frac{1}{504}\sigma_3(n)$$
$$-\frac{1}{496}c(n)-\frac{2}{31}d(n),$$

(iii) $$T_{3,5}(n)=\frac{1}{7440}\sigma_9(n)+\frac{4}{1953}\sigma_9(n/2)-\frac{1}{240}\sigma_5(n)+\frac{1}{504}\sigma_3(n/2)$$
$$+\frac{1}{248}c(n)+\frac{4}{31}d(n),$$

(iv) $$T_{1,7}(n)=\frac{17}{2976}\sigma_9(n)+\frac{8}{465}\sigma_9(n/2)+\frac{(4-3n)}{96}\sigma_7(n)-\frac{1}{480}\sigma(n/2)$$
$$-\frac{1}{62}c(n)-\frac{16}{31}d(n),$$

(v) $$U_{7,1}(n)=\frac{1}{3809280}\sigma_9(n)+\frac{17}{253952}\sigma_9(n/2)+\frac{17}{744}\sigma_9(n/4)+\frac{(2-3n)}{48}\sigma_7(n/4)$$
$$-\frac{1}{480}\sigma(n)+\frac{529}{253952}c(n)+\frac{625}{15872}d(n)+\frac{109}{992}e(n),$$

(vi) $$U_{5,3}(n)=\frac{1}{1999872}\sigma_9(n)+\frac{1}{31744}\sigma_9(n/2)+\frac{1}{465}\sigma_9(n/4)-\frac{1}{240}\sigma_5(n/4)$$
$$+\frac{1}{504}\sigma_3(n)-\frac{63}{31744}c(n)-\frac{99}{1984}d(n)-\frac{33}{124}e(n),$$

(vii) $U_{3,5}(n) = \frac{1}{119040}\sigma_9(n) + \frac{1}{7936}\sigma_9(n/2) + \frac{4}{1953}\sigma_9(n/4) - \frac{1}{240}\sigma_5(n)$
$+ \frac{1}{504}\sigma_3(n/4) + \frac{33}{7936}c(n) + \frac{99}{496}d(n) + \frac{63}{31}e(n),$

(viii) $U_{1,7}(n) = \frac{17}{11904}\sigma_9(n) + \frac{17}{3968}\sigma_9(n/2) + \frac{8}{465}\sigma_9(n/4) + \frac{(8-3n)}{192}\sigma_7(n)$
$- \frac{1}{480}\sigma(n/4) - \frac{109}{3968}c(n) - \frac{625}{248}d(n) - \frac{1058}{31}e(n).$

7. Evaluation of $T_{e,f}(n)$ and $U_{e,f}(n)$ for $e + f = 10$

We first determine $L(q)M(q^2)N(q^2)$, $M(q)M^2(q^2)$, $N(q)N(q^2)$, $M^2(q)M(q^2)$, $M(q)N(q)L(q^2)$, $L(q)M(q^4)N(q^4)$, $M(q)M^2(q^4)$, $N(q)N(q^4)$, $M^2(q)M(q^4)$ and $M(q)N(q)L(q^4)$. In order to do this we introduce the constants $f(n)$ $(n \in \mathbb{N})$ defined by

$$(7.1) \qquad F(q) = \sum_{n=1}^{\infty} f(n)q^n = \left(\frac{\Delta(q^4)^4}{\Delta(q)}\right)^{1/3} = \frac{x^5w^{12}}{2^{20}}.$$

The first twenty values of $f(n)$ are given in the following table.

n	$f(n)$	n	$f(n)$	n	$f(n)$	n	$f(n)$
1	0	6	8	11	6296	16	388608
2	0	7	44	12	16384	17	756822
3	0	8	192	13	39569	18	1419200
4	0	9	694	14	89424	19	2572328
5	1	10	2208	15	191028	20	4521984

Similarly to the proof of Theorem 3.1, we obtain the following result.

THEOREM 7.1.

(i) $L(q)M(q^2)N(q^2) = 2L(q^2)M(q^2)N(q^2) - \frac{7}{520}M(q)^3 - \frac{56}{65}M(q^2)^3$
$+ \frac{31}{2184}N(q)^2 - \frac{38}{273}N(q^2)^2,$

(ii) $M(q)M^2(q^2) = \frac{11}{78}M(q)^3 + \frac{196}{39}M(q^2)^3 - \frac{25}{182}N(q)^2 - \frac{1100}{273}N(q^2)^2,$

(iii) $N(q)N(q^2) = -\frac{147}{520}M(q)^3 - \frac{1176}{65}M(q^2)^3 + \frac{31}{104}N(q)^2 + \frac{248}{13}N(q^2)^2,$

(iv) $M^2(q)M(q^2) = \frac{49}{156}M(q)^3 + \frac{1408}{39}M(q^2)^3 - \frac{275}{1092}N(q)^2 - \frac{3200}{91}N(q^2)^2,$

(v) $$M(q)N(q)L(q^2) = \frac{1}{2}L(q)M(q)N(q) - \frac{14}{65}M(q)^3 - \frac{896}{65}M(q^2)^3 - \frac{19}{546}N(q)^2 + \frac{3968}{273}N(q^2)^2,$$

(vi) $$L(q)M(q^4)N(q^4) = 4L(q^4)M(q^4)N(q^4) - \frac{683}{49920}M(q)^3 - \frac{1727}{4160}M(q^2)^3 - \frac{22138}{195}M(q^4)^3 + \frac{327}{23296}N(q)^2 + \frac{605}{1456}N(q^2)^2 + \frac{10058}{91}N(q^4)^2 - 1520640F(q),$$

(vii) $$M(q)M^2(q^4) = \frac{57}{416}M(q)^3 + \frac{515}{104}M(q^2)^3 + \frac{13932}{13}M(q^4)^3 - \frac{175}{1248}N(q)^2 - \frac{450}{91}N(q^2)^2 - \frac{292300}{273}N(q^4)^2 + 13824000F(q),$$

(viii) $$N(q)N(q^4) = -\frac{4851}{16640}M(q)^3 - \frac{69237}{4160}M(q^2)^3 - \frac{77616}{65}M(q^4)^3 + \frac{971}{3328}N(q)^2 + \frac{3465}{208}N(q^2)^2 + \frac{15536}{13}N(q^4)^2,$$

(ix) $$M^2(q)M(q^4) = \frac{129}{416}M(q)^3 + \frac{3765}{104}M(q^2)^3 - \frac{89664}{13}M(q^4)^3 - \frac{2225}{8736}N(q)^2 - \frac{13175}{364}N(q^2)^2 + \frac{1883200}{273}N(q^4)^2 - 221184000F(q),$$

(x) $$M(q)N(q)L(q^4) = \frac{1}{4}L(q)M(q)N(q) - \frac{853}{3120}M(q)^3 - \frac{3971}{130}M(q^2)^3 + \frac{2651392}{195}M(q^4)^3 - \frac{1}{208}N(q)^2 + \frac{2805}{91}N(q^2)^2 - \frac{1237248}{91}N(q^4)^2 + 389283840F(q).$$

Appealing to (2.1), (2.2), (2.3), (2.6), (2.7), (2.11), (2.12), (7.1) and Theorem 7.1, we obtain the following result.

THEOREM 7.2.

(i) $$T_{9,1}(n) = \frac{1}{91212}\sigma_{11}(n) + \frac{31}{2073}\sigma_{11}(n/2) + \frac{(5-6n)}{120}\sigma_9(n/2) + \frac{1}{264}\sigma(n) - \frac{21}{5528}\tau(n) - \frac{282}{3455}\tau(n/2),$$

(ii) $$T_{7,3}(n) = \frac{1}{331680}\sigma_{11}(n) + \frac{17}{20730}\sigma_{11}(n/2) - \frac{1}{240}\sigma_7(n/2) - \frac{1}{480}\sigma_3(n) + \frac{23}{11056}\tau(n) + \frac{91}{1382}\tau(n/2),$$

(iii) $$T_{5,5}(n) = \frac{1}{174132}\sigma_{11}(n) + \frac{16}{43533}\sigma_{11}(n/2) + \frac{1}{504}\sigma_5(n) + \frac{1}{504}\sigma_5(n/2) - \frac{11}{5528}\tau(n) - \frac{88}{691}\tau(n/2),$$

(iv) $$T_{3,7}(n) = \frac{17}{331680}\sigma_{11}(n) + \frac{8}{10365}\sigma_{11}(n/2) - \frac{1}{240}\sigma_7(n) - \frac{1}{480}\sigma_3(n/2) + \frac{91}{22112}\tau(n) + \frac{368}{691}\tau(n/2),$$

(v) $$T_{1,9}(n) = \frac{31}{8292}\sigma_{11}(n) + \frac{256}{22803}\sigma_{11}(n/2) + \frac{(5-3n)}{120}\sigma_9(n) + \frac{1}{264}\sigma(n/2) - \frac{141}{6910}\tau(n) - \frac{2688}{691}\tau(n/2),$$

(vi) $$U_{9,1}(n) = \frac{31}{5837568}\sigma_{11}(n) + \frac{1}{176896}\sigma_{11}(n/2) + \frac{31}{2073}\sigma_{11}(n/4) + \frac{(5-6n)}{120}\sigma_9(n/4) + \frac{1}{264}\sigma(n) - \frac{671}{176896}\tau(n) - \frac{2505}{22112}\tau(n/2) - \frac{105314}{3455}\tau(n/4) - 240f(n),$$

(vii) $$U_{7,3}(n) = -\frac{7}{2653440}\sigma_{11}(n) + \frac{1}{17689}\sigma_{11}(n/2) + \frac{17}{20730}\sigma_{11}(n/4) - \frac{1}{240}\sigma_7(n/4) - \frac{1}{480}\sigma_3(n) + \frac{369}{176896}\tau(n) + \frac{1641}{22112}\tau(n/2) + \frac{22203}{1382}\tau(n/4) + 120f(n),$$

(viii) $$U_{5,5}(n) = \frac{1}{11144448}\sigma_{11}(n) + \frac{1}{176896}\sigma_{11}(n/2) + \frac{16}{43533}\sigma_{11}(n/4) + \frac{1}{504}\sigma_5(n) + \frac{1}{504}\sigma_5(n/4) - \frac{351}{176896}\tau(n) - \frac{2505}{22112}\tau(n/2) - \frac{5616}{691}\tau(n/4),$$

$$\text{(ix)}\ U_{3,7}(n) = \frac{121}{2653440}\sigma_{11}(n) + \frac{1}{176896}\sigma_{11}(n/2) + \frac{8}{10365}\sigma_{11}(n/4) - \frac{1}{240}\sigma_7(n) - \frac{1}{480}\sigma_3(n/4) + \frac{729}{176896}\tau(n) + \frac{6003}{11056}\tau(n/2) - \frac{71496}{691}\tau(n/4) - 1920f(n),$$

$$\text{(x)}\ U_{1,9}(n) = -\frac{7}{16584}\sigma_{11}(n) + \frac{23}{5528}\sigma_{11}(n/2) + \frac{256}{22803}\sigma_{11}(n/4) + \frac{(10-3n)}{240}\sigma_9(n) + \frac{1}{264}\sigma(n/4) - \frac{1589}{55280}\tau(n) - \frac{5790}{691}\tau(n/2) + \frac{2562304}{691}\tau(n/4) + 61440f(n).$$

8. Sums of 12 triangular numbers

The triangular numbers are the nonnegative integers

$$T_k = \frac{1}{2}k(k+1), \quad k = 0,1,2,\ldots,$$

so that

$$T_0 = 0, \quad T_1 = 1, \quad T_2 = 3, \quad T_3 = 6, \quad T_4 = 10, \quad T_5 = 15, \ldots.$$

We set

$$\Delta = \{T_k \mid k = 0,1,2,\ldots\}.$$

For $m \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$, we let

$$\delta_m(n) = \mathrm{card}\{(t_1, t_2, \ldots, t_m) \in \Delta \times \Delta \times \cdots \times \Delta \mid n = t_1 + t_2 + \cdots + t_m\}$$

so that $\delta_m(n)$ counts the number of representations of n as the sum of m triangular numbers. For $n \in \mathbb{N} \cup \{0\}$ we have the classical results

$$\delta_4(n) = \sigma(2n+1) \tag{8.1}$$

and

$$\delta_8(n) = \sigma_3(n+1) - \sigma_3\left(\frac{n+1}{2}\right), \tag{8.2}$$

see for example [4, Theorem 10, p. 259; Theorem 12, p. 265], [10, Theorem 3, p. 80; Theorem 5, p. 82]. We use these two formulae in conjunction with Theorem 4.2 to give a very simple proof of the following result of Ono, Robins and Wahl [10, Theorem 7, p. 85], which was originally proved using the theory of modular forms.

THEOREM 8.1. *For* $n \in \mathbb{N} \cup \{0\}$

$$\delta_{12}(n) = \frac{1}{256}(\sigma_5(2n+3) - a(2n+3)),$$

where $a(n)$ *is defined in* (4.1).

Proof. We have by (8.1) and (8.2)

$$\begin{aligned}
\delta_{12}(n) &= \sum_{m=0}^{n} \delta_4(m)\delta_8(n-m) \\
&= \sum_{m=0}^{n} \sigma(2m+1)\left(\sigma_3(n-m+1) - \sigma_3\left(\frac{n-m+1}{2}\right)\right) \\
&= A - B,
\end{aligned}$$

where

$$\begin{aligned}
A &= \sum_{m=0}^{n} \sigma(2m+1)\sigma_3(n-m+1) \\
&= \sum_{k=1}^{n+1} \sigma_3(k)\sigma(2n+3-2k) \\
&= T_{3,1}(2n+3) \\
&= \frac{1}{240}(\sigma_5(2n+3) - \sigma(2n+3)),
\end{aligned}$$

by Theorem 4.2(i), and

$$\begin{aligned}
B &= \sum_{m=0}^{n} \sigma(2m+1)\sigma_3\left(\frac{n-m+1}{2}\right) \\
&= \sum_{k<\frac{2n+3}{4}} \sigma_3(k)\sigma(2n+3-4k) \\
&= U_{3,1}(2n+3) \\
&= -\frac{1}{240}\sigma(2n+3) + \frac{1}{3840}\sigma_5(2n+3) + \frac{1}{256}a(2n+3),
\end{aligned}$$

by Theorem 4.2(iii). Hence

$$\begin{aligned}
\delta_{12}(n) &= \frac{1}{240}\sigma_5(2n+3) - \frac{1}{3840}\sigma_5(2n+3) - \frac{1}{256}a(2n+3) \\
&= \frac{1}{256}(\sigma_5(2n+3) - a(2n+3))
\end{aligned}$$

as asserted. □

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