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LAGRANGIAN H-MINIMAL SURFACES WITH 1-PARAMETER FAMILY OF PAIR OF GREAT CIRCLES IN $S^2 \times S^2$

By

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Abstract. We give a characterization of totally geodesic Lagrangian surfaces among minimal or H-minimal surfaces in $S^2 \times S^2$.

Introduction

Lagrangian submanifolds in Kähler manifolds have been studied as very interesting subjects in differential geometry. In particular, for Lagrangian submanifolds in a complex projective space with a constant holomorphic sectional curvature, many results are known, for example, the existence conditions of the Lagrangian immersion and their congruence (cf.[2]). But for higher rank cases, for general Hermitian symmetric spaces, it seems that there are no such results about Lagrangian submanifolds. On the other hand, recently it was shown [3] that a totally geodesic Lagrangian torus $S^1 \times S^1$ in $S^2 \times S^2$ has Hamiltonian volume minimizing property.

In this paper, we consider Lagrangian surfaces in $S^2 \times S^2$ which are important next to the complex projective plane CP^2 among compact Kähler surfaces. Here, we explain typical examples of Lagrangian surfaces in $S^2 \times S^2$: (i) A surface which consists of two curves γ_1, γ_2 in S^2 embedded in $S^2 \times S^2$ by a product immersion. In this case, the surface $\gamma_1 \times \gamma_2$ is minimal if and only if both γ_i $(i = 1, 2)$ are great circles. (ii) Identify S^2 with the complex projective line CP^1 and be corresponded an element z of CP^1 to a pair of z and the complex conjugate $\bar{z}$, then we can get a totally geodesic Lagrangian surface in $S^2 \times S^2$.

We first construct a surface M of $S^2 \times S^2$ which consists of 1-parameter family of pair of great circles in $S^2 \times S^2$. Then we get the condition for the surface M to be Lagrangian in $S^2 \times S^2$ (Propositions 1 and 2). Moreover we show that if such a Lagrangian surface M is minimal, then M is totally geodesic and M is locally congruent to either the example (i) or (ii) above (Theorem 6). In the last

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section, we study about Lagrangian Hamiltonian minimal (H-minimal) surfaces. For a compact Lagrangian submanifold in a Kähler manifold, the submanifold is Lagrangian H-minimal if it has extremal volume under all Hamiltonian deformations of the Lagrangian immersion (see [1] and section 1). By definition, a compact Lagrangian minimal submanifold in a Kähler manifold is H-minimal, but generally, not vice versa. We give the condition for the Lagrangian surface M to be H-minimal (Theorem 7).

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1. Preliminaries

Let $\widetilde{M}$ be a Kähler manifold of complex dimension m with Kähler form θ and complex structure J. Let M be a real m-dimensional submanifold and let $x : M \to \widetilde{M}$ be a Lagrangian immersion, i.e., $x^*\theta = 0$ on M, or equivalently, for any tangent vector X of M, JX is contained in the normal space to M.

Let S^2 be a unit sphere in $\mathbf{R}^3$. For any $p \in S^2$, we define a linear transformation J of the tangent space T_pS^2 of S^2 at p as $Jv = p \times v$ by the vector product $\times$ of $\mathbf{R}^3$, so J is a complex structure on S^2. Then the special orthogonal group $SO(3)$ acts naturally on S^2 and is the isometry group for the Riemannian metric on S^2 which is induced by the standard inner product of $\mathbf{R}^3$. Moreover $SO(3)$ preserves J. Standard symplectic form θ on S^2 is given by $\theta_p(u, v) = (p \times u) \cdot v$, where $u, v \in T_pS^2$ and $\cdot$ is the induced Riemannian metric on S^2 by the inclusion $S^2 \subset \mathbf{R}^3$.

We define a complex structure $\widetilde{J}$ on $S^2 \times S^2$ by

$$\widetilde{J}(X_1, X_2) = (JX_1, JX_2) \tag{1.1}$$

for all tangent vectors (X_1, X_2) to $S^2 \times S^2$. Let $\langle\ ,\ \rangle$ be the product metric on $S^2 \times S^2$ defined by

$$\langle (X_1, X_2), (Y_1, Y_2) \rangle = X_1 \cdot Y_1 + X_2 \cdot Y_2.$$

Then $\langle\ ,\ \rangle$ is a Hermitian metric and $S^2 \times S^2$ is a Kähler manifold with respect to the complex structure $\widetilde{J}$. $S^2 \times S^2$ is considered as a symplectic manifold with symplectic form $\widetilde{\theta} = (\mathrm{pr}_1)^*\theta + (\mathrm{pr}_2)^*\theta$, where $\mathrm{pr}_1, \mathrm{pr}_2 : S^2 \times S^2 \to S^2$ are projection maps and θ is the standard symplectic form on S^2. We also denote $\cdot$ by $\langle\ ,\ \rangle$ in below.

Finally, we review a Hamiltonian minimal (H-minimal) submanifold ([6]). A compact Lagrangian submanifold M immersed in a Kähler manifold $(\widetilde{M}, J)$ is called *Hamiltonian minimal* or H-*minimal* if the first variation for the volume of M vanishes under all Hamiltonian deformations of M of $\widetilde{M}$ (cf.[1]). For the mean curvature vector field H of M, we define the one form α_H on M as $\alpha_H(X) = g(X, JH)$ for each tangent vector field X to M, where g is the induced metric of M. It is known that a compact Lagrangian submanifold M of $\widetilde{M}$ is H-minimal if and only if $\delta\alpha_H = 0$ on M where δ is the Hodge-dual of the exterior derivative d on M with respect to the induced metric g (cf.[6], Theorem 2.4). Hence, a compact Lagrangian submanifold M of $\widetilde{M}$ is H-minimal if and only if

$$\operatorname{div}(JH) = \sum_{i=1}^{n} g(\nabla_{e_i}(JH), e_i) = 0 \tag{1.2}$$

for an orthonormal basis $e_1, \cdots, e_n$ of a tangent space to M where ∇ is the Levi-Civita connection of M.

2. Lagrangian surfaces in $S^2 \times S^2$

Let $\gamma_i(t)(i = 1,2)$ be great circles in S^2. Then $(\gamma_1, \gamma_2) : t \to (\gamma_1(t), \gamma_2(t))$ is a geodesic in $S^2 \times S^2$. We put the set of all such pairs (γ_1, γ_2) as $\mathcal{M}$. We denote that $SO(n)$ is the special orthogonal group and $\mathfrak{o}(n)$ is the orthogonal Lie algebra. Because $SO(3) \times SO(3)$ acts transitively on $\mathcal{M}$, $\mathcal{M}$ is a homogeneous space of $SO(3) \times SO(3)$. Let $\gamma = \gamma(t)$ be an element of $\mathcal{M}$ satisfying

$$\gamma(t) = ((\cos t, \sin t, 0), (\cos t, \sin t, 0)).$$

If K is a set of all elements of $SO(3) \times SO(3)$ which preserve $\gamma \in \mathcal{M}$, then $K = \{(g, g) | g \in SO(2)\}$ and we identify $SO(3) \times SO(3)/K$ with $\mathcal{M}$. We define the natural projection by

$$\pi : SO(3) \times SO(3) \to \mathcal{M}.$$

Moreover, when we give naturally the two-sided invariant Riemannian metric for $SO(3) \times SO(3)$, we can introduce the Riemannian metric on $\mathcal{M}$ such that π is a Riemannian submersion. Let φ be a curve from an open interval I into $SO(3) \times SO(3)/K$ and $\widetilde{\varphi}(s) = (g_1(s), g_2(s)) \in SO(3) \times SO(3)$ be a horizontal lift of φ with respect to π. Then, we define a map $\Phi : I \times S^1 \to S^2 \times S^2$ by $\Phi(s,t) = \widetilde{\varphi}(s)\gamma(t)$. We denote the velocity vector of $g_i(s)$ as $g_i'(s)$ $(i = 1,2)$. We can easily see that $g_i^{-1}(s)g_i'(s)$ $(i = 1,2)$ are skew-symmetric matrices of degree 3. We put for some functions a_i, b_i and $c_i(i = 1,2)$ with respect to s

$$g_1^{-1}(s)g_1'(s) = \begin{pmatrix} 0 & a_1(s) & b_1(s) \\ -a_1(s) & 0 & c_1(s) \\ -b_1(s) & -c_1(s) & 0 \end{pmatrix},$$

$$g_2^{-1}(s)g_2'(s) = \begin{pmatrix} 0 & a_2(s) & b_2(s) \\ -a_2(s) & 0 & c_2(s) \\ -b_2(s) & -c_2(s) & 0 \end{pmatrix}.$$

Any element of K is expressed as

$$\left(\begin{pmatrix} \cos s & -\sin s & 0 \\ \sin s & \cos s & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \cos s & -\sin s & 0 \\ \sin s & \cos s & 0 \\ 0 & 0 & 1 \end{pmatrix} \right),$$

and the Lie algebra of K is spanned by

$$\left(\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right). \tag{2.1}$$

Since $\widetilde{\varphi}(s) = (g_1(s), g_2(s)) \in SO(3) \times SO(3)$ is horizontal, $(g_1^{-1}(s)g_1'(s), g_2^{-1}(s)g_2'(s)) \in \mathfrak{o}(3) \times \mathfrak{o}(3)$ is orthogonal to (2.1). Hence we have $a_1(s) + a_2(s) = 0$. So we put $a(s) = a_1(s) = -a_2(s)$. The map $\Phi : I \times S^1 \to S^2 \times S^2$ is written as

$$\Phi(s,t) = \left(g_1(s) \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix}, g_2(s) \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix} \right). \tag{2.2}$$

First order differentials of Φ are

$$\Phi_t := \frac{\partial \Phi}{\partial t} = \left(g_1(s) \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix}, g_2(s) \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix} \right)$$

and

$$\begin{aligned} \Phi_s := \frac{\partial \Phi}{\partial s} = \Biggl(g_1(s) & \begin{pmatrix} a(s) \sin t \\ -a(s) \cos t \\ -b_1(s) \cos t - c_1(s) \sin t \end{pmatrix}, \\ g_2(s) & \begin{pmatrix} -a(s) \sin t \\ a(s) \cos t \\ -b_2(s) \cos t - c_2(s) \sin t \end{pmatrix} \Biggr). \end{aligned} \tag{2.3}$$

So we get $\langle \Phi_t, \Phi_t \rangle = 2$ and $\langle \Phi_t, \Phi_s \rangle = 0$. Hence Φ is regular at (s,t) if and only if $\langle \Phi_s, \Phi_s \rangle \neq 0$. We get from (2.3)

$$\begin{aligned} \langle \Phi_s, \Phi_s \rangle = 2a(s)^2 &+ \{b_1(s)^2 + b_2(s)^2\} \cos^2 t + \{c_1(s)^2 + c_2(s)^2\} \sin^2 t \\ &+ \{b_1(s)c_1(s) + b_2(s)c_2(s)\} \sin 2t. \end{aligned}$$

We put the right hand of this equation as $f(s,t)$:

$$\begin{aligned} f(s,t) &= 2a(s)^2 + \{b_1(s)^2 + b_2(s)^2\}\cos^2 t \\ &\quad + \{c_1(s)^2 + c_2(s)^2\}\sin^2 t + \{b_1(s)c_1(s) + b_2(s)c_2(s)\}\sin 2t. \end{aligned} \tag{2.4}$$

Now, we try to find the condition for the immersion Φ to be Lagrangian with respect to the complex structure $\widetilde{J}$ defined by (1.1), i.e., the condition of $\langle \widetilde{J}\Phi_t, \Phi_s\rangle = 0$. We can express $\widetilde{J}\Phi_t$ by using the vector product $\times$ of $\mathbf{R}^3$ as

$$\begin{aligned} \widetilde{J}\Phi_t = \Bigg(& g_1(s)\begin{pmatrix}\cos t\\ \sin t\\ 0\end{pmatrix} \times g_1(s)\begin{pmatrix}-\sin t\\ \cos t\\ 0\end{pmatrix}, \\ & g_2(s)\begin{pmatrix}\cos t\\ \sin t\\ 0\end{pmatrix} \times g_2(s)\begin{pmatrix}-\sin t\\ \cos t\\ 0\end{pmatrix}\Bigg). \end{aligned} \tag{2.5}$$

So we get

$$\langle \widetilde{J}\Phi_t, \Phi_s\rangle = -\{b_1(s) + b_2(s)\}\cos t - \{c_1(s) + c_2(s)\}\sin t.$$

Hence, Φ is a Lagrangian immersion if and only if $b_1(s) + b_2(s) = 0$ and $c_1(s) + c_2(s) = 0$. Then we get

PROPOSITION 1. *For the map $\Phi : I \times S^1 \to S^2 \times S^2$ given by (2.2), if Φ is a Lagrangian immersion, then*

$$g_1^{-1}(s)g_1'(s) + g_2^{-1}(s)g_2'(s) = 0. \tag{2.6}$$

In the following, we put $b(s) = b_1(s) = -b_2(s)$ and $c(s) = c_1(s) = -c_2(s)$ and assume that a parameter s of the curve $\varphi : I \to \mathcal{M}$ satisfies $a(s)^2+b(s)^2+c(s)^2 = 1$. So (2.4) becomes

$$f(s,t) = 1 + a(s)^2 + \{b(s)^2 - c(s)^2\}\cos 2t + 2b(s)c(s)\sin 2t. \tag{2.7}$$

Then we can see that

PROPOSITION 2. *For the map $\Phi : I \times S^1 \to S^2 \times S^2$ given by (2.2), suppose that $g_1(s)$ and $g_2(s)$ satisfy the equation (2.6) and a, b and c are functions on I which satisfy*

$$g_1^{-1}(s)g_1'(s) = -g_2^{-1}(s)g_2'(s) = \begin{pmatrix} 0 & a(s) & b(s)\\ -a(s) & 0 & c(s)\\ -b(s) & -c(s) & 0\end{pmatrix}.$$

Then, for the function $f(s,t)$ given by (2.7),

(i) *If* $a(s) \neq 0$, *then* $f(s,t) \neq 0$ *for any* t.
(ii) *If* $a(s_0) = 0$, *then* $f(s,t) = 0$ *for some* t_0, *so* $d\Phi$ *is singular at* (s_0, t_0).

Now we review the almost product structure of $S^2 \times S^2$ (cf.[5]). The almost product structure $\widetilde{P}$ of $S^2 \times S^2$ is defined by

$$\widetilde{P}(X_1, X_2) = (X_1, -X_2) \quad \text{for} \quad (X_1, X_2) \in S^2 \times S^2.$$

If M is a Lagrangian surface of $S^2 \times S^2$, then we have the following (cf.[4]):

LEMMA 3. *Let* $x : M \to S^2 \times S^2$ *be a Lagrangian immersion and* $\widetilde{P}$ *be the almost product structure of* $S^2 \times S^2$. *If the vector* $\widetilde{P}X$ *is orthogonal to the tangent space to* M *for any tangent vector* X *of* M, *then the immersion* x *is totally geodesic and the Gauss curvature* K *of* M *satisfies* $K \equiv 1/2$.

Here, we suppose $a(s) = 0$ for the curve φ in $\mathcal{M}$. Since

$$\widetilde{P}\Phi_s = \left(g_1(s) \begin{pmatrix} 0 \\ 0 \\ -b(s)\cos t - c(s)\sin t \end{pmatrix}, g_2(s) \begin{pmatrix} 0 \\ 0 \\ -b(s)\cos t - c(s)\sin t \end{pmatrix} \right),$$

we get

$$\langle \widetilde{P}\Phi_s, \Phi_t \rangle = 0 \quad \text{and} \quad \langle \widetilde{P}\Phi_s, \Phi_s \rangle = 0.$$

So $\widetilde{P}\Phi_s$ is a normal vector to M. Hence we get from the Lemma above

PROPOSITION 4. *Define a Lagrangian immersion* $\Phi : I \times S^1 \to S^2 \times S^2$ *as*

$$\Phi(s,t) = \left(g_1(s) \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix}, g_2(s) \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix} \right)$$

where $g_1(s), g_2(s) \in SO(3)$ *satisfy*

$$g_1^{-1}(s) g_1'(s) = -g_2^{-1}(s) g_2'(s) = \begin{pmatrix} 0 & 0 & b(s) \\ 0 & 0 & c(s) \\ -b(s) & -c(s) & 0 \end{pmatrix}$$

for functions b *and* c *on* I. *Then the immersion* Φ *is totally geodesic.*

3. Lagrangian minimal surfaces in $S^2 \times S^2$

First we consider examples of which functions a, b and c are constant and satisfy

$$g_1^{-1}(s) g_1'(s) = -g_2^{-1}(s) g_2'(s) = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} =: A.$$

Note that solutions of $g_1'(s) = g_1(s)A$ and $g_2'(s) = g_2(s)(-A)$ are

$$g_1(s) = g_1(0)\exp(sA) \qquad \text{and} \qquad g_2(s) = g_2(0)\exp(-sA).$$

We suppose that $g_1(0)$ and $g_2(0)$ are unit matrices of degree 3. Then

$$g_1(s) = \exp(sA) \qquad \text{and} \qquad g_2(s) = \exp(-sA). \tag{3.1}$$

Example 1. The case of $a = 1$ and $b = c = 0$.

In this case, we have

$$\exp(sA) = \exp\begin{pmatrix} 0 & s & 0 \\ -s & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cos s & \sin s & 0 \\ -\sin s & \cos s & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We calculate similarly for $\exp(-sA)$. So we get

$$(g_1(s), g_2(s)) = \left(\begin{pmatrix} \cos s & \sin s & 0 \\ -\sin s & \cos s & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \cos s & -\sin s & 0 \\ \sin s & \cos s & 0 \\ 0 & 0 & 1 \end{pmatrix} \right).$$

Then (2.2) is

$$\Phi(s,t) = \left(\begin{pmatrix} \cos(t-s) \\ \sin(t-s) \\ 0 \end{pmatrix}, \begin{pmatrix} \cos(t+s) \\ \sin(t+s) \\ 0 \end{pmatrix} \right).$$

Since we can regard $t - s$ and $t + s$ as independent variable, Φ is a product immersion and totally geodesic immersion. In fact, we put

$$\Phi_1 := \left(\begin{pmatrix} \cos(t-s) \\ \sin(t-s) \\ 0 \end{pmatrix}, 0 \right), \quad \Phi_2 := \left(0, \begin{pmatrix} \cos(t+s) \\ \sin(t+s) \\ 0 \end{pmatrix} \right),$$

then they are unit normal vectors to $S^2 \times S^2 \subset \mathbf{R}^3 \times \mathbf{R}^3$. Hence for $X \in \mathbf{R}^3 \times \mathbf{R}^3$, if we denote the normal component of X to $S^1 \times S^1 \subset S^2 \times S^2$ as $X^\perp$ and $\partial^2\Phi/\partial t^2$ as Φ_{tt}, similarly to Φ_{ts}, Φ_{ss} , then

$$\begin{aligned} \sigma(\partial/\partial t, \partial/\partial t) &= (\Phi_{tt})^\perp = (-\Phi)^\perp = 0, \\ \sigma(\partial/\partial t, \partial/\partial s) &= (\Phi_{ts})^\perp = (\Phi_1 - \Phi_2)^\perp = 0, \\ \sigma(\partial/\partial s, \partial/\partial s) &= (\Phi_{ss})^\perp = (-\Phi)^\perp = 0 \end{aligned}$$

where σ is the second fundamental form of Φ. Therefore we have $\sigma \equiv 0$, i.e., Φ is a totally geodesic immersion.

EXAMPLE 2. The case of $a = c = 0$ and $b = 1$.

By the same calculation as Example 1, we get

$$\Phi(s,t) = \left(\begin{pmatrix} \cos s \cos t \\ \sin t \\ -\sin s \cos t \end{pmatrix}, \begin{pmatrix} \cos s \cos t \\ \sin t \\ \sin s \cos t \end{pmatrix} \right).$$

Hence $\Phi : S^2 \to S^2 \times S^2$ satisfies

$$\Phi \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \left(\begin{pmatrix} x \\ y \\ -z \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right).$$

Then we have

$$\Phi_t = \left(\begin{pmatrix} -\cos s \sin t \\ \cos t \\ \sin s \sin t \end{pmatrix}, \begin{pmatrix} -\cos s \sin t \\ \cos t \\ -\sin s \sin t \end{pmatrix} \right),$$

$$\Phi_s = \left(\begin{pmatrix} -\sin s \cos t \\ 0 \\ -\cos s \cos t \end{pmatrix}, \begin{pmatrix} -\sin s \cos t \\ 0 \\ \cos s \cos t \end{pmatrix} \right),$$

so we get $\langle \Phi_t, \Phi_t \rangle = 2, \langle \Phi_t, \Phi_s \rangle = 0$ and $\langle \Phi_s, \Phi_s \rangle = 2\cos^2 t$. Hence, Φ is regular if $t \neq \pi/2 (\mathrm{mod}\, \pi)$. Then, we have $\sigma(\partial/\partial t, \partial/\partial t) = \sigma(\partial/\partial t, \partial/\partial s) = \sigma(\partial/\partial s, \partial/\partial s) = 0$ by the similar calculation to Example 1. So Φ is totally geodesic.

EXAMPLE 3. The case of a, b and c are general constant real numbers satisfying $a^2 + b^2 + c^2 = 1$.

We have from (2.2) and (3.1)

$$\Phi(s,t) = \left(\exp(sA) \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix}, \exp(-sA) \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix} \right),$$

so we get

$$\Phi_t = \left(\exp(sA) \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix}, \exp(-sA) \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix} \right),$$

$$\Phi_s = \left(\exp(sA) \begin{pmatrix} a \sin t \\ -a \cos t \\ -b \cos t - c \sin t \end{pmatrix}, \exp(-sA) \begin{pmatrix} -a \sin t \\ a \cos t \\ b \cos t + c \sin t \end{pmatrix} \right).$$

Since A and $-A$ are skew-symmetric matrices, $\exp(sA)$ and $\exp(-sA)$ are orthogonal matrices. Then these matrices preserve the Riemannian metric $\langle\ ,\ \rangle$. Therefore, we get $\langle \Phi_t, \Phi_t \rangle = 2, \langle \Phi_t, \Phi_s \rangle = 0$ and $\langle \Phi_s, \Phi_s \rangle = 1 + a^2 + (b^2 - c^2)\cos 2t + 2bc \sin 2t$. We put also Example 1 that

$$\Phi_1 := \left(\exp(sA) \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix}, 0 \right), \quad \Phi_2 := \left(0, \exp(-sA) \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix} \right),$$

and so Φ_1 and Φ_2 are unit normal vectors to $S^2 \times S^2 \subset \mathbf{R}^3 \times \mathbf{R}^3$. Then we consider the condition for Φ to be a minimal immersion. Since $\sigma(\partial/\partial t, \partial/\partial t) = 0$, Φ is minimal if and only if $\sigma(\partial/\partial s, \partial/\partial s) = 0$. Since

$$\begin{aligned} &\Phi_{ss} \\ &= \left(\exp(sA) \begin{pmatrix} (c^2-1)\cos t - bc \sin t \\ (b^2-1)\sin t - bc\cos t \\ -ab \sin t + ac \cos t \end{pmatrix}, \exp(-sA) \begin{pmatrix} (c^2-1)\cos t - bc \sin t \\ (b^2-1)\sin t - bc\cos t \\ -ab \sin t + ac \cos t \end{pmatrix} \right), \end{aligned}$$

we have

$$\begin{aligned} \langle \Phi_{ss}, \Phi_1 \rangle &= -\frac{1}{2}\{1 + a^2 + (b^2 - c^2)\cos 2t + 2bc \sin 2t\}, \\ \langle \Phi_{ss}, \Phi_2 \rangle &= -\frac{1}{2}\{1 + a^2 + (b^2 - c^2)\cos 2t + 2bc \sin 2t\}, \\ \langle \Phi_{ss}, \Phi_t \rangle &= (b^2 - c^2)\sin 2t - 2bc \cos 2t, \\ \langle \Phi_{ss}, \Phi_s \rangle &= 0. \end{aligned}$$

Hence, we get

$$\begin{aligned} &\sigma(\partial/\partial s, \partial/\partial s) \\ &= \Phi_{ss} + \frac{1}{2}\{1 + a^2 + (b^2 - c^2)\cos 2t + 2bc \sin 2t\}\Phi \\ &\quad -\frac{1}{2}\{(b^2 - c^2)\sin 2t - 2bc\cos 2t\}\Phi_t \\ &= \left(\exp(sA) \begin{pmatrix} 0 \\ 0 \\ a(c\cos t - b \sin t) \end{pmatrix}, \exp(-sA) \begin{pmatrix} 0 \\ 0 \\ a(c \cos t - b \sin t) \end{pmatrix} \right). \end{aligned}$$

So the immersion Φ is minimal if and only if $a(c\cos t - b\sin t) = 0$, i.e., $a = 0$ or $a \neq 0$ and $b = c = 0$.

Therefore, we have from Proposition 4 and Example 1

PROPOSITION 5. *Let $\Phi : I \times S^1 \to S^2 \times S^2$ be a Lagrangian immersion defined by*

$$\Phi(s,t) = \left(g_1(s) \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix}, g_2(s) \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix} \right)$$

where $g_1(s), g_2(s) \in SO(3)$ satisfy

$$g_1^{-1}(s) g_1'(s) = -g_2^{-1}(s) g_2'(s) = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$$

for some constant real numbers a, b and c. Then the immersion Φ is minimal if and only if either $a = 0$ or $a \neq 0$ and $b = c = 0$. Furthermore, the immersion Φ is totally geodesic.

Next, we try to find the condition for which the Lagrangian immersion $\Phi(s,t)$ defined by functions $a(s), b(s)$ and $c(s)$, which are not necessarily constant, is minimal. By a similar calculation above, we get $\sigma(\partial/\partial t, \partial/\partial t) = 0$. Hence, Φ is a minimal immersion if and only if $\sigma(\partial/\partial s, \partial/\partial s) = 0$, i.e., $\langle \Phi_{ss}, \widetilde{J}\Phi_t \rangle = \langle \Phi_{ss}, \widetilde{J}\Phi_s \rangle = 0$. By a straightforward computation, we get

$$\Phi_{ss} = \left(g_1(s) \begin{pmatrix} -\{a(s)^2 + b(s)^2\} \cos t + \{-b(s)c(s) + a'(s)\} \sin t \\ -\{b(s)c(s) + a'(s)\} \cos t - \{(a(s)^2 + c(s)^2\} \sin t \\ \{a(s)c(s) - b'(s)\} \cos t - \{a(s)b(s) + c'(s)\} \sin t \end{pmatrix}, \right.$$
$$\left. g_2(s) \begin{pmatrix} -\{a(s)^2 + b(s)^2\} \cos t - \{b(s)c(s) + a'(s)\} \sin t \\ \{-b(s)c(s) + a'(s)\} \cos t - \{a(s)^2 + c(s)^2\} \sin t \\ \{a(s)c(s) + b'(s)\} \cos t + \{-a(s)b(s) + c'(s)\} \sin t \end{pmatrix} \right).$$

So we have from (2.5)

$$\langle \Phi_{ss}, \widetilde{J}\Phi_t \rangle = 2a(s)\{-b(s) \sin t + c(s) \cos t\}. \tag{3.2}$$

Hence $\langle \Phi_{ss}, \widetilde{J}\Phi_t \rangle = 0$ if and only if $a(s) = 0$ or $a(s) \neq 0$ and $b(s) = c(s) = 0$. For

$$\widetilde{J}\Phi_s = \left(g_1(s) \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix} \times g_1(s) \begin{pmatrix} a(s) \sin t \\ -a(s) \cos t \\ -b(s) \cos t - c(s) \sin t \end{pmatrix}, \right.$$
$$\left. g_2(s) \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix} \times g_2(s) \begin{pmatrix} -a(s) \sin t \\ a(s) \cos t \\ b(s) \cos t + c(s) \sin t \end{pmatrix} \right),$$

we get

$$\begin{aligned}&\langle \Phi_{ss}, \widetilde{J}\Phi_s\rangle \\ &= -2a'(s)\{b(s)\cos t + c(s)\sin t\} + 2a(s)\{b'(s)\cos t + c'(s)\sin t\}. \quad (3.3)\end{aligned}$$

If $a(s) = 0$ or $b(s) = c(s) = 0$, then $\langle \Phi_{ss}, \widetilde{J}\Phi_s\rangle = 0$.

Hence, we get the following result.

THEOREM 6 ([7]). *Lagrangian minimal surfaces in $S^2 \times S^2$ which consist of 1- parameter family of pair (γ_1, γ_2) where γ_1 and γ_2 are great circles in S^2 are totally geodesic and they are locally congruent to either (a) $S^1 \times S^1 \subset S^2 \times S^2$ or (b) $S^2 \subset S^2 \times S^2$.*

4. Lagrangian H-minimal surfaces in $S^2 \times S^2$

In this section, we consider the condition for which the Lagrangian immersion $\Phi : I \times S^1 \to S^2 \times S^2$ with $a(s) \neq 0$ is H-minimal. Because of (1.2), we want to define a Lagrangian immersion $\Phi : I \times S^1 \to S^2 \times S^2 \subset \mathbf{R}^3 \times \mathbf{R}^3$ satisfying

$$\begin{aligned}\operatorname{div}(\widetilde{J}H) &= \langle D_{\Phi_s}(\widetilde{J}H), \Phi_s\rangle/\|\Phi_s\|^2 + \langle D_{\Phi_t}(\widetilde{J}H), \Phi_t\rangle/\|\Phi_t\|^2 \\ &= 0 \quad (4.1)\end{aligned}$$

where D is a connection of $\mathbf{R}^3 \times \mathbf{R}^3$. Here, from $\sigma(\partial/\partial t, \partial/\partial t) = 0$, the mean curvature H of Φ is $\sigma(\partial/\partial s, \partial/\partial s)/2f(s,t)$ where $f(s,t)$ is the function satisfying (2.7) (with $a(s) \neq 0$). We denote the functions a, b, c and f for simplicity in below. Then

$$\sigma(\partial/\partial s, \partial/\partial s) = \langle \Phi_{ss}, \widetilde{J}\Phi_s\rangle \widetilde{J}\Phi_s/\|\Phi_s\|^2 + \langle \Phi_{ss}, \widetilde{J}\Phi_t\rangle \widetilde{J}\Phi_t/\|\Phi_t\|^2$$

and so we have from $\|\Phi_s\|^2 = f$, $\|\Phi_t\|^2 = 2$, (3.2) and (3.3)

$$\begin{aligned}&\widetilde{J}H \\ &= a(b\sin t - c\cos t)\Phi_t/2f + \{(a'b - ab')\cos t + (a'c - ac')\sin t\}\Phi_s/f^2. \quad (4.2)\end{aligned}$$

We get from (4.1) and (4.2)

$$\operatorname{div}(\widetilde{J}H) = 2\{(a''h - ah_{ss} + ah)(a^2 + h^2) + 3h_t h_{st}(a'h - ah_s)\}/f^3$$

where $h = h(s,t) = b(s)\cos t + c(s)\sin t$, $h_s = \partial h/\partial s$ and similarly for h_t, h_{ss} and h_{st}. We can express the right hand of this equation by only terms of

$\cos t, \sin t, \cos 3t$ and $\sin 3t$. Since functions $\cos t, \sin t, \cos 3t$ and $\sin 3t$ are linearly independent, we get from $\operatorname{div}(\widetilde{J}H) = 0$

$$\begin{cases} (a^2+3)ba'' - (4a^2+3b^2+c^2)ab'' - 2abcc'' \\ \quad = -(a^2+3)ab - 3(bb'+3cc')(a'b-ab') + 3(bc'+b'c)(a'c-ac'), \\ (a^2+3)ca'' - 2abcb'' - (4a^2+b^2+3c^2)ac'' \\ \quad = -(a^2+3)ac - 3(3bb'+cc')(a'c-ac') + 3(bc'+b'c)(a'b-ab'), \\ (b^2-3c^2)ba'' + (-b^2+c^2)ab'' + 2abcc'' \\ \quad = (-b^2+3c^2)ab + 3(bb'-cc')(a'b-ab') - 3(bc'+b'c)(a'c-ac'), \\ (3b^2-c^2)ca'' - 2abcb'' + (-b^2+c^2)ac'' \\ \quad = (-3b^2+c^2)ac + 3(bb'-cc')(a'c-ac') + 3(bc'+b'c)(a'b-ab'). \end{cases}$$

We put

$$a = \cos\alpha, \quad b = \sin\alpha\cos\beta, \quad c = \sin\alpha\sin\beta \tag{4.3}$$

for these equations where $\alpha = \alpha(s) \in [0, \pi]$ and $\beta = \beta(s) \in [0, 2\pi]$ are some functions, then we have

$$\begin{aligned} &\alpha''(\cos^2\alpha+3)\cos\beta - \beta''\cos\alpha\sin\alpha(3\cos^2\alpha+1)\sin\beta \\ &+ 3(\alpha')^2\cos\alpha\sin\alpha\cos\beta - 2(\beta')^2\cos\alpha\sin\alpha(\sin^2\alpha+2)\cos\beta \\ &- \alpha'\beta'(8\cos^2\alpha - 3\sin^4\alpha)\sin\beta - \cos\alpha\sin\alpha(\cos^2\alpha+3)\cos\beta = 0, \end{aligned} \tag{4.4}$$

$$\begin{aligned} &\alpha''(\cos^2\alpha+3)\sin\beta + \beta''\cos\alpha\sin\alpha(3\cos^2\alpha+1)\cos\beta \\ &+ 3(\alpha')^2\cos\alpha\sin\alpha\sin\beta - 2(\beta')^2\cos\alpha\sin\alpha(\sin^2\alpha+2)\sin\beta \\ &+ \alpha'\beta'(8\cos^2\alpha - 3\sin^4\alpha)\cos\beta - \cos\alpha\sin\alpha(\cos^2\alpha+3)\sin\beta = 0, \end{aligned} \tag{4.5}$$

$$\begin{aligned} &\alpha''\sin^2\alpha\cos3\beta - \beta''\cos\alpha\sin^3\alpha\sin3\beta \\ &- 3(\alpha')^2\cos\alpha\sin\alpha\cos3\beta + 2(\beta')^2\cos\alpha\sin^3\alpha\cos3\beta \\ &+ \alpha'\beta'\sin^2\alpha(\cos^2\alpha+3)\sin3\beta - \cos\alpha\sin^3\alpha\cos3\beta = 0, \end{aligned} \tag{4.6}$$

$$\begin{aligned} &\alpha''\sin^2\alpha\sin3\beta + \beta''\cos\alpha\sin^3\alpha\cos3\beta \\ &- 3(\alpha')^2\cos\alpha\sin\alpha\sin3\beta + 2(\beta')^2\cos\alpha\sin^3\alpha\sin3\beta \\ &- \alpha'\beta'\sin^2\alpha(\cos^2\alpha+3)\cos3\beta - \cos\alpha\sin^3\alpha\sin3\beta = 0. \end{aligned} \tag{4.7}$$

Notice that $\alpha \neq \pi/2$ because $a \neq 0$. We get from (4.6) and (4.7)

$$\alpha'' = 3(\alpha')^2\cot\alpha - 2(\beta')^2\cos\alpha\sin\alpha + \cos\alpha\sin\alpha,$$

$$\beta'' \cos\alpha \sin\alpha = (\cos^2\alpha + 3)\alpha'\beta'$$

and we substitute these equations for (4.4) and (4.5). Then we get

$$-(\alpha')^2 \cos\beta / \sin\alpha + (\beta')^2 \sin\alpha \cos\beta + 2\alpha'\beta' \cos\alpha \sin\beta = 0,$$
$$-(\alpha')^2 \sin\beta / \sin\alpha + (\beta')^2 \sin\alpha \sin\beta - 2\alpha'\beta' \cos\alpha \cos\beta = 0.$$

Hence we have from these equations

$$-(\alpha')^2 + (\beta')^2 \sin^2\alpha = 0, \tag{4.8}$$
$$\alpha'\beta' \cos\alpha = 0. \tag{4.9}$$

If $\sin\alpha \neq 0$, we get $\alpha' = \beta' = 0$ (i.e., α and β are constant) because of (4.8), (4.9) and $\cos\alpha \neq 0$. Then equations (4.4) and (4.5) are $\cos\beta = 0$ and $\sin\beta = 0$. So this case does not occur. If $\sin\alpha = 0$, we get $a = 1$ and $b, c = 0$ for (4.3). Then this is the case of Example 1.

Finally, we consider the case of $\cos\alpha = 0$. Then we get $a = 0$ for (4.3). By Proposition 4, the immersion Φ is totally geodesic.

Hence we have

THEOREM 7. *Lagrangian H-minimal surfaces in $S^2 \times S^2$ which consist of* 1-*parameter family of pair (γ_1, γ_2) where γ_1 and γ_2 are great circles in S^2 are totally geodesic and they are locally congruent to either* (*a*) $S^1 \times S^1 \subset S^2 \times S^2$ *or* (*b*) $S^2 \subset S^2 \times S^2$.

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