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EXPLICIT CONSTRUCTION OF CONTROLLED-U GATES AND UNITARY OPERATORS IN TWO-QUDIT

By

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Abstract. We concretely construct extensions of controlled-Ugates in multivalued algebra from some elementary gates. We also construct unitary operators in two-qudit by means of the extended controlled-Ugates and show its universality.

1. Introduction

To build a theory of quantum computation a qubit, which is a state in a Hilbert space $\mathbb{C}^2$, has been used as a unit of computation[5, 6]. The logic in the qubit theory is simple and is compared to that in a theory of classical computation.

However there are some disadvantages in it. First number of steps in calculation tends to become large. It is, of course, favored to be small. Also in the point of view of the decoherence (which is the suffering of the systems by any unwanted interactions with the outside of them) problem[16, 7], this is desired. Next to realize quantum computers in actual physical systems, energy levels of atoms are utilized, while many physical systems have more than two-level. To apply binary logic many levels are discarded wastefully.

Instead of binary algebra, multivalued algebra can be considered. A counterpart of the qubit is a qudit, which is a state in a Hilbert space $\mathbb{C}^d$ $(d > 2)$ and an n-qudit is a state in the tensor product $(\mathbb{C}^d)^{\otimes n}$.

Computation in multivalied algebra has some advantages to that in binary one. First use of multivalued unit decreases number of steps. This is favorable for the decoherence problem. Next we can use high excited states in physical systems effectively.

Quantum gates in the qudit theory are unitary operators on multiqudits. Such gates have been devised by some researchers[13, 3, 12, 8, 9, 10, 11, 4, 2]. However, as far as we know, there is no work on construction of unitary operators from elementary gates concretely. Also there is no proof of its universality, of

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which meaning is that any unitary operator can be constructed. It seems that the universality is approved on the analogy of the qubit theory, however, it should be proved strictly. In constructing a controlled-Ugate (a gate that operates a unitary operator U under a specific condition) in the qubit theory, the Euler decomposition is used. However it seems to be complicated in the qudit one where number of states in one-qudit is large[14, 15]. So, instead of it, we use the diagonalization method of unitary operators. An advantage of it is that, even when number of states is large, networks are relatively easy built.

This paper is organized as follows. In Sec.2 we give some elementary gates. We extend controlled-Ugates, which were introduced in [1] for the qubit theory, to construct unitary operators in Sec. 3. In Sec. 4 we show that we can construct any unitary operator in two-qudit using the controlled-Ugates constructed in the previous section. The last section is devoted to the discussion.

2. Elementary Gates for Qudit

We begin by introducing some notations and elementary gates in our method. A single qudit has d-states, labeled

$$|k\rangle \qquad (k \in \mathbb{Z}_d)$$

and identified, in matrix notation, with d-dimensional vector

$$|k\rangle = {}^t(0, 0, \dots, 0, \overset{\overset{k}{\vee}}{1}, 0, \dots, 0),$$

where t means the transposition. The following is assumed:

1. any unitary gate can be used in a single qudit,
2. there exists at least one gate which interconnects two qudits.

2.1 Elementary gates in a single qudit

By assumption, no primary gate in a single qudit exists, however, there are some important gates.

First we introduce the exchange gate P_{ab} which operates

$$P_{ab}|c\rangle = \begin{cases} |b\rangle, & \text{if } |c\rangle = |a\rangle \\ |a\rangle, & \text{if } |c\rangle = |b\rangle \\ |c\rangle, & \text{otherwise} \end{cases} \qquad (a, b, c \in \mathbb{Z}_d),$$

and its matrix form is

$$(P_{ab})_{ij} = \delta_{ij} + \delta_{ia}(-\delta_{ja} + \delta_{jb}) + \delta_{ib}(\delta_{ja} - \delta_{jb}).$$

Next we introduce a gate which is an extension of the Walsh-Hadamard gate, which operates

$$H_{ab}|c\rangle = \begin{cases} \frac{1}{\sqrt{2}}(|a\rangle + |b\rangle), & \text{if } |c\rangle = |a\rangle \\ \frac{1}{\sqrt{2}}(|a\rangle - |b\rangle), & \text{if } |c\rangle = |b\rangle \\ |c\rangle, & \text{otherwise} \end{cases}$$

and its matrix form is

$$(H_{ab})_{ij} = \delta_{ij} + \delta_{ia}\left\{(-1+\frac{1}{\sqrt{2}})\delta_{ja} + \frac{1}{\sqrt{2}}\delta_{jb}\right\} + \delta_{ib}\left\{\frac{1}{\sqrt{2}}\delta_{ja} - (1+\frac{1}{\sqrt{2}})\delta_{jb}\right\}.$$

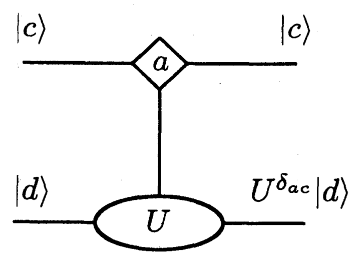
2.2 Controlled gates

A controlled gate has two bits. One of them is known as the control bit and the other is the target bit. This gate performs a unitary operation on the target bit if the control bit is set to a specific value and otherwise leaves the target bit alone.

We introduce a symbol $\tilde{C}^a(U)$ to express a controlled gate. C means controlled operation and exponent a indicates a state of the control bit in which a unitary operator U is applied. The tilde over C means two-qudit operation. By this symbol an operation of $\tilde{C}^a(U)$ is expressed as

$$\tilde{C}^a(U)|c\rangle|d\rangle = \begin{cases} |a\rangle(U|d\rangle), & \text{if } |c\rangle = |a\rangle \\ |c\rangle|d\rangle, & \text{otherwise} \end{cases},$$

where $|c\rangle|d\rangle$ is an abbreviation of the tensor product $|c\rangle \otimes |d\rangle$ and hereafter $\otimes$ is often omitted for simplicity. Also we introduce the diagram of $\tilde{C}^a(U)$ as follows:



The controlled-NOT (the controlled-σ_x) and the controlled-σ_z gate are used in interconnecting two-qubit in many times, where σ_x and σ_z are the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We adopt an extended controlled-σ_z gate which acts

> If the control bit is set to $|a\rangle$, then the sign of $|b\rangle$ in the target bit is reversed, otherwise the target bit is left alone.

This operation is written by

$$\tilde{C}^a(M_b)|c\rangle|d\rangle = \begin{cases} -|a\rangle|b\rangle, & \text{if } |c\rangle|d\rangle = |a\rangle|b\rangle \\ |c\rangle|d\rangle, & \text{otherwise} \end{cases}, \quad (a, b, c, d \in \mathbb{Z}_d), \tag{2.1}$$

where M_b is a single qudit operation which reverses the sign of $|b\rangle$:

$$M_b|c\rangle = \begin{cases} -|b\rangle, & \text{if } |c\rangle = |b\rangle \\ |c\rangle, & \text{otherwise} \end{cases}, (b, c \in \mathbb{Z}_d).$$

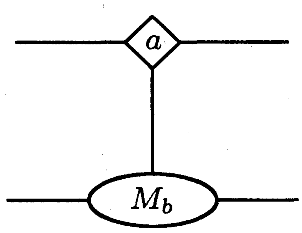
The matrix form of $\tilde{C}^a(M_b)$ is

$$(\tilde{C}^a(M_b))_{ij} = \delta_{ij}(1 - 2\delta_{i,ad+b}), \quad (i, j \in \mathbb{Z}_{d^2}), \tag{2.2}$$

where

$$M_b = \mathrm{diag}(1, 1, \dots, \overset{\overset{b}{\vee}}{-1}, \dots, 1), \quad (i, j \in \mathbb{Z}_d),$$

and the diagram is

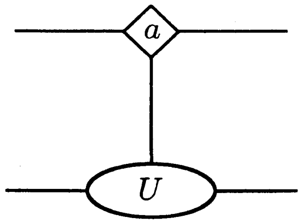


3. Construction of a Controlled-U Gate

In this section we construct a controlled-U gate which operates

$$\tilde{C}^a(U)|c\rangle|b\rangle = \begin{cases} |a\rangle(U|b\rangle), & \text{if } |c\rangle = |a\rangle \\ |c\rangle|b\rangle, & \text{otherwise} \end{cases},$$

of which diagram is



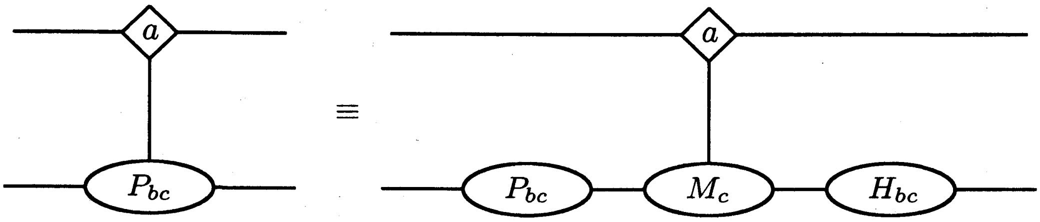
First we construct a controlled-exchange gate $\tilde{C}^a(P_{bc})$ which operates

$$\tilde{C}^a(P_{bc})|d\rangle|e\rangle = \begin{cases} |a\rangle|c\rangle, & \text{if } |d\rangle|e\rangle = |a\rangle|b\rangle \\ |a\rangle|b\rangle, & \text{if } |d\rangle|e\rangle = |a\rangle|c\rangle \\ |d\rangle|e\rangle, & \text{otherwise} \cdot \end{cases}$$

This gate is built by

$$\tilde{C}^a(P_{bc}) = (\mathbf{1} \otimes H_{bc})\tilde{C}^a(M_c)(\mathbf{1} \otimes H_{bc})$$

and the diagram is



Indeed, if the control bit is set to $|a\rangle$

$$\tilde{C}^a(P_{bc})(|a\rangle \sum_{k=0}^{d-1} \alpha_k |k\rangle) = |a\rangle(\alpha_c|b\rangle + \alpha_b|c\rangle + \sum_{k=0}^{d-1^{\vee b,c}} \alpha_k|k\rangle)$$

and otherwise

$$\tilde{C}^a(P_{bc})(|l\rangle \sum_{k=0}^{d-1} \alpha_k|k\rangle) = (\mathbf{1} \otimes \mathbf{1})|l\rangle \sum_{k=0}^{d-1} \alpha_k|k\rangle = |l\rangle \sum_{k=0}^{d-1} \alpha_k|k\rangle$$

where $\sum^{\vee b,c}$ means a sum except for b, c.

Second by means of $\tilde{C}^a(P_{bc})$ we construct a gate which operates

$$\tilde{C}^a(\Theta_b(\theta))|c\rangle|d\rangle = \begin{cases} e^{-i\theta}|a\rangle|0\rangle, & \text{if } |c\rangle|d\rangle = |a\rangle|0\rangle \\ e^{i\theta}|a\rangle|b\rangle, & \text{if } |c\rangle|d\rangle = |a\rangle|b\rangle \\ |c\rangle|d\rangle, & \text{otherwise} , \end{cases}$$

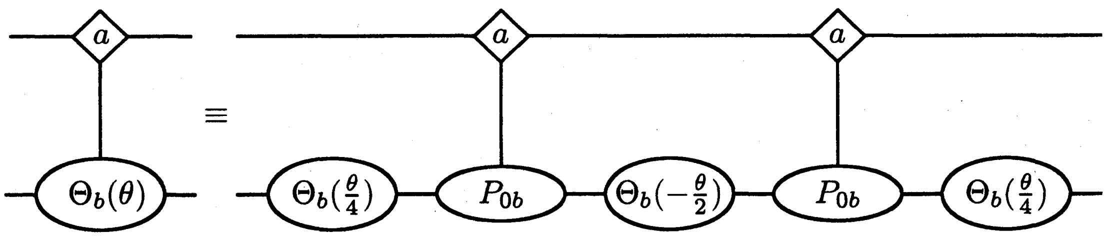
where $\Theta_b(\theta)$ is a single qudit operation of which matrix form is

$$\Theta_b(\theta) \equiv \operatorname{diag}(\overset{\overset{0}{\vee}}{e^{-i\theta}}, 1, \dots, \overset{\overset{b}{\vee}}{e^{i\theta}}, \dots, 1).$$

This gate is realized by

$$\tilde{C}^a(\Theta_b(\theta)) \equiv (\mathbf{1} \otimes \Theta_b(\frac{\theta}{4}))\tilde{C}^a(P_{0b})(\mathbf{1} \otimes \Theta_b(-\frac{\theta}{2})) \otimes \tilde{C}^a(P_{0b})(\mathbf{1} \otimes \Theta_b(\frac{\theta}{4}))$$

and the diagram is



Next we consider a single qudit operation

$$\Theta(\boldsymbol{\theta})|b\rangle = \begin{cases} e^{-i(\theta_1+\theta_2+\cdots+\theta_{d-1})}|0\rangle, & \text{if } |b\rangle = |0\rangle \\ e^{i\theta_b}|b\rangle, & \text{otherwise} \end{cases},$$

where we abbreviate

$$\boldsymbol{\theta} = (\theta_1, \theta_2, \dots, \theta_{d-1}).$$

The matrix form of this operation is

$$\Theta(\boldsymbol{\theta}) = \operatorname{diag}(e^{-i(\theta_1+\theta_2+\cdots+\theta_{d-1})}, e^{i\theta_1}, \dots, e^{i\theta_{d-1}}).$$

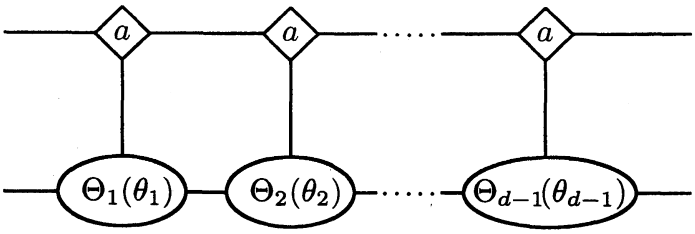
Then the controlled-$\Theta(\boldsymbol{\theta})$ operation:

$$\tilde{C}^a(\Theta(\boldsymbol{\theta}))|c\rangle|b\rangle = \begin{cases} |a\rangle\Theta(\boldsymbol{\theta})|b\rangle, & \text{if } |c\rangle = |a\rangle \\ |c\rangle|b\rangle, & \text{otherwise} \end{cases}$$

is realized by

$$\tilde{C}^a(\Theta(\boldsymbol{\theta})) = \prod_{b=1}^{d-1} \tilde{C}^a(\Theta_b(\theta_b))$$

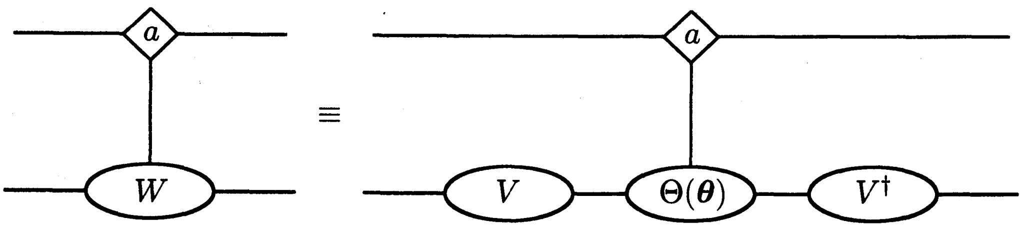
and the diagram is

$\tilde{C}^a(\Theta(\boldsymbol{\theta})) \equiv$ 

Now we construct a controlled-U for a special unitary operator $W \in \mathrm{SU}(d)$. For any $W \in \mathrm{SU}(d)$, there exists $V \in \mathrm{SU}(d)$ which satisfies

$$W = V^\dagger \Theta(\boldsymbol{\theta}) V,$$
$$\Theta(\boldsymbol{\theta}) \equiv \mathrm{diag}(e^{-i(\theta_1+\theta_2+\cdots+\theta_{d-1})}, e^{i\theta_1}, e^{i\theta_2}, \cdots, e^{i\theta_{d-1}}),$$

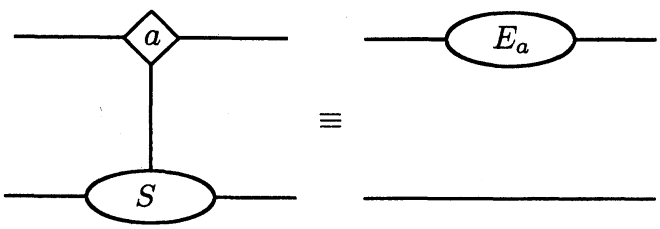
where $\Theta(\boldsymbol{\theta})$ is an appropriate diagonal matrix. By this fact we obtain the diagram of the controlled-U for $\mathrm{SU}(d)$ as follows:



To extend the above result to $\mathrm{U}(d)$, we introduce a phase gate $\tilde{C}^a(S)$:

$$\tilde{C}^a(S)|c\rangle|b\rangle = \begin{cases} e^{i\delta}|a\rangle|b\rangle, & \text{if } |c\rangle = |a\rangle \\ |c\rangle|b\rangle, & \text{otherwise} \end{cases}$$

following [1]. In the similar way to the qubit case, the diagram is given by



where

$$E_a \equiv \mathrm{diag}(1, 1, \ldots, 1, \overset{\overset{a}{\vee}}{e^{i\delta}}, 1, \ldots, 1) \in \mathrm{SU}(d).$$

Indeed, in two-qudit representation

$$E_a \otimes \mathbf{1} = \mathrm{diag}(\mathbf{1}, \mathbf{1}, \dots, \mathbf{1}, \overset{\overset{a}{\vee}}{e^{i\delta}\mathbf{1}}, \mathbf{1}, \dots, \mathbf{1}) = \tilde{C}^a(S).$$

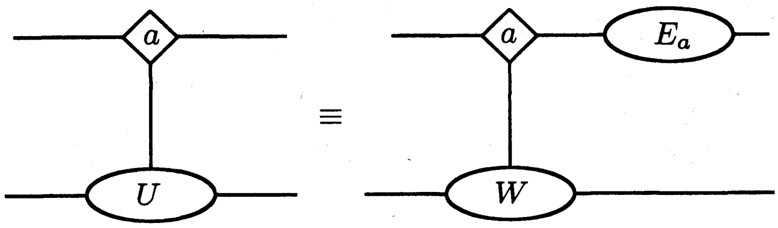
Making use of the phase gate, we can construct a controlled-U for $\mathrm{U}(d)$. Any $U \in \mathrm{U}(d)$ is decomposed to

$$U = e^{i\delta}W \qquad (W \in \mathrm{SU}(d)).$$

By this fact a controlled-U for $U(d)$ is given by

$$\tilde{C}^a(U) = \tilde{C}^a(W)\tilde{C}^a(S),$$

and the diagram is



4. Construction of Unitary Operators in two-Qudit

In this section we construct a unitary operator $\tilde{U} \in \mathrm{U}(d^2)$ in two-qudit making use of the controlled-U in the previous section. We follow the method by Deutch[5].

4.1 Transformation to a basis vector in a single qudit

A state in a single qudit is written by

$$|\boldsymbol{x}\rangle = \sum_{k=0}^{d-1} c_k|k\rangle, \quad (c_k \in \mathbb{C},\ k \in \mathbb{Z}_d)$$

and, in the matrix form, identified by

$$\boldsymbol{x} = {}^t(c_0, c_1, \dots, c_{d-1}).$$

For any state $|\boldsymbol{x}\rangle$, there exists a unitary operator U which satisfies

$$U|\boldsymbol{x}\rangle = N_0|0\rangle$$

and, in the matrix form

$$Ux = {}^t(N_0, 0, \dots, 0)$$

where

$$N_n \equiv \Big(\sum_{i=n}^{d-1} |c_i|^2\Big)^{1/2}.$$

U is concretely (but not necessarily efficiently) constructed as follows. Put

$$h_k = \begin{array}{c c} & \begin{array}{cccccccc} & & & k & k+1 & & & \end{array} \\ \begin{array}{c} \\ \\ \\ k \\ k+1 \\ \\ \\ \\ \end{array} & \begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & \frac{c_k}{N_k} & -\frac{N_{k+1}}{N_k} & & & \\ & & & \frac{N_{k+1}}{N_k} & -\frac{c_k^*}{N_k} & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix} \end{array}, \quad (k \in \mathbb{Z}_{d-2}),$$

$$h_{d-2} = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \frac{c_{d-2}}{N_{d-2}} & -\frac{c_{d-1}^*}{N_{d-2}} \\ & & & \frac{c_{d-1}}{N_{d-2}} & -\frac{c_{d-2}^*}{N_{d-2}} \end{pmatrix},$$

then we obtain

$$(h_{d-2}h_{d-3}\cdots h_1h_0)^\dagger x = {}^t(N_0, 0, \dots, 0).$$

We also obtain an arbitrary basis vector as follows:

$${}^t(0, 0, , \dots, \overset{k}{N_0}, 0, \dots, 0) = Ux. \tag{4.1}$$

Indeed, if we multiply an exchange operator after operating h's, we obtain

$$P_{0k}(h_{d-2}h_{d-3}\cdots h_1h_0)^\dagger x = P_{0k}{}^t(N_0, 0, \dots, 0) = {}^t(0, 0, , \dots, \overset{k}{N_0}, 0, \dots, 0).$$

We temporarily call this operator as $T_k(x)$. By this symbol the above equation is written by

$$T_k(x)x = {}^t(0, 0, , \dots, \overset{k}{N_0}, 0, \dots, 0) \quad (k \in \mathbb{Z}_d).$$

4.2 Transformation to a basis vector in two-qudit

Making use of $T_k(\boldsymbol{x})$, we construct a unitary operator $\tilde{S}_b^a(\tilde{\boldsymbol{x}})$, which transforms a state $\tilde{\boldsymbol{x}}$ in two-qudit to a basis vector.

Any state in two-qudit is written by

$$|\tilde{\boldsymbol{x}}\rangle = \sum_{i,j=0}^{d-1} c_{ij}|i\rangle|j\rangle, \qquad (\sum_{i,j=0}^{d-1} |c_{ij}|^2 = 1),$$

or, in the matrix form, identifies

$$\begin{aligned}\tilde{\boldsymbol{x}} &= {}^t(c_{00}, c_{01}, \dots, c_{0,d-1}, c_{10}, c_{11}, \dots, c_{1,d-1}, \dots, c_{d-1,0}, \dots, c_{d-1,d-1}) \\ &= {}^t(\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_{d-1})\end{aligned}$$

where

$$\boldsymbol{x}_i \equiv (c_{i0}, c_{i1}, \dots, c_{i,d-1}) \qquad (i \in \mathbb{Z}_d).$$

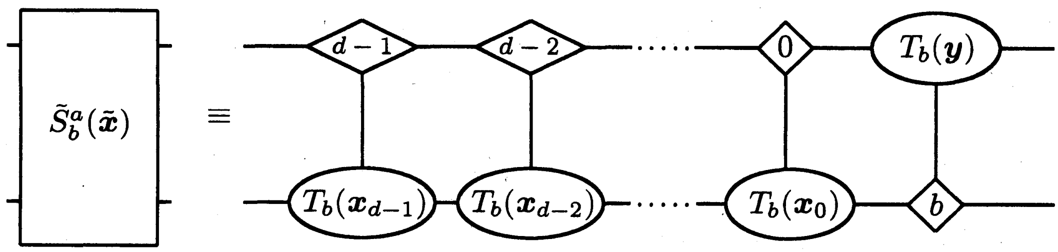
Then $\tilde{S}_b^a(\tilde{\boldsymbol{x}})$ is obtained by

$$\tilde{S}_b^a(\tilde{\boldsymbol{x}}) \equiv \tilde{C}_b(T_a(\boldsymbol{y}))\tilde{C}^0(T_b(\boldsymbol{x}_0))\tilde{C}^1(T_b(\boldsymbol{x}_1)) \cdots \tilde{C}^{d-1}(T_b(\boldsymbol{x}_{d-1}))$$

where

$$\boldsymbol{y} \equiv (\|\boldsymbol{x}_0\|, \|\boldsymbol{x}_1\|, \dots, \|\boldsymbol{x}_{d-1}\|) \in V_d(\mathbb{C})$$

and $\tilde{C}_a$ means controlled operation in which the control bit and the target bit are reversed. The diagram is



In this circuit we find

$$\tilde{S}_b^a(\tilde{\boldsymbol{x}})|\tilde{\boldsymbol{x}}\rangle = |a\rangle|b\rangle.$$

4.3 Construction of unitary operators in two-qudit

Finally we construct a unitary operators in two-qudit. Let $\tilde{U} \in \mathrm{U}(d^2)$ be a unitary operator of which eigenvalues are $e^{i\sigma(a,b)}$ $(a, b \in \mathbb{Z}_d)$ and the corresponding eigenstates $|\sigma(a,b)\rangle$.

We introduce $\tilde{X}(a,b)$ which operates

$$\tilde{X}(a,b)|c\rangle|d\rangle = \begin{cases} e^{i\sigma(a,b)}|a\rangle|b\rangle, & \text{if } |c\rangle|d\rangle = |a\rangle|b\rangle \\ |c\rangle|d\rangle, & \text{otherwise} \end{cases} \cdot$$

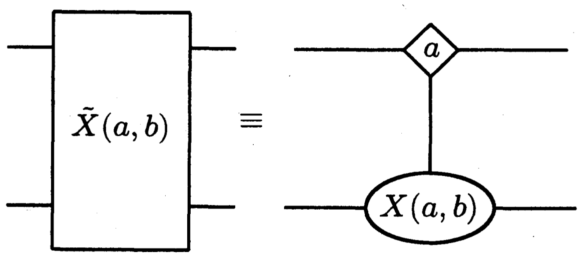
$\tilde{X}(a,b)$ is constructed by

$$\tilde{X}(a,b) = \tilde{C}^a(X(a,b))$$

where $X(a,b)$ is a single qudit gate:

$$\tilde{X}(a,b)|c\rangle|d\rangle = \begin{cases} e^{i\sigma(a,b)}|a\rangle|b\rangle, & \text{if } |c\rangle|d\rangle = |a\rangle|b\rangle \\ |c\rangle|d\rangle, & \text{otherwise} \end{cases} \cdot$$

The diagram is



In the matrix form

$$\tilde{X}(a,b) \equiv \mathrm{diag}(\mathbf{1}, \mathbf{1}, \dots, \mathbf{1}, X(a,b), \mathbf{1})$$

$$= \begin{array}{cc} & da+b \\ da+b & \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & e^{i\sigma(a,b)} & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \end{array}$$

with

$$
\begin{aligned}
X(a,b) &\equiv \mathrm{diag}(1,1,\ldots,1,\underset{b}{e^{i\sigma(a,b)}},1,\ldots,1) \\
&= \begin{array}{c}
\\
b
\end{array}
\begin{pmatrix}
1 & & & & & & \\
& \ddots & & & & & \\
& & 1 & & & & \\
& & & e^{i\sigma(a,b)} & & & \\
& & & & 1 & & \\
& & & & & \ddots & \\
& & & & & & 1
\end{pmatrix}
\end{aligned}
$$

or, in component,

$$
\begin{aligned}
(\tilde{X}(a,b))_{ij} &= \delta_{ij}\{1+\delta_{i,da+b}(-1+e^{i\sigma(a,b)})\}, \\
(X(a,b))_{ij} &= \delta_{ij}\{1+\delta_{ib}(-1+e^{i\sigma(a,b)})\}.
\end{aligned}
$$

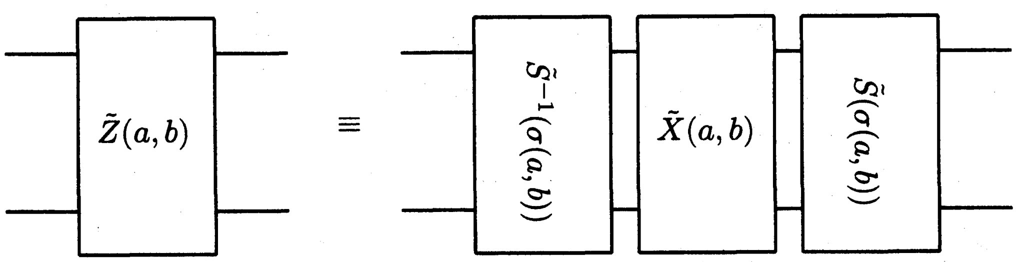
By the result of Sec. 4.2, there exists $\tilde{S}^a_b(\sigma(a,b))$ which satisfies

$$\tilde{S}^a_b(\sigma(a,b))|\sigma(a,b)\rangle = |a\rangle|b\rangle, \quad (a,b\in\mathbb{Z}_d).$$

Then we introduce an operator

$$\tilde{Z}(a,b) \equiv \tilde{S}^{-1}(\sigma(a,b))\tilde{X}(a,b)\tilde{S}(\sigma(a,b))$$

of which diagram is



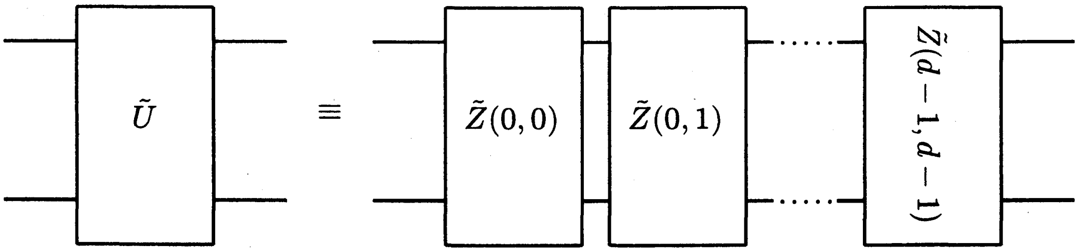
This operator satisfies

$$
\tilde{Z}(a,b)|\sigma(c,d)\rangle = \begin{cases} e^{i\sigma(a,b)}|\sigma(a,b)\rangle, & |\sigma(c,d)\rangle = |\sigma(a,b)\rangle \\ |\sigma(c,d)\rangle, & |\sigma(c,d)\rangle \neq |\sigma(a,b)\rangle \end{cases}.
$$

Finally we construct $\tilde{U}$ by

$$\tilde{U} = \prod_{a,b=0}^{d-1} \tilde{Z}(a,b).$$

The diagram is



Indeed, this operator satisfies

$$\tilde{U}|\sigma(a,b)\rangle = e^{i\sigma(a,b)}|\sigma(a,b)\rangle, \quad (a,b \in \mathbb{Z}_d),$$

to be diagonal in the eigenstates of $\tilde{U}$. Thus we find that this is a gate which perform $\tilde{U}$.

5. Discussion

We have constructed controlled-U gates and a unitary operators in two-qudit. In the similar way to extend controlled-U gates to controlled^n -U in the qubit theory[1], controlled-U in the qudit one will be extended. However, as well as in the qubit case, the larger n becomes the larger number of steps grows exponentially. To avoid this problem some quite new ideas will be required.

When we construct controlled-U gates, we do not use the Euler decomposition of unitary matrices but use the diagonalization of unitary operators. The former may fit the property of laser and has advantage to construct networks with laser operations, while latter may not fit the property of the laser, however, they are not necessarily needed to construct networks and there may exist physical systems suitable for the diagonalization.

We adopt controlled-exchange gates as elementary gates of controlled operation. This choice stems from the notion that in physical systems manipulation is allowed between only two-state at a time. However if there exist physical systems in which manipulation between more than two-state is possible at a time, some other gates may be used to decrease steps of calculation.

As stated in the introduction, there is the decoherence problem through interaction with environment. Taking this fact into consideration, the qudit theory have advantage to the qubit one. However, to realize qudits in physical systems, for example, if we make use of energy levels of electron in an atom, the energy differences between high excited states are very small, thus not so many levels may be used in actual construction.

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