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IGUSA LOCAL ZETA FUNCTION OF THE POLYNOMIAL $f(x) = x_1^m + x_2^m + \cdots + x_n^m$

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Abstract. We determine an explicit formula for the Igusa local zeta function corresponding to the character $\chi = 1$ and the polynomial $f(x) = x_1^m + x_2^m + \cdots + x_n^m$ over the p-adic field $\mathbb{Q}_p$, for an arbitrary rational prime p and a positive rational integer m satisfying $\gcd(m,p) = \gcd(m,p-1) = 1$.

1. Introduction

Let p be a prime rational integer, let $\mathbb{Q}_p$ be the field of p-adic numbers, let $\mathbb{Z}_p$ be the ring of p-adic integers, and let $\mathbb{Z}_p^\times = \mathbb{Z}_p - p\mathbb{Z}_p$ be the multiplicative group of units in $\mathbb{Z}_p$. Each nonzero p-adic number $x \in \mathbb{Q}_p - \{0\}$ may be expressed in the form $x = p^a \mathrm{ac}(x)$, for a unique rational integer $a = \mathrm{ord}_p(x)$, called the p-adic ordinal of x, and a unique unit $\mathrm{ac}(x) \in \mathbb{Z}_p^\times$, called the angular component of x. The p-adic norm of an element $x \in \mathbb{Q}_p - \{0\}$ is defined as $|x|_p = p^{-a}$, where $a = \mathrm{ord}_p(x)$. Let n be a positive rational integer, and let $dx = dx_1 \cdots dx_n$ be a product Haar measure on the set $\mathbb{Q}_p^n$, normalized so that the measure of the subset $\mathbb{Z}_p^n$ is 1. For any multiplicative character χ from $\mathbb{Q}_p^\times = \mathbb{Q}_p - \{0\}$ to the complex unit circle and any nonconstant polynomial $f(x) \in \mathbb{Q}_p[x]$, where $x = (x_1, \dots, x_n)$, the function $Z_\chi : \{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq 0\} \to \mathbb{C}$ defined by

$$Z_\chi(s) = \int_{\mathbb{Z}_p^n} \chi(\mathrm{ac}(f(x)))\, |f(x)|_p^s \, dx$$

is called the Igusa local zeta function over $\mathbb{Q}_p$ associated with χ and $f(x)$.

In 1999, Hosokawa [2] determined, for an arbitrary odd prime p, formulas for the Igusa local zeta function associated with the polynomial $f(x) = x_1^2 + x_2^2 + \cdots + x_n^2$. Hosokawa's work was in the more general setting of an arbitrary finite-degree extension K of the field $\mathbb{Q}_p$, and for an arbitrary character χ defined on $K^\times$.

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In this article we work over the field $\mathbb{Q}_p$ and take $\chi = 1$, and we prove the following result.

THEOREM. *Fix any prime rational integer p and any positive rational integers m and n such that* $\gcd(m,p) = \gcd(m,p-1) = 1$. *Let $Z = Z_\chi$ be the Igusa local zeta function over $\mathbb{Q}_p$ associated with the character $\chi = 1$ and the polynomial $f(x) = x_1^m + x_2^m + \cdots + x_n^m$. Let $s \in \mathbb{C}$ such that* $\mathrm{Re}(s) \geq 0$, *and write $t = p^{-s}$. Then*

$$Z(s) = \frac{(p-1)(p^n - t)}{(p-t)(p^n - t^m)}.$$

Observe that our hypothesis $\gcd(m,p) = \gcd(m,p-1) = 1$ implies that $m \neq 2$, and so the case treated here is disjoint from that of Hosokawa.

We mention that the argument presented here can be generalized, with only a few (essentially notational) modifications and a bit of additional explanation, to yield an explicit formula for the the Igusa local zeta function Z_K of the same polynomial over any finite-degree extension K of the field $\mathbb{Q}_p$, just as Hosokawa did with his polynomial. In this more general setting, if we let O_K be the ring of integers in K, let P_K be the unique maximal ideal of O_K, let q denote the cardinality of the finite field O_K/P_K of characteristic p, and assume that $\gcd(m,p) = \gcd(m,q-1) = 1$, then the resulting formula is

$$Z_K(s) = \frac{(q-1)(q^n - t)}{(q-t)(q^n - t^m)},$$

where we have written $t = q^{-s}$. In case $K = \mathbb{Q}_p$, we have, of course, $O_K = \mathbb{Z}_p$ and $P_K = p\mathbb{Z}_p$ and $q = p$.

Igusa local zeta functions are related to the number of solutions of congruences modulo various powers of the prime p [1]. Suppose $f(x)$ has coefficients in O_K, and let N_ℓ be the number of solutions of the congruence $f(x) \equiv 0 \pmod{P_K^\ell}$ in O_K/P_K^ℓ. Then the Poincaré series $P(t) = \sum_{\ell=0}^{\infty} q^{-n\ell} N_\ell t^\ell$ is related to the Igusa local zeta function Z_K, for the trivial character $\chi = 1$, by the formula $P(t) = [1 - tZ_K(s)]/(1-t)$.

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2. Preliminaries

The following lemma shows how we make use of the hypothesis $\gcd(m,p) = \gcd(m,p-1) = 1$. We use this lemma frequently throughout this paper to re-write certain integrals using change of variables.

LEMMA. *Fix any prime rational integer p and any positive rational integer m such that* $\gcd(m,p) = \gcd(m,p-1) = 1$. *Then the map* $g : \mathbb{Z}_p^{\times} \to \mathbb{Z}_p^{\times}$ *defined by* $g(x) = x^m$ *is a measure-preserving bijection.*

Proof. We show first that g is surjective. Fix $\alpha \in \mathbb{Z}_p^{\times}$. It suffices to find a root in $\mathbb{Z}_p^{\times}$ for the polynomial $f(x) = x^m - \alpha$. Define $\alpha_0 \in \{1,\dots,p-1\}$ by $\alpha - \alpha_0 \in p\mathbb{Z}_p$. As $\gcd(m,p-1) = 1$, there exists $\beta_0 \in \{1,\dots,p-1\}$ such that $\beta_0^m \equiv \alpha_0 \pmod p$. Choose $\beta \in \mathbb{Z}_p^{\times}$ such that $\beta - \beta_0 \in p\mathbb{Z}_p$. Thus $f(\beta) = \beta^m - \alpha \in p\mathbb{Z}_p$. As $f'(x) = mx^{m-1}$ with $\gcd(m,p) = 1$ and $\beta \in \mathbb{Z}_p^{\times}$, we have $f'(\beta) = m\beta^{m-1} \in \mathbb{Z}_p^{\times}$. Now by Hensel's Lemma, there exists $\gamma \in \mathbb{Z}_p$ such that $f(\gamma) = 0$ and $\gamma - \alpha \in p\mathbb{Z}_p$. As $\alpha \in \mathbb{Z}_p^{\times}$, it follows that $\gamma \in \mathbb{Z}_p^{\times}$, as desired.

We now show that g is injective. As g is a homomorphism from the abelian multiplicative group $\mathbb{Z}_p^{\times}$ to itself, it suffices to show that the kernel of g contains only the identity element. Fix $\alpha \in \mathbb{Z}_p^{\times}$ such that $\alpha^m = 1$. We may write $\alpha = \alpha_0 + s$ where $\alpha_0 \in \{1,\dots,p-1\}$ and $s \in p\mathbb{Z}_p$. Thus $1 = \alpha^m = \alpha_0^m + s'$ where $s' \in p\mathbb{Z}_p$, forcing $\alpha_0^m = 1$. As $\gcd(m,p-1) = 1$, it follows that $\alpha_0 = 1$, and so $\alpha = 1 + s$. Now we suppose that $s \neq 0$ and work for a contradiction. Then for some integer $i \geq 1$, we have $s = \alpha_i p^i + t$ where $\alpha_i \in \{1,\dots,p-1\}$ and $t \in p^{i+1}\mathbb{Z}_p$. Thus $1 = \alpha^m = 1 + m\alpha_i p^i + t'$ where $t' \in p^{i+1}\mathbb{Z}_p$. This forces $m\alpha_i \equiv 0 \pmod p$. As $\gcd(m,p) = 1$ and $\alpha_i \in \{1,\dots,p-1\}$, this is a contradiction, so g is injective.

The condition $\gcd(m,p) = 1$ implies that $|m|_p = 1$, and so for each unit $x \in \mathbb{Z}_p^{\times}$, we have

$$|g'(x)|_p = \left|mx^{m-1}\right|_p = |m|_p \cdot \left|x^{m-1}\right|_p = 1.$$

Thus, by Proposition 7.4.1 in [3], the function g is locally measure-preserving. But since g is injective, it follows that g is a (globally) measure-preserving function, as claimed. The proof is now complete.

We compute the Igusa local zeta function Z using a method introduced by Weil [7] in 1965, which is a three-step process. The first step is to compute the so-called Generalized Exponential Sum. This is the function $F^* : \mathbb{Q}_p \to \mathbb{C}$ defined as

$$F^*(i^*) = \int_{\mathbb{Z}_p^n} \Psi\left(i^{\ ast} f(x)\right) dx,$$

where Ψ is an arbitrary additive complex-valued character on $\mathbb{Q}_p$ such that $\Psi(x) = 1$ if and only if $x \in \mathbb{Z}_p$. (The particular choice of Ψ does not affect the value $F^*(i^*)$.) The second step is to take an inverse Fourier Transform of F^* to compute the so-called Local Singular Series. This is the function $F : \mathbb{Z}_p \to \mathbb{C}$

defined as

$$F(i) = \int_{\mathbb{Q}_p} F^*(i^*)\,\Psi(-ii^*)\,di^*.$$

In the third step, we take the Mellin Transform of the Local Singular Series to obtain the Igusa local zeta function as

$$Z(s) = \int_{\mathbb{Z}_p} F(i)\,|i|_p^s\,di.$$

Our main tool for evaluating these integrals will be the so-called Orthogonality Relations, established by Igusa [4] in 1987, which assert that if m is any rational integer and Ψ is any additive complex-valued character on $\mathbb{Q}_p$ with the property that $\Psi(x) = 1$ if and only if $x \in \mathbb{Z}_p$, then

$$\int_{\mathbb{Z}_p^\times} \Psi\left(p^{-m}y\right) dy = \begin{cases} 1-p^{-1} & \text{if} \quad m \leq 0 \\ -p^{-1} & \text{if} \quad m = 1 \\ 0 & \text{if} \quad m > 1. \end{cases}$$

Given any real number r, we shall use the notation $\lfloor r \rfloor$ to denote the greatest rational integer less than or equal to r, and the notation $\lceil r \rceil$ to denote the least rational integer greater than or equal to r.

3. Computation of F^*

3.1 Computation of F^* for $g(x) = x^m$

We begin by computing the Generalized Exponential Sum F^* corresponding to $g(x) = x^m$. Fix an arbitrary nonzero p-adic number $i^* \in \mathbb{Q}_p - \{0\}$. We may write $i^* = p^{-e}v$ for a unique rational integer e and a unique unit $v \in \mathbb{Z}_p^\times$. We will see that $F^*(i^*)$ depends on the value e, but not on the value v. Observe that

$$F^*(i^*) = \int_{\mathbb{Z}_p} \Psi(i^* x^m)\,dx = \sum_{j=0}^{\infty} \int_{p^j(\mathbb{Z}_p^\times)} \Psi\left(p^{-e}vx^m\right) dx.$$

We make the change of variables $p^j y = x$ and $p^{-j}dy = dx$, where $y \in \mathbb{Z}_p^\times$, to obtain

$$F^*(i^*) = \sum_{j=0}^{\infty} p^{-j} \int_{\mathbb{Z}_p^\times} \Psi\left(p^{-(e-mj)}vy^m\right) dy.$$

Since $\gcd(m,p) = \gcd(m,p-1) = 1$, we know that $y \mapsto y^m$ is a bijection on $\mathbb{Z}_p^\times$. This allows us to make the change of variables $y' = y^m$ and $dy' = dy$, where $y' \in \mathbb{Z}_p^\times$, to obtain

$$F^*(i^*) = \sum_{j=0}^{\infty} p^{-j} \int_{\mathbb{Z}_p^\times} \Psi\left(p^{-(e-mj)} v y'\right) dy'.$$

Now $vy' \mapsto y'$ is also a bijection on $\mathbb{Z}_p^\times$, and this allows us to make the change of variables $y'' = vy'$ and $dy'' = dy'$, where $y'' \in \mathbb{Z}_p^\times$, to obtain

$$F^*(i^*) = \sum_{j=0}^{\infty} p^{-j} \int_{\mathbb{Z}_p^\times} \Psi\left(p^{-(e-mj)} y''\right) dy''.$$

By the Orthogonality Relations, we have

$$\int_{\mathbb{Z}_p^\times} \Psi\left(p^{-(e-mj)} y''\right) dy'' = \begin{cases} 1-p^{-1} & \text{if} \quad j \geq e/m \\ -p^{-1} & \text{if} \quad j = (e-1)/m \\ 0 & \text{if} \quad j < (e-1)/m. \end{cases}$$

With the use of infinite geometric series, it follows that

$$F^*(i^*) = \begin{cases} 1 & \text{if} \quad e \leq 0 \\ 0 & \text{if} \quad e > 0 \text{ and } e \equiv 1 \,(\operatorname{mod} m) \\ p^{-\lfloor e/m \rfloor} & \text{if} \quad e > 0 \text{ and } e \not\equiv 1 \,(\operatorname{mod} m). \end{cases}$$

3.2 Computation of F^* for $f(x) = x_1^m + x_2^m + \cdots + x_n^m$

We illustrate the situation using case $n = 2$. By the additivity of the function Ψ and by Fubini's Theorem,

$$\begin{aligned} \int_{\mathbb{Z}_p^2} \Psi\left(i^*\left(y_1^m + y_2^m\right)\right) dy_1\, dy_2 &= \int_{\mathbb{Z}_p^2} \Psi\left(i^* y_1^m\right) \cdot \Psi\left(i^* y_2^m\right) dy_1\, dy_2 \\ &= \int_{\mathbb{Z}_p} \Psi\left(i^* y_1^m\right) dy_1 \cdot \int_{\mathbb{Z}_p} \Psi\left(i^* y_2^m\right) dy_2 \\ &= \left(\int_{\mathbb{Z}_p} \Psi\left(i^* y^m\right) dy\right)^2. \end{aligned}$$

This observation allows us to compute F^* corresponding to the polynomial $f(x) = x_1^m + x_2^m + \cdots + x_n^m$ by simply taking the function F^* corresponding to $g(x) = x^m$ and raising it to the n^{th} power. Thus, for the polynomial $f(x) = x_1^m + x_2^m + \cdots + x_n^m$, we obtain

$$F^*(i^*) = \begin{cases} 1 & \text{if} \quad e \leq 0 \\ 0 & \text{if} \quad e > 0 \text{ and } e \equiv 1 \,(\operatorname{mod} m) \\ p^{-n\lfloor e/m \rfloor} & \text{if} \quad e > 0 \text{ and } e \not\equiv 1 \,(\operatorname{mod} m). \end{cases}$$

4. Computation of F

We now compute the Local Singular Series. Fix an arbitrary nonzero p-adic integer $i \in \mathbb{Z}_p - \{0\}$. We may write $i = p^k u$ for a unique nonnegative integer k and a unique unit $u \in \mathbb{Z}_p^\times$. We will see that $F(i)$ depends on the value k, but not on the value u. Define the value $r \in \{0, 1, \ldots, m-1\}$ by $k \equiv r \pmod{m}$. Thus $(k-r)/m$ is a nonnegative integer. We now define

$$X = \begin{cases} 0 & \text{if} \quad k = r \\ \sum_{a=0}^{(k-r)/m-1} p^{ma-na} & \text{if} \quad k > r \end{cases} \qquad \text{and} \qquad Y = \sum_{a=0}^{(k-r)/m} p^{ma-na}.$$

Observe that $Y - 1 = p^{m-n} X$, a fact that will be used later. The goal of this section is to show that

$$F(i) = \begin{cases} 1 + p^{-n}(p^m - p) X & \text{if} \quad r = 0 \\ 1 + p^{-n}\left[(p^m - p^r) X + (p^r - p) Y\right] - p^{k-n((k-r)/m+1)} & \text{if} \quad r \neq 0. \end{cases}$$

To begin this computation, recall that when $i^* \in \mathbb{Z}_p$, we have $F^*(i^*) = 1 = \Psi(-ii^*)$. Thus

$$\int_{\mathbb{Z}_p} F^*(i^*) \Psi(-ii^*)\, di^* = \int_{\mathbb{Z}_p} di^* = 1.$$

This last fact allows us to express $F(i)$ in the following manner.

$$\begin{aligned} F(i) &= \int_{\mathbb{Q}_p} F^*(i^*) \Psi(-ii^*)\, di^* \\ &= \int_{\mathbb{Z}_p} F^*(i^*) \Psi(-ii^*)\, di^* + \int_{\mathbb{Q}_p - \mathbb{Z}_p} F^*(i^*) \Psi(-ii^*)\, di^* \\ &= 1 + \int_{\mathbb{Q}_p - \mathbb{Z}_p} F^*(i^*) \Psi(-ii^*)\, di^* = 1 + \sum_{e=1}^{\infty} \int_{p^{-e}(\mathbb{Z}_p^\times)} F^*(i^*) \Psi(-ii^*)\, di^* \\ &= 1 + \sum_{\substack{e=1 \\ e \not\equiv 1 \pmod{m}}}^{\infty} \int_{p^{-e}(\mathbb{Z}_p^\times)} p^{-n\lfloor e/m \rfloor} \Psi(-ii^*)\, di^*, \end{aligned}$$

where the final form of this expression uses the computed value of $F^*(i^*)$ from the preceding section.

In case $m = 1$, we clearly have $e \equiv 1 \pmod{m}$ for all $e \in \{1, 2, 3, \ldots\}$, and so $F(i) = 1$ in this case.

Now suppose that $m \neq 1$. Recall that $m \geq 1$. The conditions $\gcd(m, p) = \gcd(m, p-1) = 1$ guarantee that $m \neq 2$. Hence $m \geq 3$. Further expanding the previous expression for $F(i)$, we obtain

$$F(i)=1+\sum_{\substack{e=1\\ e\equiv 0 (\mathrm{mod}\, m)}}^{\infty}\int_{p^{-e}(\mathbb{Z}_p^\times)} p^{-ne/m}\Psi(-ii^*)\,di^*$$

$$+\sum_{c=2}^{m-1}\left(\sum_{\substack{e=1\\ e\equiv c (\mathrm{mod}\, m)}}^{\infty}\int_{p^{-e}(\mathbb{Z}_p^\times)} p^{-n(e+m-c)/m}\Psi(-ii^*)\,di^*\right).$$

Define the set $\mathcal{C}=\{2,3,\ldots m-1\}$. For each $c\in\mathcal{C}\cup\{0\}$, we make the change of variables $p^{-(ma+c)}v=i^*$ and $p^{ma+c}dv=di^*$, where $v\in\mathbb{Z}_p^\times$, on the integral corresponding to $e\equiv c\,(\mathrm{mod}\, m)$. This leads us to

$$F(i)=1+\sum_{a=1}^{\infty}p^{ma}\int_{\mathbb{Z}_p^\times}p^{-na}\Psi\left(-p^k u p^{-ma}v\right)dv$$

$$+\sum_{c=2}^{m-1}\left(\sum_{a=0}^{\infty}p^{ma+c}\int_{\mathbb{Z}_p^\times}p^{-n(a+1)}\Psi\left(-p^k u p^{-(ma+c)}v\right)dv\right).$$

We now make another change of variables, namely $u'=-uv$ and $du'=dv$, where $u'\in\mathbb{Z}_p^\times$. For each value $c\in\mathcal{C}\cup\{0\}$ and integer $a\geq 0$, for notational convenience we write

$$I(a,c)=\int_{\mathbb{Z}_p^\times}\Psi\left(p^{-(-k+ma+c)}u'\right)du'.$$

As an immediate consequence of the Orthogonality Relations, we know that

$$I(a,c)=\begin{cases}1-p^{-1} & \text{if} \quad a\leq (k-c)/m\\ -p^{-1} & \text{if} \quad a=(k-c+1)/m\\ 0 & \text{if} \quad a>(k-c+1)/m.\end{cases}$$

After this latest change of variables, and using this new notation, we obtain

$$F(i)=1+\sum_{a=1}^{\infty}p^{(m-n)a}I(a,0)+\sum_{c=2}^{m-1}\left(\sum_{a=0}^{\infty}p^{-n+c}p^{(m-n)a}I(a,c)\right).$$

For each value $c\in\mathcal{C}\cup\{0\}$, for notational convenience we define

$$S(c)=\sum_{a=0}^{\infty}p^{(m-n)a}I(a,c).$$

It then follows that

$$F(i)=1+S(0)-I(0,0)+p^{-n}\sum_{c=2}^{m-1}p^c S(c).$$

We now determine $F(i)$ separately for the cases $0\leq k\leq m-1$ and $k\geq m$.

4.1 Computation of F in case $0 \le k \le m-1$

Let $c \in \mathcal{C} \cup \{0\}$. We now determine $S(c)$.

First suppose that $c > k+1$. Then $(k-c+1)/m < 0$, and so for each nonnegative integer a we have $a > (k-c+1)/m$, forcing $I(a,c) = 0$. Hence $S(c) = 0$ in this case.

Now suppose that $c = k+1$. Then $(k-c+1)/m = 0$, and so $I(0,c) = -p^{-1}$. For each positive integer a we have $a > (k-c+1)/m$, forcing $I(a,c) = 0$. Hence $S(c) = p^0 I(0,c) = -p^{-1}$ in this case.

Finally, suppose that $c < k+1$, which is equivalent to $c \le k$. Thus $(k-c)/m \ge 0$. However, since $k \le m-1$ while c is nonnegative, we have $(k-c)/m < 1$. So $0 \le (k-c)/m < 1$. Thus $a = 0$ is the only nonnegative integer satisfying $a \le (k-c)/m$. Hence $I(0,c) = 1 - p^{-1}$. The next two paragraphs continue to address the case $c < k+1$.

Now suppose in particular that $c = 0$ and $k = m-1$. Then $(k-c+1)/m = 1$ is an integer, and so $I(1,c) = -p^{-1}$ while $I(a,c) = 0$ for all $a \ge 2$. Hence $S(c) = p^0\left(1-p^{-1}\right) + p^{m-n}\left(-p^{-1}\right) = 1 - p^{-1} - p^{m-n-1}$ in this case.

Now suppose that either $c \ne 0$ or $k \ne m-1$. If $k \ne m-1$ then we have $0 \le k < m-1$, and so the fact that c is nonnegative yields $(k-c+1)/m < 1$. If $c \ne 0$, then we know $c \in \mathcal{C}$, and so the fact that $0 \le k \le m-1$ clearly yields $(k-c+1)/m < 1$. So in either case we see that $(k-c+1)/m < 1$. Thus for each integer $a \ge 1$ we have $a > (k-c+1)m$, forcing $I(a,c) = 0$. Hence $S(c) = p^0 I(0,c) = 1 - p^{-1}$ in this case.

In summary, we have shown for each value $c \in \mathcal{C} \cup \{0\}$ that

$$S(c) = \begin{cases} 1 - p^{-1} - p^{m-n-1} & \text{if} \quad c = 0 \text{ and } k = m-1 \\ 1 - p^{-1} & \text{if} \quad c < k+1 \text{ and either } c \ne 0 \text{ or } k \ne m-1 \\ -p^{-1} & \text{if} \quad c = k+1 \\ 0 & \text{if} \quad c > k+1. \end{cases}$$

Using the fact $I(0,0) = 1 - p^{-1}$, along with the above summary with $c = 0$, we deduce that

$$1 + S(0) - I(0,0) = \begin{cases} 1 & \text{if} \quad 0 \le k < m-1 \\ 1 - p^{m-n-1} & \text{if} \quad k = m-1. \end{cases}$$

Write $W = p^{-n} \sum_{c=2}^{m-1} p^c S(c)$. We now determine an explicit formula for W.

In case $k = 0$, then for all values $c \in \mathcal{C}$ we have $c > k+1$, and so $S(c) = 0$, which forces $W = 0$.

In case $k = 1$, we have $S(2) = -p^{-1}$, while $S(c) = 0$ for $3 \le c \le m-1$, so $W = p^{-n} p^2 \left(-p^{-1}\right) = -p^{1-n}$.

In case $1 < k < m-1$, we see that $S(c) = 1 - p^{-1}$ for $2 \le c \le k$ and $S(k+1) = -p^{-1}$ and $S(c) = 0$ for $k+1 < c \le m-1$. So $W = p^{-n}\left[\sum_{c=2}^{k} p^c(1 - p^{-1}) + p^{k+1}(-p^{-1})\right] = p^{-n}\left[(1-p^{-1})\left(\frac{p^{k+1}-p^2}{p-1}\right) - p^k\right] = p^{-n}\left[(p^k - p) - p^k\right] = -p^{1-n}$ in this case.

In case $k = m-1$, we see that $S(c) = 1 - p^{-1}$ for all $c \in \{2, \dots, m-1\}$. So $W = p^{-n}\sum_{c=2}^{m-1} p^c(1-p^{-1}) = p^{-n}(1-p^{-1})\left(\frac{p^m - p^2}{p-1}\right) = p^{-n}(p^m - p) = p^{m-n-1} - p^{1-n}$ in this case.

In summary, we have now shown that

$$p^{-n}\sum_{c=2}^{m-1} p^c S(c) = \begin{cases} 0 & \text{if} \quad k = 0 \\ -p^{1-n} & \text{if} \quad 1 \le k < m-1 \\ p^{m-n-1} - p^{1-n} & \text{if} \quad k = m-1. \end{cases}$$

It now follows easily that

$$F(i) = \begin{cases} 1 & \text{if} \quad k = 0 \\ 1 - p^{1-n} & \text{if} \quad 1 \le k \le m-1. \end{cases}$$

Note that this is consistent with the explicit form of $F(i)$ given earlier, since we are currently working in the special case $k = r$, which forces $X = 0$ and $Y = 1$.

4.2 Computation of F in case $k \ge m$

Now assume that $k \ge m$. For each value $c \in \mathcal{C} \cup \{0\}$, clearly

$$\left\lfloor \frac{k-c}{m} \right\rfloor = \begin{cases} (k-r)/m & \text{if} \quad r \ge c \\ (k-r)/m - 1 & \text{if} \quad r < c. \end{cases}$$

For each $c \in \mathcal{C} \cup \{0\}$, we define

$$T(c) = (1 - p^{-1}) \sum_{a=0}^{\lfloor \frac{k-c}{m} \rfloor} p^{(m-n)a}.$$

The condition $k \ge m$ guarantees that $k - c \ge 0$, and so clearly $\lfloor (k-c)/m \rfloor \ge 0$.

First suppose that $c \in \mathcal{C} \cup \{0\}$ satisfies $c \equiv r + 1 \pmod{m}$. Hence $(k-c+1)/m$ is an integer. In fact, the condition $k \ge m$ implies that $(k-c+1)/m$ is a positive integer. Thus for $a = (k-c+1)/m$ we have $I(a,c) = -p^{-1}$. Note that $\lfloor (k-c)/m \rfloor + 1 = (k-c+1)/m$. Hence we see that

$$S(c) = \sum_{a=0}^{\lfloor \frac{k-c}{m} \rfloor} p^{(m-n)a}(1 - p^{-1}) + (p^{m-n})^{\left(\frac{k-c+1}{m}\right)}(-p^{-1})$$

$$= T(c) - p^{(m-n)\left(\frac{k-c+1}{m}\right)-1}.$$

Now suppose that $c \in \mathcal{C} \cup \{0\}$ satisfies $c \not\equiv r+1 \pmod{m}$. Hence $(k-c+1)/m$ is an not an integer, and indeed $\lfloor (k-c+1)/m \rfloor = \lfloor (k-c)/m \rfloor \geq 0$. Hence we see that

$$S(c) = \sum_{a=0}^{\left\lfloor \frac{k-c}{m} \right\rfloor} p^{(m-n)a}\left(1-p^{-1}\right) = T(c).$$

We may summarize the last two paragraphs to say for each value $c \in \mathcal{C} \cup \{0\}$ that

$$S(c) = \begin{cases} T(c) - p^{(m-n)\left(\frac{k-c+1}{m}\right)-1} & \text{if} \quad c \equiv r+1 \pmod{m} \\ T(c) & \text{if} \quad c \not\equiv r+1 \pmod{m} \end{cases}$$

In particular, for $c = 0$ we obtain

$$1 + S(0) - I(0,0) = \begin{cases} 1 + T(0) - I(0,0) - p^{k-n\left(\frac{k+1}{m}\right)} & \text{if} \quad r = m-1 \\ 1 + T(0) - I(0,0) & \text{if} \quad r \neq m-1. \end{cases}$$

Recall that $\lfloor k/m \rfloor = (k-r)/m$. Thus $T(0) = \left(1-p^{-1}\right)Y$. Observe that $I(0,0) = 1 - p^{-1}$. We thus have $T(0) - I(0,0) = \left(1-p^{-1}\right)Y - \left(1-p^{-1}\right) = \left(1-p^{-1}\right)(Y-1)$. Recall that $Y - 1 = p^{m-n}X$. Hence $T(0) - I(0,0) = \left(1-p^{-1}\right)p^{m-n}X = p^{-n}\left(p^m - p^{m-1}\right)X$, and it follows that

$$1 + S(0) - I(0,0) = \begin{cases} 1 + p^{-n}\left(p^m - p^{m-1}\right)X & \text{if} \quad r \neq m-1 \\ 1 + p^{-n}\left(p^m - p^{m-1}\right)X - p^{k-n\left(\frac{k+1}{m}\right)} & \text{if} \quad r = m-1. \end{cases}$$

For notational convenience, we now define

$$A = \sum_{\substack{c \in \mathcal{C} \\ c \leq r}} p^c T(c) \qquad \text{and} \qquad B = \sum_{\substack{c \in \mathcal{C} \\ c > r}} p^c T(c).$$

First suppose that $r \in \{0, m-1\}$. Then for each value $c \in \mathcal{C}$ we have $c \not\equiv r+1 \pmod{m}$, and so we have $S(c) = T(c)$, forcing $\sum_{c=2}^{m-1} p^c S(c) = \sum_{c=2}^{m-1} p^c T(c) = A + B$.

Now suppose that $1 \leq r \leq m-2$. Then $r+1 \in \mathcal{C}$. Thus for each value $c \in \mathcal{C}$ we have

$$S(c) = \begin{cases} T(c) & \text{if} \quad c \neq r+1 \\ T(c) - p^{(m-n)\left(\frac{k-c+1}{m}\right)-1} & \text{if} \quad c = r+1, \end{cases}$$

and so $\sum_{c=2}^{m-1} p^c S(c) = \sum_{c=2}^{m-1} p^c T(c) - p^{r+1}\left[p^{(m-n)\left(\frac{k-r}{m}\right)-1}\right] = A + B - p^{k-n\left(\frac{k-r}{m}\right)}$.

In summary, we have

$$\sum_{c=2}^{m-1} p^c S(c) = \begin{cases} A+B & \text{if} \quad r \in \{0, m-1\} \\ A+B-p^{k-n\left(\frac{k-r}{m}\right)} & \text{if} \quad 1 \leq r < m-1. \end{cases}$$

We now calculate A explicitly. In case $r \in \{0,1\}$, there is no value $c \in \mathcal{C}$ satisfying $c \leq r$, and so $A = 0$. Now assume that $r \in \{2, 3, \ldots, m-1\}$. When $c \leq r$, we have $\lfloor (k-c)/m \rfloor = (k-r)/m$ and so $T(c) = \left(1 - p^{-1}\right) Y$. Thus

$$A = \sum_{c=2}^{r} p^c T(c) = \left(1 - p^{-1}\right) Y \sum_{c=2}^{r} p^c = \left(\frac{p-1}{p}\right) Y \left(\frac{p^{r+1} - p^2}{p-1}\right) = (p^r - p) Y.$$

In summary then,

$$A = \begin{cases} 0 & \text{if} \quad r = 0 \\ (p^r - p) Y & \text{if} \quad r \in \{1, 2, \ldots, m-1\}. \end{cases}$$

We now calculate B explicitly. In case $r = m-1$, there is no value $c \in \mathcal{C}$ satisfying $c > r$, and so $B = 0$. Now assume that $r \in \{0, 1, \ldots, m-2\}$. When $c > r$, we have $\lfloor (k-c)/m \rfloor = (k-r)/m - 1$, and so $T(c) = \left(1 - p^{-1}\right) X$. Thus $B = \left(1 - p^{-1}\right) X \sum_{\substack{c \in \mathcal{C} \\ c > r}} p^c$. If $r \in \{1, \ldots, m-1\}$, then $\sum_{\substack{c \in \mathcal{C} \\ c > r}} p^c = \sum_{c=r+1}^{m-1} p^c = \frac{p^m - p^{r+1}}{p-1}$, and so $B = \left(\frac{p-1}{p}\right) X \left(\frac{p^m - p^{r+1}}{p-1}\right) = \left(p^{m-1} - p^r\right) X$. If $r = 0$, then $\sum_{\substack{c \in \mathcal{C} \\ c > r}} p^c = \sum_{c=2}^{m-1} p^c = (p^m - p^2)/(p-1)$. Thus $B = \left(\frac{p-1}{p}\right) X \left(\frac{p^m - p^2}{p-1}\right) = \left(p^{m-1} - p\right) X$. In summary then,

$$B = \begin{cases} \left(p^{m-1} - p\right) X & \text{if} \quad r = 0 \\ \left(p^{m-1} - p^r\right) X & \text{if} \quad r \in \{1, 2, \ldots, m-1\}. \end{cases}$$

5. Computation of the Igusa Local Zeta Function

Throughout this section, we set $t = p^{-s}$. Recall the fact stated earlier about the Igusa Local Zeta Function:

$$Z(s) = \int_{\mathbb{Z}_p} F(i) \, |i|_p^s di = \int_{\mathbb{Z}_p - \{0\}} F(i) \, |i|_p^s di = \sum_{k=0}^{\infty} \int_{p^k\left(\mathbb{Z}_p^\times\right)} F(i) \, |i|_p^s di.$$

In case $m = 1$, we have seen that $F(i) = 1$ for all values $i \in \mathbb{Z}_p - \{0\}$, and so the change of variables $p^k u = i$ and $p^{-k} du = di$, where $u \in \mathbb{Z}_p^\times$, yields

$$Z(s) = \sum_{k=0}^{\infty} \int_{p^k(\mathbb{Z}_p^\times)} |i|_p^s \, di = \sum_{k=0}^{\infty} \int_{\mathbb{Z}_p^\times} |p^k u|_p^s \, p^{-k} du = \sum_{k=0}^{\infty} p^{-ks-k} \int_{\mathbb{Z}_p^\times} du$$
$$= \sum_{k=0}^{\infty} p^{-k(s+1)} \left(1 - p^{-1}\right) = \frac{p-1}{p-p^{-s}} = \frac{p-1}{p-t},$$

which conforms to the expression for $Z(s)$ stated in the introduction when we let $m = 1$.

Now suppose that $m \neq 1$. Recall that this forces $m \geq 3$. We expand our earlier expression for $Z(s)$ to obtain

$$Z(s) = \sum_{\substack{k=0 \\ k \equiv 0 (\mathrm{mod}\, m)}}^{\infty} \int_{p^k(\mathbb{Z}_p^\times)} F(i) \, |i|_p^s di + \sum_{r=1}^{m-1} \left(\sum_{\substack{k=0 \\ k \equiv r (\mathrm{mod}\, m)}}^{\infty} \int_{p^k(\mathbb{Z}_p^\times)} F(i) \, |i|_p^s di \right).$$

The expression for $F(i)$ computed in the preceding section involves the variables X and Y. But the expressions for X and Y both depend on whether n is equal to m. Thus we treat the cases $n = m$ and $n \neq m$ separately. For convenience, write $H = p - t$ and $I = p^n - t^m$ and $J = p^m - t^m$ and $K = p^m t - pt^m$ and $L = \sum_{r=1}^{m-1} t^r$. Note that L is a number whose value we do not need to know.

5.1 The case $m \neq n$.

In this case we have

$$X = \frac{p^{(m-n)\left(\frac{k-r}{m}\right)} - 1}{p^{m-n} - 1} \quad \text{and} \quad Y = \frac{p^{(m-n)\left(\frac{k-r}{m}+1\right)} - 1}{p^{m-n} - 1}.$$

Hence, using the expressions for $F(i)$ computed in preceding section, we obtain

$$F(i) = \begin{cases} 1 + \left[\frac{p^m - p}{p^m - p^n}\right] \left(p^{(m-n)\left(\frac{k}{m}\right)} - 1\right) & \text{if} \quad r = 0 \\ 1 + \left[\frac{p^m - p^r}{p^m - p^n}\right] \left(p^{(m-n)\left(\frac{k-r}{m}\right)} - 1\right) & \\ \quad + \left[\frac{p^r - p}{p^m - p^n}\right] \left(p^{(m-n)\left(\frac{k-r}{m}+1\right)} - 1\right) & \text{if} \quad 1 \leq r \leq m-1, \\ \quad - p^{k-n\left(\frac{k-r}{m}+1\right)} & \end{cases}$$

and these expressions may be substitued for $F(i)$ in our earlier expression for $Z(s)$ as a sum of integrals. For each integer $k \geq 0$, the Division Algorithm yields unique integers $a \geq 0$ and $r \in \{0, 1, \ldots, m-1\}$ such that $k = ma + r$. On the integrals in the expression for $Z(s)$ above, we now make the change of variables $p^{ma+r}u = i$ and $p^{-(ma+r)}du = di$, where $u \in \mathbb{Z}_p^\times$ and $0 \leq r \leq m-1$. Thus we obtain

$$Z(s) = \sum_{a=0}^{\infty} \left(p^{-1}t\right)^{ma} \left(1 + \frac{p^m - p}{p^m - p^n}\left(p^{(m-n)a} - 1\right)\right) \int_{\mathbb{Z}_p^\times} du \tag{1}$$

$$+ \sum_{r=1}^{m-1} \left(\sum_{a=0}^{\infty} \left(p^{-1}t\right)^{ma+r} \left(1 + \frac{p^m - p^r}{p^m - p^n}\left(p^{(m-n)a} - 1\right) \right.\right.$$

$$\left.\left. + \frac{p^r - p}{p^m - p^n}\left(p^{(m-n)(a+1)} - 1\right) - p^{ma+r-n(a+1)}\right)\right) \int_{\mathbb{Z}_p^\times} du. \tag{2}$$

This expression for $Z(s)$ is a sum of two terms, labelled as (1) and (2). We refer to $\sum_{a=0}^{\infty} \left(p^{-1}t\right)^{ma} \cdot 1$ as the first term of (1), $\sum_{a=0}^{\infty} \left(p^{-1}t\right)^{ma} \frac{p^m - p}{p^m - p^n}\left(p^{(m-n)a} - 1\right)$ as the second term of (1), $\sum_{a=0}^{\infty} \left(p^{-1}t\right)^{ma+r} \cdot 1$ as the first term of (2), and so on. Let S_α denote the sum of the first terms of (1) and (2), let S_β denote the second term of (1), let S_γ denote the second term of (2), let S_δ denote the third term of (2), and let S_ε denote the fourth term of (2). Thus, if we write $S = S_\alpha + S_\beta + S_\gamma + S_\delta + S_\varepsilon$, it then follows that

$$Z(s) = S \int_{\mathbb{Z}_p^\times} du = \left(1 - p^{-1}\right) S.$$

We now determine simplified expressions for each of S_α, S_β, S_γ, S_δ, and S_ε. Observe that

$$S_\alpha = \sum_{r=0}^{m-1} \sum_{a=0}^{\infty} \left(p^{-1}t\right)^{ma+r} = \sum_{r=0}^{m-1} \left(p^{-1}t\right)^r \sum_{a=0}^{\infty} \left(p^{-m}t^m\right)^r$$

$$= \left[\frac{\left(p^{-1}t\right)^m - 1}{p^{-1}t - 1}\right] \left[\frac{1}{1 - \left(p^{-1}t\right)^m}\right] = \frac{-1}{p^{-1}t - 1} = \frac{p}{p - t} = \frac{p}{H}.$$

Further,

$$S_\beta = \left[\frac{p^m - p}{p^m - p^n}\right] \sum_{a=0}^{\infty} \left(p^{-1}t\right)^{ma} \left(p^{(m-n)a} - 1\right)$$

$$= \left[\frac{p^m - p}{p^m - p^n}\right] \left[\frac{t^m\left(p^m - p^n\right)}{IJ}\right] = \frac{\left(p^m - p\right)t^m}{IJ}.$$

Further,

$$S_\gamma = \sum_{r=1}^{m-1} \left[\frac{p^m - p^r}{p^m - p^n}\right] \sum_{a=0}^{\infty} \left(p^{-1}t\right)^{ma+r} \left(p^{(m-n)a} - 1\right)$$

$$
\begin{aligned}
&= \sum_{r=1}^{m-1} \left(p^{-1}t\right)^{r} \left[\frac{p^{m}-p^{r}}{p^{m}-p^{n}}\right] \sum_{a=0}^{\infty} \left(p^{-1}t\right)^{ma} \left(p^{(m-n)a}-1\right) \\
&= \sum_{r=1}^{m-1} \left(p^{-1}t\right)^{r} \left[\frac{p^{m}-p^{r}}{p^{m}-p^{n}}\right] \left[\frac{t^{m}\left(p^{m}-p^{n}\right)}{IJ}\right] \\
&= \frac{t^{m}}{IJ} \sum_{r=1}^{m-1} \left(p^{-1}t\right)^{r} \left(p^{m}-p^{r}\right) = \frac{t^{m}}{IJ} \left[p^{m} \sum_{r=1}^{m-1} \left(p^{-1}t\right)^{r} - \sum_{r=1}^{m-1} t^{r}\right] \\
&= \frac{t^{m}}{IJ} \left[p^{m} \frac{K}{p^{m}H} - L\right] = \frac{t^{m}}{IJ} \left[\frac{K}{H} - L\right].
\end{aligned}
$$

Further,

$$
\begin{aligned}
S_{\delta} &= \sum_{r=1}^{m-1} \left[\frac{p^{r}-p}{p^{m}-p^{n}}\right] \sum_{a=0}^{\infty} \left(p^{-1}t\right)^{ma+r} \left(p^{(m-n)(a+1)}-1\right) \\
&= \sum_{r=1}^{m-1} \left(p^{-1}t\right)^{r} \left[\frac{p^{r}-p}{p^{m}-p^{n}}\right] \sum_{a=0}^{\infty} \left(p^{-1}t\right)^{ma} \left(p^{(m-n)(a+1)}-1\right) \\
&= \sum_{r=1}^{m-1} \left(p^{-1}t\right)^{r} \left[\frac{p^{r}-p}{p^{m}-p^{n}}\right] \left[\frac{p^{m}\left(p^{m}-p^{n}\right)}{IJ}\right] \\
&= \frac{p^{m}}{IJ} \sum_{r=1}^{m-1} \left(p^{-1}t\right)^{r} \left(p^{r}-p\right) = \frac{p^{m}}{IJ} \left[\sum_{r=1}^{m-1} t^{r} - p \sum_{r=1}^{m-1} \left(p^{-1}t\right)^{r}\right] \\
&= \frac{p^{m}}{IJ} \left[L - p\left(\frac{K}{p^{m}H}\right)\right] = \frac{1}{IJ} \left[p^{m}L - \frac{pK}{H}\right].
\end{aligned}
$$

And finally,

$$
\begin{aligned}
S_{\varepsilon} &= \sum_{r=1}^{m-1} \sum_{a=0}^{\infty} \left(p^{-1}t\right)^{ma+r} \left(-p^{ma+r-n(a+1)}\right) \\
&= -p^{-n} \sum_{r=1}^{m-1} t^{r} \sum_{a=0}^{\infty} p^{-na} t^{ma} = -p^{-n} L \left(\frac{p^{n}}{I}\right) = \frac{-L}{I}.
\end{aligned}
$$

5.2 The case $m = n$.

In this case we have $X = (k-r)/m$ and $Y = (k-r)/m + 1$. Hence, using the expressions for $F(i)$ computed in preceding section, and replacing each occurrence of n by m, we obtain

$$
F(i) = \begin{cases}
1 + p^{-m}\left(p^{m}-p\right)\left(\frac{k-r}{m}\right) & \text{if} \quad r = 0 \\
1 + p^{-m}\left[\left(p^{m}-p^{r}\right)\left(\frac{k-r}{m}\right) + \left(p^{r}-p\right)\left(\frac{k-r}{m}+1\right)\right] & \\
\quad - p^{k-m\left(\frac{k-r}{m}+1\right)} & \text{if} \quad 1 \leq r \leq m-1,
\end{cases}
$$

and these expressions may be substitued for $F(i)$ in our earlier expression for $Z(s)$ as a sum of integrals. Now, using the same change of variables that was used in Section 5.1, we obtain

$$Z(s) = \sum_{a=0}^{\infty} \left(p^{-1}t\right)^{ma} \left(1 + p^{-m}\left(p^m - p\right) a\right) \int_{\mathbb{Z}_p^\times} du \tag{3}$$

$$+ \sum_{r=1}^{m-1} \left(\sum_{a=0}^{\infty} (p^{-1}t)^{ma+r} \Big(1 + p^{-m}(p^m - p^r)a + p^{-m}(p^r - p)(a+1) - p^{ma+r-n(a+1)} \Big) \right) \int_{\mathbb{Z}_p^\times} du. \tag{4}$$

This expression for $Z(s)$ is a sum of two terms, labelled as (3) and (4). We refer to $\sum_{a=0}^{\infty} \left(p^{-1}t\right)^{ma} \cdot 1$ as the first term of (3), $\sum_{a=0}^{\infty} \left(p^{-1}t\right)^{ma} p^{-m} \left(p^m - p\right) a$ as the second term of (3), $\sum_{a=0}^{\infty} \left(p^{-1}t\right)^{ma+r} \cdot 1$ as the first term of (4), and so on. Let S'_α denote the sum of the first terms of (3) and (4), let S'_β denote the second term of (3), let S'_γ denote the second term of (4), let S'_δ denote the third term of (4), and let S'_ε denote the fourth term of (4). Thus, if we write $S' = S'_\alpha + S'_\beta + S'_\gamma + S'_\delta + S'_\varepsilon$, it then follows that

$$Z(s) = S' \int_{\mathbb{Z}_p^\times} du = \left(1 - p^{-1}\right) S'.$$

Clearly $S'_\alpha = S_\alpha$ and $S'_\varepsilon = S_\varepsilon$. We now show that $S'_\beta = S_\beta$ and $S'_\gamma = S_\gamma$ and $S'_\delta = S_\delta$. From this it will follow that $S' = S$, and so our expression for $Z(s)$ is the same for the cases $m \neq n$ and $m = n$. Note that

$$S'_\beta = p^{-m}\left(p^m - p\right) \sum_{a=0}^{\infty} \left(p^{-1}t\right)^{ma} a = p^{-m}\left(p^m - p\right)\left[\frac{p^m t^m}{IJ}\right] = \frac{\left(p^m - p\right) t^m}{IJ} = S_\beta.$$

Further,

$$\begin{aligned} S'_\gamma &= p^{-m} \sum_{r=1}^{m-1} \left(p^m - p^r\right) \sum_{a=0}^{\infty} \left(p^{-1}t\right)^{ma+r} a \\ &= p^{-m} \sum_{r=1}^{m-1} \left(p^{-1}t\right)^r \left(p^m - p^r\right) \sum_{a=0}^{\infty} \left(p^{-1}t\right)^{ma} a \\ &= p^{-m} \left[p^m \sum_{r=1}^{m-1} \left(p^{-1}t\right)^r + \sum_{r=1}^{m-1} t^r \right] \sum_{a=0}^{\infty} \left(p^{-1}t\right)^{ma} a \\ &= p^{-m} \left[\frac{K}{H} - L\right] \left[\frac{p^m t^m}{IJ}\right] = \frac{t^m}{IJ} \left[\frac{K}{H} - L\right] = S_\gamma. \end{aligned}$$

And finally,

$$\begin{aligned}
S'_\delta &= p^{-m}\sum_{r=1}^{m-1}(p^r-p)\sum_{a=0}^{\infty}\left(p^{-1}t\right)^{ma+r}(a+1)\\
&= p^{-m}\left[\sum_{r=1}^{m-1}\left(p^{-1}t\right)^r(p^r-p)\right]\left[\sum_{a=0}^{\infty}\left(p^{-1}t\right)^{ma}a+\sum_{a=0}^{\infty}\left(p^{-1}t\right)^{ma}\right]\\
&= p^{-m}\left[\sum_{r=1}^{m-1}t^r-p\sum_{r=1}^{m-1}\left(p^{-1}t\right)^r\right]\left[\frac{p^mt^m}{IJ}+\frac{p^m}{J}\right]\\
&= p^{-m}\left[L-p\left(\frac{K}{p^mH}\right)\right]\left[\frac{p^m\left(t^m+I\right)}{IJ}\right]\\
&= \frac{1}{IJ}\left[L-p\left(\frac{K}{p^mH}\right)\right]p^m=\frac{1}{IJ}\left[p^mL-\frac{pK}{H}\right]=S_\delta.
\end{aligned}$$

5.3 The simplification

Recalling that $J=p^m-t^m$, we observe that

$$S_\gamma+S_\delta=\frac{1}{IJ}\left\{\frac{(t^m-p)K}{H}+JL\right\}=\frac{(t^m-p)K+HJL}{HIJ}.$$

As $S_\varepsilon=-HJL/HIJ$, it follows that

$$S_\gamma+S_\delta+S_\varepsilon=\frac{(t^m-p)K}{HIJ}.$$

Using $S_\beta=(p^m-p)\,t^mH/HIJ$, we then obtain

$$S_\beta+S_\gamma+S_\delta+S_\varepsilon=\frac{(p^m-p)\,t^mH+(t^m-p)K}{HIJ}.$$

It is tedious but straightforward to show that $(p^m-p)\,t^mH+(t^m-p)K=p\,(t^m-t)\,J$. Hence

$$S_\beta+S_\gamma+S_\delta+S_\varepsilon=\frac{p\,(t^m-t)\,J}{HIJ}=\frac{p\,(t^m-t)}{HI}.$$

In the case $t=1$ (which corresponds to $s=0$), we have $t^m-t=0$, and so $S_\beta+S_\gamma+S_\delta+S_\varepsilon=0$. Hence in this case, $S=S_\alpha=p/H$. But the condition $t=1$ also forces $H=p-t=p-1$, and so $S=p/H=p/(p-1)$. Thus

$$Z(s)=\left(\frac{p-1}{p}\right)S=\left(\frac{p-1}{p}\right)\left(\frac{p}{p-1}\right)=1.$$

Now suppose that $t \neq 1$. In this case we have

$$S = S_\alpha + (S_\beta + S_\gamma + S_\delta + S_\varepsilon) = \frac{pI}{HI} + \frac{p\,(t^m - t)}{HI}.$$

But observe that $I + (t^m - t) = (p^n - t^m) + (t^m - t) = p^n - t$. Hence $S = p\,(p^n - t)\,/HI$, and so

$$Z\,(s) = \left(1 - p^{-1}\right) S = \left(\frac{p-1}{p}\right)\left[\frac{p\,(p^n - t)}{HI}\right] = \frac{(p-1)\,(p^n - t)}{(p-t)\,(p^n - t^m)}.$$

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References

[1] J. Denef, Report on Igusa's local zeta function, Seminaire Bourbaki, Vol. 1990/1991. *Asterisque*, No. 201-203 (1991), Exp. No. 741 (1992), 359–386.

[2] H. Hosokawa, Igusa local zeta function of the quadratic form $x_1^2+x_2^2+\cdots+x_n^2$, *Yokohama Mathematics Journal*, **46** (1999), 181–189.

[3] J.-I. Igusa, *An Introduction to the Theory of Local Zeta Functions.* American Mathematical Society, International Press, 2000.

[4] J.-I. Igusa, *Lectures on Forms of Higher Degree.* Tata Institute of Fundamental Research, Bombay, Springer-Verlag, New York, 1978.

[5] N. Koblitz, *p-adic Numbers, p-adic Analysis, and Zeta Functions.* Springer-Verlag, New York, 1984.

[6] B. Marko and A. Wiswell, Igusa local zeta function of the cubic polynomial $x_1^3 + x_2^3 + \cdots + x_n^3$, unpublished.

[7] A. Weil, Sur la formule de Siegel dans la théorie des groupes classiques, *Acta Mathematica*, **113** (1965), 1–87.

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