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QUATERNIONIC FRENET CURVES AND TOTALLY GEODESIC IMMERSIONS

By

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Abstract. In a quaternionic Kähler manifold M, we introduce a notion of quaternionic Frenet curves on M which is closely related to the quaternionic Kähler structure of M and give a chracterization of totally geodesic immersions of M into an ambient real space form $\widetilde{M}^N(\tilde{c};\mathbb{R})$ of constant sectional curvature $\tilde{c}$ by the extrinsic shape of such curves.

1. Introduction

A smooth curve $\gamma=\gamma(s)$ in a Riemannian manifold M parametrized by its arclength s is called a *Frenet curve of proper order* 2 if there exist a smooth unit vector field $V=V(s)$ along γ and a positive smooth function $\kappa=\kappa(s)$ satisfying that

$$(1.1)\qquad \nabla_{\dot\gamma}\dot\gamma(s)=\kappa(s)V(s) \quad\text{and}\quad \nabla_{\dot\gamma}V(s)=-\kappa(s)\dot\gamma(s),$$

where $\dot\gamma$ denotes the unit tangent vector of γ and $\nabla_{\dot\gamma}$ denotes the covariant differentiation along γ with respect to the Riemannian connection ∇ of M. The function κ and the orthonormal frame $\{\dot\gamma, V\}$ are called the *curvature* and the *Frenet frame* of γ, respectively. A Frenet curve of proper order 2 with constant curvature $k(>0)$ is called a *circle* of curvature k. We regard a geodesic as a circle of null curvature.

By observing the extrinsic shape of such curves on a submanifold M, we can study the properties of the immersion of M into an ambient Riemannian manifold $\widetilde{M}$ in some cases. In their paper [6], Nomizu and Yano proved a well-known theorem which states that a submanifold M is an extrinsic sphere of $\widetilde{M}$, namely M is a totally umbilic submanifold with parallel mean curvature vector in $\widetilde{M}$, if and only if all circles of some positive curvature k in M are circles in the ambient space $\widetilde{M}$. In [3], Kôzaki and Maeda improved this theorem, that is, they show that M is an extrinsic sphere of $\widetilde{M}$ if and only if all circles of some positive

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curvature k in M are Frenet curves of proper order 2 in $\widetilde{M}$. On the other hand, Suizu, Maeda and Adachi gave the characterizations of parallel imbbedings of complex and quaternionic projective spaces into a real space forms using the notions of *Kähler circles* and *quaternionic circles* (see [9]).

In this context, it is natural to pose the following problem: If an isometric immersion $f : M \to \widetilde{M}$ maps some Frenet curves of proper order 2 on M to Frenet curves of proper order 2 in ambient space $\widetilde{M}$, what can we say about the immersion f? From this point of view, S. Maeda and the author characterized totally geodesic immersions into an arbitrary Riemannian manifold, parallel isometric immersions of complex projective spaces into a real space form in terms of the extrinsic shapes of some kind of Frenet curves of order 2 ([5, 11]).

In this paper, we introduce a notion of quaternionic Frenet curves in a quaternionic Kähler manifold M, which is a particular class of Frenet curves of order 2 closely related to the quaternionic Kähler structure of M. By observing the extrinsic shape of quaternionic Frenet curves, we provide a chracterization of totally geodesic immersions of M into an ambient real space form $\widetilde{M}^N(\tilde{c};\mathbb{R})$ of constant sectional curvature $\tilde{c}$ (Theorem 1). We also characterize every parallel isometric immersion of an n-dimensional quaternionic space form $M^n(c;\mathbb{H})$ of quaternionic sectional curvature c into $\widetilde{M}^N(\tilde{c};\mathbb{R})$ from this point of view (Theorem 2). These are quaternionic versions of our preceding results in [5].

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2. Quaternionic Frenet curves in quaternionic Kähler manifolds

A *quaternionic Kähler structure* $\mathcal{J}$ on a Riemannian manifold M of real dimension $4n$ is a rank 3 vector subbundle of the bundle of endmorphism of the tangent bundle TM with the following properties:

1. For each point $x \in M$ there exists an open neighborhood U of x in M and sections J_1, J_2, J_3 of the restriction $\mathcal{J}|_U$ over U such that
   (a) each J_i is an almost Hermitian structure on U, that is, ${J_i}^2 = -id$ and $\langle J_iX, Y\rangle + \langle X, J_iY\rangle = 0$ for all vector fields X and Y on U, where $\langle\ ,\ \rangle$ is the Riemannian metric of M.
   (b) $J_iJ_{i+1} = J_{i+2} = -J_{i+1}J_i$ (i mod 3) for $i = 1, 2, 3$.
2. The condition that $\nabla_X J$ is a section of $\mathcal{J}$ holds for each vector field X on M and section J of the bundle $\mathcal{J}$, where ∇ denotes the Riemannian connection of M.

This triple $\{J_1, J_2, J_3\}$ is called a canonical local basis of $\mathcal{J}$. For each canonical local basis of quaternionic structure, there exist three 1-forms q_1, q_2 and q_3 on U satisfying

$$\nabla_X J_i = q_{i+2}(X)J_{i+1} - q_{i+1}(X)J_{i+2} \quad (i \bmod 3) \tag{2.1}$$

for each vector field X on U and $i = 1, 2, 3$.

We say that an n-dimensional connected quaternionic Kähler manifold M is an *n-dimensional quaternionic space form* of quaternionic sectional curvature c ($\in \mathbb{R}$) if the Riemannian sectional curvature of M is equal to c for all tangent 2-planes spanned by $v \in T_xM$ and Jv with $J \in \mathcal{J}_x$ at each point $x \in M$. We donote it by $M^n(c;\mathbb{H})$. The standard model of a quaternionic space form is locally congruent to one of a quaternionic projective space $\mathbb{H}P^n(c)$ of quaternionic sectional curvature c (> 0), a quaternionic Euclidean space $\mathbb{H}^n$ and a quaternionic hyperbolic space $\mathbb{H}H^n(c)$ of quaternionic sectional curvature c (< 0).

Let $\gamma = \gamma(s)$ be a Frenet curve of proper order 2 in a quaternionic Kähler manifold M which satisfies (1.1). For this curve γ we put

$$\tau_\gamma := \sqrt{\langle\dot\gamma, J_1V\rangle^2 + \langle\dot\gamma, J_2V\rangle^2 + \langle\dot\gamma, J_3V\rangle^2}\,.$$

We can see from (2.1) and (1.1) that τ_γ is constant along γ. We call τ_γ *structure torsion* of γ (see [1]). Then it is easy to prove

Proposition 1. *For the structure torsion τ_γ of γ satisfying (1.1), the following two conditions are mutually equivalent:*

(1) $\tau_\gamma = 1$,
(2) *there exist a smooth section J of $\mathcal{J}$ with $J^2 = -id$ such that $V(s) = J_{\gamma(s)}\dot\gamma(s)$ for each s.*

A Frenet curve γ of proper order 2 in a quaternionic Kähler manifold M is said to be a *quaternionic Frenet curve* if it satisfies one (hence both) of the conditions in Proposition 1. A quaternionic Frenet curve of constant curvature $k (> 0)$ is called a *quaternionic circle* of curvature k. We regard a geodesic as a quaternionic circle of null curvature. Thus the notion of quaternionic Frenet curves is a natural extension of that of quaternionic circles.

Since τ_γ is constant along γ, using Proposition 1, we can get following proposition.

Proposition 2. *Let x be an arbitrary point of a quaternionic Kähler manifold M and v an arbitrary unit vector in T_xM. For arbitrary $J \in \mathcal{J}_x$ with $J^2 = -id$, there exists a unique quaternionic Frenet curve $\gamma = \gamma(s)$ defined on* *some open interval* $(-\varepsilon, \varepsilon) \subset \mathbb{R}$ *such that*

$$\gamma(0) = x, \quad \dot{\gamma}(0) = v \quad \textit{and} \quad V(0) = Jv.$$

3. Isotropic immersions

We first recall a few fundamental notions in submanifold theory. Let M, $\widetilde{M}$ be Riemannian manifolds and $f : M \to \widetilde{M}$ an isometric immersion. The Riemannian metrics on M, $\widetilde{M}$ are denoted by the same notation $\langle\ ,\ \rangle$. We denote by ∇ and $\widetilde{\nabla}$ the covariant differentiations of M and $\widetilde{M}$, respectively. Then the formulae of Gauss and Weingarten are

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \quad \widetilde{\nabla}_X \xi = -A_\xi X + D_X \xi,$$

where σ, A_ξ and D denote the second fundamental form of f, the shape operator in the direction of ξ and the covariant differentiation in the normal bundle, respectively. We define the covariant differentiation $\bar{\nabla}$ of the second fundamental form σ with respect to the connection in (tangent bundle) $\oplus$ (normal bundle) as follows:

$$(\bar{\nabla}_X \sigma)(Y, Z) = D_X(\sigma(Y, Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z).$$

If $\bar{\nabla}\sigma = 0$, an isometric immersion f is called *parallel.*

An isometric immersion f is said to be *isotropic* at $x \in M$ if $\|\sigma(v, v)\|/\|v\|^2$ does not depend on the choice of $v\ (\neq 0) \in T_x M$. In this case we put the number as $\lambda(x)$. If the immersion is isotropic at every point, then the immersion is said to be isotropic. When the function $\lambda = \lambda(x)$ is constant on M, we say that M is *constant isotropic* in the ambient space $\widetilde{M}$. Note that a totally umbilic immersion is isotropic, but not *vice versa.*

The following is well-known([7]):

LEMMA 1. *Let f be an isometric immersion of M into $\widetilde{M}$. Then f is isotropic at $x \in M$ if and only if the second fundamental form σ satisfies $\langle\sigma(v, v), \sigma(v, u)\rangle = 0$ for an arbitrary orthogonal pair $v, u \in T_x M$.*

4. Main results

A curve $\gamma = \gamma(s)$ on a Riemannian manifold M is said to be a *plane curve* if the curve γ is locally contained in some 2-dimensional totally geodesic submanifold of M. As a matter of course, every plane curve with positive curvature function is a Frenet curve of proper order 2. But in general, the converse does not hold. In case that the space M is a real space form $\widetilde{M}^N(\tilde{c};\mathbb{R})$ of constant sectional curvature $\tilde{c}$ (that is, $\widetilde{M}^N(\tilde{c};\mathbb{R})$ is locally congruent to either a Euclidean space $\mathbb{R}^N$, a standard sphere $S^N(\tilde{c})$ or a real hyperbolic space $H^N(\tilde{c})$ according as the curvature $\tilde{c}$ is zero, positive, or negative), it is easy to see that a curve γ is a Frenet curve of proper order 2 if and only if the curve γ is a plane curve with positive curvature function.

Now, we give the following theorem.

THEOREM 1. *Let M be a quaternionic Kähler manifold of quaternionic dimension $n(\geq 2)$ and f an isometric immersion of M into a real space form $\widetilde{M}^N(\tilde{c};\mathbb{R})$. Assume that there exists a non constant positive smooth function $\kappa = \kappa(s)$ satisfying that f maps every quaternionic Frenet curve $\gamma = \gamma(s)$ of curvature κ on M to a plane curve in $\widetilde{M}^N(\tilde{c};\mathbb{R})$. Then f is a totally geodesic immersion.*

The idea of proof is similar to that of Theorem 2 in [5]. But for readers we explain it in detail.

First, relaxing the condition that κ is a non constant positive smooth function to that it is a positive smooth function, we shall prove the following proposition.

PROPOSITION 3. *Let M be a quaternionic Kähler manifold of quaternionic dimension $n(\geq 2)$ and f an isometric immersion of M into a real space form $\widetilde{M}^N(\tilde{c};\mathbb{R})$. Assume that there exists a positive smooth function $\kappa = \kappa(s)$ satisfying that f maps every quaternionic Frenet curve $\gamma = \gamma(s)$ of curvature κ on M to a plane curve in $\widetilde{M}^N(\tilde{c};\mathbb{R})$. Then f is paralell and constant isotropic.*

Proof. Let x be an arbitrary point of M, $v \in T_xM$ an arbitrary unit vector and J an arbitrary element of $\mathcal{J}_x$ with $J^2 = -id$. We consider a quaternionic Frenet curve $\gamma = \gamma(s)$ $(s \in (-\varepsilon, \varepsilon))$ satisfying equations (1.1) and the initial condition $\gamma(0) = x$, $\dot\gamma(0) = v$ and $V(0) = Jv$. Since the curve $f \circ \gamma$ is a plane curve in $\widetilde{M}^N(\tilde{c};\mathbb{R})$ by assumption, there exist a (nonnegative) function $\tilde\kappa = \tilde\kappa(s)$ and a field of unit vectors $\widetilde{V} = \widetilde{V}(s)$ along $f \circ \gamma$ in $\widetilde{M}^N(\tilde{c};\mathbb{R})$ which satisfy

$$\widetilde{\nabla}_{\dot\gamma}\dot\gamma = \tilde\kappa\widetilde{V}, \quad \widetilde{\nabla}_{\dot\gamma}\widetilde{V} = -\tilde\kappa\dot\gamma. \tag{4.1}$$

Then by the formula of Gauss we have

$$\tilde\kappa\widetilde{V} = \kappa V + \sigma(\dot\gamma,\dot\gamma), \tag{4.2}$$

so that

$$\tilde\kappa^2 = \kappa^2 + \|\sigma(\dot\gamma,\dot\gamma)\|^2. \tag{4.3}$$

We here note that the function $\tilde{\kappa}$ is positive because $\kappa > 0$.

Differentiating the left-hand side of (4.2), we see

$$\tilde{\kappa}\widetilde{\nabla}_{\dot{\gamma}}(\tilde{\kappa}\widetilde{V}) = \tilde{\kappa}\{\dot{\tilde{\kappa}}\widetilde{V} + \tilde{\kappa}\widetilde{\nabla}_{\dot{\gamma}}\widetilde{V}\} = \tilde{\kappa}\dot{\tilde{\kappa}}\widetilde{V} - \tilde{\kappa}^3\dot{\gamma} = \dot{\tilde{\kappa}}\{\kappa V + \sigma(\dot{\gamma},\dot{\gamma})\} - \tilde{\kappa}^3\dot{\gamma} \tag{4.4}$$

by use of (4.1) and (4.2). On the other hand, differentiating the right-hand side of (4.2), by the formulae of Gauss and Weingarten we have

(4.5)

$$\begin{aligned}
&\tilde{\kappa}\widetilde{\nabla}_{\dot{\gamma}}\{\kappa V + \sigma(\dot{\gamma},\dot{\gamma})\}\\
&\quad = \tilde{\kappa}\Big\{\dot{\kappa}V + \kappa\widetilde{\nabla}_{\dot{\gamma}}V - A_{\sigma(\dot{\gamma},\dot{\gamma})}\dot{\gamma} + D_{\dot{\gamma}}(\sigma(\dot{\gamma},\dot{\gamma}))\Big\}\\
&\quad = \tilde{\kappa}\Big\{\dot{\kappa}V + \kappa\left(\nabla_{\dot{\gamma}}V + \sigma(\dot{\gamma},V)\right) - A_{\sigma(\dot{\gamma},\dot{\gamma})}\dot{\gamma} + (\bar{\nabla}_{\dot{\gamma}}\sigma)(\dot{\gamma},\dot{\gamma}) + 2\sigma(\nabla_{\dot{\gamma}}\dot{\gamma},\dot{\gamma})\Big\}\\
&\quad = \tilde{\kappa}\Big\{\dot{\kappa}V - \kappa^2\dot{\gamma} + 3\kappa\sigma(\dot{\gamma},V) - A_{\sigma(\dot{\gamma},\dot{\gamma})}\dot{\gamma} + (\bar{\nabla}_{\dot{\gamma}}\sigma)(\dot{\gamma},\dot{\gamma})\Big\}.
\end{aligned}$$

We compare the tangential components and the normal components for the submanifold M in (4.4) and (4.5), respectively. Then we get the following:

$$\dot{\tilde{\kappa}}\kappa V - \tilde{\kappa}^3\dot{\gamma} = \tilde{\kappa}\{\dot{\kappa}V - \kappa^2\dot{\gamma} - A_{\sigma(\dot{\gamma},\dot{\gamma})}\dot{\gamma}\}, \tag{4.6}$$

$$\dot{\tilde{\kappa}}\sigma(\dot{\gamma},\dot{\gamma}) = \tilde{\kappa}\{3\kappa\sigma(\dot{\gamma},V) + (\bar{\nabla}_{\dot{\gamma}}\sigma)(\dot{\gamma},\dot{\gamma})\}. \tag{4.7}$$

Equation (4.7) implies

$$\tilde{\kappa}\dot{\tilde{\kappa}}\sigma(\dot{\gamma},\dot{\gamma}) = \tilde{\kappa}^2\{3\kappa\sigma(\dot{\gamma},V) + (\bar{\nabla}_{\dot{\gamma}}\sigma)(\dot{\gamma},\dot{\gamma})\}. \tag{4.8}$$

On the other hand, from (4.3) we have

$$\begin{aligned}
\tilde{\kappa}\dot{\tilde{\kappa}} &= \frac{1}{2}\frac{d}{ds}\tilde{\kappa}^2\\
&= \kappa\dot{\kappa} + \frac{1}{2}\frac{d}{ds}\langle\sigma(\dot{\gamma},\dot{\gamma}),\sigma(\dot{\gamma},\dot{\gamma})\rangle\\
&= \kappa\dot{\kappa} + \langle D_{\dot{\gamma}}(\sigma(\dot{\gamma},\dot{\gamma})),\sigma(\dot{\gamma},\dot{\gamma})\rangle\\
&= \kappa\dot{\kappa} + \langle(\bar{\nabla}_{\dot{\gamma}}\sigma)(\dot{\gamma},\dot{\gamma}),\sigma(\dot{\gamma},\dot{\gamma})\rangle + 2\kappa\langle\sigma(V,\dot{\gamma}),\sigma(\dot{\gamma},\dot{\gamma})\rangle.
\end{aligned} \tag{4.9}$$

Substituting (4.3) and (4.9) into (4.8), at $s = 0$ we obtain

$$\begin{aligned}
&\Big\{\kappa(0)\dot{\kappa}(0) + \langle(\bar{\nabla}_v\sigma)(v,v),\sigma(v,v)\rangle + 2\kappa(0)\langle\sigma(v,v),\sigma(v,Jv)\rangle\Big\}\sigma(v,v)\\
&= \Big\{\kappa(0)^2 + \|\sigma(v,v)\|^2\Big\}\Big\{3\kappa(0)\sigma(v,Jv) + (\bar{\nabla}_v\sigma)(v,v)\Big\}.
\end{aligned} \tag{4.10}$$

Since Proposition 2 guarantees the existence of another quaternionic Frenet curve $\gamma_1 = \gamma_1(s)$ $(s \in (-\varepsilon_1, \varepsilon_1))$ of the same curvature κ in M satisfying $\nabla_{\dot{\gamma}_1}\dot{\gamma}_1 =$

κV_1 and $\nabla_{\dot\gamma_1} V_1 = -\kappa\dot\gamma_1$ with initial condition $\gamma_1(0) = x$, $\dot\gamma_1(0) = v$ and $V_1(0) = -Jv$, we can change the vector Jv into $-Jv$ in (4.10). Then the equality (4.10) for γ_1 turns to

$$\begin{aligned}\Big\{\kappa(0)\dot\kappa(0) + \langle(\bar\nabla_v\sigma)(v,v), \sigma(v,v)\rangle - 2\kappa(0)\langle\sigma(v,v), \sigma(v,Jv)\rangle\Big\}\sigma(v,v) \\ = \Big\{\kappa(0)^2 + \|\sigma(v,v)\|^2\Big\}\Big\{-3\kappa(0)\sigma(v,Jv) + (\bar\nabla_v\sigma)(v,v)\Big\}.\end{aligned} \tag{4.10'}$$

Therefore, from (4.10) and (4.10') we have

$$2\kappa(0)\langle\sigma(v,v), \sigma(v,Jv)\rangle\sigma(v,v) = 3\kappa(0)\Big\{\kappa(0)^2 + \|\sigma(v,v)\|^2\Big\}\sigma(v,Jv),$$

so that

$$2\langle\sigma(v,v), \sigma(v,Jv)\rangle\sigma(v,v) = 3\Big\{\kappa(0)^2 + \|\sigma(v,v)\|^2\Big\}\sigma(v,Jv). \tag{4.11}$$

Taking the inner product of both sides of this with $\sigma(v,v)$, we get

$$2\langle\sigma(v,v), \sigma(v,Jv)\rangle\|\sigma(v,v)\|^2 = 3\Big\{\kappa(0)^2 + \|\sigma(v,v)\|^2\Big\}\langle\sigma(v,v), \sigma(v,Jv)\rangle$$

hence

$$\Big\{3\kappa(0)^2 + \|\sigma(v,v)\|^2\Big\}\langle\sigma(v,v), \sigma(v,Jv)\rangle = 0.$$

So we have $\langle\sigma(v,v), \sigma(v,Jv)\rangle = 0$, because $3\kappa(0)^2 + \|\sigma(v,v)\|^2 > 0$. It follows from (4.11) again that

$$\sigma(v, Jv) = 0 \tag{4.12}$$

for any $v \in T_xM$ at any point $x \in M$ and any $J \in \mathcal{J}_x$ with $J^2 = -id$. Replacing v by $v + Jv$ in (4.12), we get

$$\sigma(Jv, Jv) = \sigma(v, v). \tag{4.13}$$

Using (4.13), we have

$$\sigma(Jv, J'v) = 0 \tag{4.14}$$

for $J, J' \in \mathcal{J}_x$ with $J^2 = (J')^2 = -id$ and $JJ' = -J'J$. By making use of equations (4.12), (4.13), (4.14) and Codazzi's equation in a space of constant curvature we see that the immersion f is parallel (see [4]).

Next, taking the inner product of both sides of (4.6) with V, we have

$$\begin{aligned}\dot{\tilde\kappa}\kappa &= \tilde\kappa\dot\kappa - \tilde\kappa\langle A_{\sigma(\dot\gamma,\dot\gamma)}\dot\gamma, V\rangle \\ &= \tilde\kappa\dot\kappa - \tilde\kappa\langle\sigma(\dot\gamma,\dot\gamma), \sigma(\dot\gamma, V)\rangle.\end{aligned}$$

On the other hand, from (4.12) we know that

$$\sigma(\dot\gamma, V) = 0 \quad \text{for each } s \in (-\varepsilon, \varepsilon). \tag{4.15}$$

Hence the above equation becomes

$$\dot{\tilde\kappa}\kappa = \tilde\kappa\dot\kappa, \tag{4.16}$$

so that equation (4.6) reduces to

$$A_{\sigma(\dot\gamma,\dot\gamma)}\dot\gamma = (\tilde\kappa^2 - \kappa^2)\dot\gamma.$$

Therefore

$$\langle\sigma(v,v), \sigma(v,u)\rangle = \langle A_{\sigma(v,v)}v, u\rangle = 0$$

for any orthonormal pair of vectors $v, u \in T_xM$ at each point $x \in M$. Thus, by virtue of Lemma 1, the immersion f is isotropic. Besides, we can see that f is constant isotropic as follows: Let $c = c(s)$ be an arbitrary geodesic on M parametrized by its arclength s. Then, from the fact that $\bar\nabla\sigma = 0$, we have

$$\frac{d}{ds}\|\sigma(\dot c, \dot c)\|^2 = 2\langle(\bar\nabla_{\dot c}\sigma)(\dot c, \dot c), \sigma(\dot c, \dot c)\rangle + 4\langle\sigma(\nabla_{\dot c}\dot c, \dot c), \sigma(\dot c, \dot c)\rangle = 0. \tag{4.17}$$

Thus $\|\sigma(\dot c, \dot c)\|$ is constant along the curve $c{=}c(s)$. Hence our assertion follows. □

We shall now prove Theorem 1. Suppose that the curvature function κ is not constant. Then there exists some $s_0 \in (-\varepsilon, \varepsilon)$ with $\dot\kappa(s_0) \neq 0$. Since $\kappa, \tilde\kappa > 0$, it follows from (4.16) that $\dot{\tilde\kappa}(s_0) \neq 0$. We know the fact that $\bar\nabla\sigma = 0$. So equation (4.7), combined with (4.15), yields $\sigma(\dot\gamma(s_0), \dot\gamma(s_0)) = 0$. Moreover, we can see that $\|\sigma(\dot\gamma, \dot\gamma)\|$ is constant along the curve γ because the same equation as (4.17) holds for γ. Thus we conclude $\sigma(v,v) = 0$ for an arbitrary unit vector $v \in T_xM$ at each point $x \in M$. Hence our immersion $f : M \to \widetilde{M}^N(\tilde c; \mathbb{R})$ is totally geodesic.

Theorem 1 does not hold without the condition that κ is not constant. Indeed we have following theorem:

THEOREM 2. *Let M^n be a quaternionic Kähler manifold of quaternionic dimension n (≥ 2) and f an isometric immersion of M^n into a real space form $\widetilde{M}^{4n+p}(\tilde c; \mathbb{R})$. Suppose that there exists a positive smooth function κ satisfying that f maps every quaternionic Frenet curve γ of curvature κ on M^n to a plane curve in $\widetilde{M}^{4n+p}(\tilde c; \mathbb{R})$. Then f is a parallel immersion and locally equivalent to one of the following :*

(1) *f is a totally geodesic immersion of $M^n = \mathbb{H}^n = \mathbb{R}^{4n}$ into $\widetilde{M}^{4n+p}(\tilde{c};\mathbb{R}) = \mathbb{R}^{4n+p}$, where $\tilde{c} = 0$.*

(2) *f is a totally umbilic immersion of $M^n = \mathbb{H}^n = \mathbb{R}^{4n}$ into $\widetilde{M}^{4n+p}(\tilde{c};\mathbb{R}) = \mathbb{R}H^{4n+p}(\tilde{c})$, where $\tilde{c} < 0$.*

(3) *f is a parallel immersion defined by*

$$f = f_2 \circ f_1 : M^n = \mathbb{H}P^n(c) \xrightarrow{f_1} S^{2n^2+3n-1}((n+1)c/(2n)) \xrightarrow{f_2} \widetilde{M}^{4n+p}(\tilde{c};\mathbb{R}),$$

where f_1 is the first standard minimal immersion, f_2 is a totally umbilic immersion and $(n+1)c/(2n) \geq \tilde{c}$.

Proof. By Proposition 3 the immersion f is parallel and constant isotropic. Let R denote the curvature tensor of M^n. For arbitrary $J \in \mathcal{J}_x$ with $J^2 = -id$, from (4.12), (4.13) and equation of Gauss, we have

$$\begin{aligned}\langle R(v, Jv)Jv, v\rangle &= \tilde{c} + \langle \sigma(v,v), \sigma(Jv, Jv)\rangle - \|\sigma(v, Jv)\|^2 \\ &= \tilde{c} + \|\sigma(v,v)\|^2\end{aligned}$$

for an arbitrary unit vector $v \in T_xM$ at any point x of M^n. Since M^n is constant isotropic, this implies that M^n is a quaternonic space form. Then we can see that the submanifold M^n is one of (1), (2) and (3) (cf.[2, 10]).

In order to prove our assertion, we must check the examples (1), (2) and (3) satisfy the hypothesis of theorem. If the function κ is not constant, we obtain only the case (1). If κ is constant, we get the cases (1), (2) and (3). In the case (1), the hypothesis is obviously satisfied. In the case of (2), for each circle γ of curvature $k(>0)$ on M^n the curve $f \circ \gamma$ is a circle of curvature $\sqrt{k^2 - \tilde{c}}$ (see page 169 in [6]), hence it is a plane curve in the ambient space $\widetilde{M}^{4n+p}(\tilde{c};\mathbb{R})$.

In the case of (3), the isometric immersion f given by (3) is $\sqrt{c - \tilde{c}}$-isotropic and the parallel second fundamental form σ of f satisfies $\sigma(JX, JY) = \sigma(X, Y)$ for all vector fields $X, Y \in TM^n$ and all $J \in \mathcal{J}$. Let $\gamma = \gamma(s)$ be a quaternionic circle of curvature k (> 0). Then we can see that the curve $f \circ \gamma$ is a circle of curvature $\sqrt{k^2 + c - \tilde{c}}$ in $\widetilde{M}^{4n+p}(\tilde{c};\mathbb{R})$ as follows: The curve $f \circ \gamma$ satisfies $\widetilde{\nabla}_{\dot{\gamma}}\dot{\gamma} = kV + \sigma(\dot{\gamma}, \dot{\gamma})$, so that

$$\|\widetilde{\nabla}_{\dot{\gamma}}\dot{\gamma}\| = \sqrt{k^2 + \|\sigma(\dot{\gamma}, \dot{\gamma})\|^2} = \sqrt{k^2 + c - \tilde{c}}.$$

We write

$$\widetilde{V} = \frac{1}{\sqrt{k^2 + c - \tilde{c}}}\{kV + \sigma(\dot{\gamma}, \dot{\gamma})\}.$$

Since $\sigma(\dot\gamma, V) = \sigma(\dot\gamma, J\dot\gamma) = 0$, we have

$$\begin{aligned}
\widetilde\nabla_{\dot\gamma}\widetilde V &= \frac{1}{\sqrt{k^2+c-\tilde c}}\widetilde\nabla_{\dot\gamma}\{kV+\sigma(\dot\gamma,\dot\gamma)\}\\
&= \frac{1}{\sqrt{k^2+c-\tilde c}}\Big\{k\left(\nabla_{\dot\gamma}V+\sigma(\dot\gamma,V)\right)-A_{\sigma(\dot\gamma,\dot\gamma)}\dot\gamma+D_{\dot\gamma}(\sigma(\dot\gamma,\dot\gamma))\Big\}\\
&= \frac{1}{\sqrt{k^2+c-\tilde c}}\Big\{-k^2\dot\gamma-\|\sigma(\dot\gamma,\dot\gamma)\|^2\dot\gamma+(\bar\nabla_{\dot\gamma}\sigma)(\dot\gamma,\dot\gamma)+2\sigma(\nabla_{\dot\gamma}\dot\gamma,\dot\gamma)\Big\}\\
&= \frac{1}{\sqrt{k^2+c-\tilde c}}\Big\{-(k^2+c-\tilde c)\dot\gamma+2k\sigma(V,\dot\gamma)\Big\}\\
&= -\sqrt{k^2+c-\tilde c}\;\dot\gamma\,.
\end{aligned}$$

Thus the curve $f\circ\gamma$ is a plane curve in $\widetilde M^{4n+p}(\tilde c\,;\mathbb{R})$. □

Remark. Theorem 2 also holds under the condition $\kappa \equiv 0$ (see [8]).

Added in proof. We recently obtain similar results in the case of Cayley projective plane (see [12]).

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