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VARIATIONAL STABILITY AND LOCAL RIGIDITY OF EINSTEIN METRICS

By

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Abstract. Certain curvature conditions for variational stability of Einstein metrics are given. The argument of Besson, Courtois and Gallot, developed in [2], is improved in terms of the Weyl curvature operator and the scalar curvature. For compact Kähler-Einstein manifolds (M, g, J) we can show that the infinitesimal rigidity of complex structures J is equivalent to the variational stability of Einstein metrics g.

1. Introduction

An Einstein metric g on a compact manifold M is called *locally rigid*, if it gives an isolated point $[g]$ in the moduli space of Einstein metrics on M. Further, an Einstein metric g is called *variationally stable*, when the quadratic form associated to the second variation of the total scalar curvature functional $S(g') = \mathrm{Vol}(g')^{2/n-1} \int_M s_{g'} dv_{g'}$ is positive definite; i.e., there exists $\lambda > 0$ such that

$$-S''_g(h, h) \geq \lambda ||h||^2$$

for any $h \in \Gamma(M; S^2(M))$ satisfying $\mathrm{tr}_g h = 0$ and $\delta_g h = 0$. See Definition 4.63 in [1].

For the precise definition of local rigidity refer to [10], where (and also in [1]) the terminology *rigidity* is used. Notice also that by the result of [9] any variationally stable Einstein metric is locally rigid, since any variationally stable Einstein metric admits no non-trivial infinitesimal Einstein deformation.

Several examples of locally rigid Einstein metrics are given in [1]; the n-sphere S^n with the standard metric, the complex projective space $\mathbf{C}P^n$ with the Fubini-Study metric and a compact Einstein manifold of negative sectional curvature.

For Einstein manifolds of negative sectional curvature Besson, Courtois and

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Gallot considered in [2] the functional $g' \mapsto K(g') = \int_X |s_{g'}|^{n/2} dv_{g'}$ which is closely related to the functional $S = S(g')$ and obtained the following variational stability theorem.

THEOREM (Besson, Courtois and Gallot [2]). *Let (X, g) be a compact, connected oriented n-manifold X with an Einstein metric g of negative scalar curvature. If g is a metric of negative sectional curvature or a locally symmetric metric of non-compact type, then $K : \Sigma \to \mathbf{R}$ is locally minimal at $g' = g$. Here Σ denotes the set of all smooth metrics g' with* $\mathrm{vol}(g') = 1$ *and of constant scalar curvature. Namely, such an Einstein metric is variationally stable and hence locally rigid.*

The local rigidity of Einstein manifolds of negative curveture is also stated in [1], Corollary 12.73.

The following theorem, known as a global rigidity theorem, states that the moduli of Einstein metrics on hyperbolic manifolds consists of a single point.

THEOREM (Besson, Courtois and Gallot [3], LeBrun [13]). *Let X be a compact, connected oriented 4-manifold admitting a real or complex hyperbolic metric g_0. Then any Einstein metric g on X is homothetic to g_0 up to diffeomorphisms of X.*

The aim of this article is to relax the strictly negative curvature condition of the local rigidity of Einstein metrics in Theorem(Besson, Courtois and Gallot [2]). Since the Ricci tensor is a multiple of the metric, the Riemannian curvature tensor R is expressed as the sum of the Weyl conformal curvature part W and the scalar curvature part and we obtain the following variational stability theorems.

THEOREM 1. *Let (M, g) be a compact, connected oriented Einstein n-manifold with $s_g < 0$. If*

$$\sup_{x \in M} w(x) + \frac{1}{n(n-1)} s_g < 0,$$

then g is variationally stable and then locally rigid. Here $w(x)$ denotes the largest eigenvalue of the Weyl curvature operator $W : \Lambda^2(M) \longrightarrow \Lambda^2(M)$ at x.

THEOREM 2. *Let (M, g) be a compact, connected oriented Einstein 4-manifold. If $s_g < 0$ and*

$$\sup_{x \in M} \{w^+(x) + w^-(x)\} + \frac{s_g}{6} < 0,$$

then g is variationally stable and hence locally rigid.

Here $w^+ = w^+(x)$ and $w^- = w^-(x)$ denote the largest eigenvalue of the (anti-)self-dual Weyl curvature operators $W^\pm : \Lambda^2_\pm(M) \longrightarrow \Lambda^2_\pm(M)$ at x, respectively. Here $\Lambda^2_\pm(M)$ are the bundles of self-dual, or anti-self-dual 2-forms.

COROLLARY 1. *Let (M,g) be a compact, connected oriented Einstein 4-manifold. Suppose that $s_g < 0$ and g is self-dual, i.e., $W^- = 0$ identically. If $\sup_M w^+ + \frac{s_g}{6} < 0$, then g is variationally stable and then locally rigid.*

This corollary supports strongly the following conjecture; a compact, oriented self-dual Einstein 4-manifold (M,g) with $s_g < 0$ must be real hyperbolic or complex hyperbolic.

Now, let (M,g) be a compact Kähler-Einstein manifold of real dimension four with $s_g < 0$. Then from Proposition 2, [4] the largest eigenvalue of W^+ is exactly $w^+ = -\frac{1}{12}s_g$.

COROLLARY 2. *Let (M,g) be a compact, connected Kähler-Einstein manifold of real dimension four. Suppose that $s_g < 0$ and $\sup_M w^- + \frac{s_g}{12} < 0$. Then g is variationally stable and hence locally rigid.*

For compact Kähler-Einstein manifolds of $s_g < 0$ and of dimension n for arbitrary n we are able to present a similar stability theorem in terms of the Bochner curvature tensor(indeed, as Theorem 4.1 in §4). However, due to Calabi-Aubin-Yau's argument, deformation of Einstein metrics on a complex manifold is directly related to the deformation of complex structures. By the result of Koiso, if a compact Kähler-Einstein manifold of $s_g < 0$ is variationally stable, then the first Kodaira-Spencer cohomology group vanishes, i.e., $H^1(M;\Theta) = 0$, that is, the complex structure is infinitesimally rigid(see [8]). See for this statement [11] and Proposition 12.98, [1]. We derive the following theorem which is the converse implication.

THEOREM 3. *Let (M,g,J) be a compact Kähler-Einstein manifold of $s_g < 0$. If $H^1(M;\Theta) = 0$, then g is variationally stable.*

2. The functionals K and S

Let $\mathcal{R}$ be the space of all smooth metrics on a compact oriented n-manifold M. Define functionals K, S : $\mathcal{R} \longrightarrow \mathbf{R}$ by

$$K(g') = \int_M |s_{g'}|^{n/2} dv_{g'},$$

$$S(g') = \mathrm{vol}(g')^{2/n-1} \int_M s_{g'} dv_{g'}, \quad g' \in \mathcal{R}.$$

These functionals K, S are significantly important for studying geometry of Riemannian manifolds and appear for instance in the Yamabe problem associated to scalar curvature (see [12], and [15]).

THEOREM 2.1 (Besson, Courtois and Gallot [2]). *Let g be a smooth metric on a compact oriented n-manifold M having negative constant scalar curvature. If g' is a smooth metric and is conformal to g, then*

$$K(g') \geq K(g), \tag{1}$$

and the equality holds if and only if $g' = cg$ for some positive constant c.

Remark that the same statement is also given in [15]. In four dimension the Seiberg-Witten theory enables to estimate for certain 4-manifolds the absolute minimum of K, the square L^2-functional of scalar curvature, in terms of the topological invariants, as

$$K(g') \geq 32\pi^2(2\chi(M) + 3\tau(M)),$$

for any smooth metric g' and the equality holds when the 4-manifold is Kähler-Einstein. See for this [5], [14].

Denote by $\mathcal{R}^{vol}$ the space of smooth metrics of unit volume,

$$\mathcal{R}^{vol} = \{g' \in \mathcal{R} \mid \mathrm{vol}(g') = 1\},$$

and by $\mathcal{R}^{scal}$ the space of smooth metrics of constant scalar curvature,

$$\mathcal{R}^{scal} = \{g' \in \mathcal{R} \mid s_g \text{ is constant}\},$$

and restrict the functional K to $\Sigma = \mathcal{R}^{vol} \cap \mathcal{R}^{scal}$, the space of metrics of unit volume having constant scalar curvature. Then

$$K(g') = |s_{g'}|^{n/2} = (-s_{g'})^{n/2} = \{-S(g')\}^{n/2}.$$

Therefore, $K|_\Sigma$ is locally minimal at $g' = g$ if and only if $S|_\Sigma$ is locally maximal at $g' = g$.

The functional S is invariant under the action of the group $\mathrm{Diff}^+(M)$ of orientation preserving diffeomorphisms of M. The space $\mathcal{R}^{vol}$ is also invariant under its action. Notice that for any $g \in \mathcal{R}^{vol}$ there exists a slice $\mathcal{R}^{vol}_o \subset$

$\mathcal{R}^{vol}$ transversal to the action of $\mathrm{Diff}^+(M)$ such that $g \in \mathcal{R}_o^{vol}$ and the subset $\{\phi^*(g') \mid g' \in \mathcal{R}_o^{vol},\ \phi \in \mathrm{Diff}^+(M)\}$ contains a neighborhood of g in $\mathcal{R}^{vol}$.

Then the tangent space at g to $\mathcal{R}^{vol}$ is given by the direct sum

$$T_g\mathcal{R}^{vol} = \mathcal{L}_{\mathfrak{X}(M)} \oplus T_g\mathcal{R}_o^{vol}$$

where

$$\mathcal{L}_{\mathfrak{X}(M)} = \{h \in \Gamma(M; S^2M) \mid h = \mathcal{L}_X g \,\text{for}\, X \in \mathfrak{X}(M)\}$$

is the infinite dimensional vector space tangent to the $\mathrm{Diff}^+(M)$-orbit through g and

$$T_g\mathcal{R}_o^{vol} = \{h \in \Gamma(M; S^2M) \mid \int_M \mathrm{tr}_g h\, dv_g = 0,\ \delta_g h = 0\,\}$$

is the tangent space at g to $\mathcal{R}_o^{vol}$.

Any 2-symmetric tensor h is written as $h = h_0 + fg$, where h_0 is trace free and $f \in C^\infty(M)$. The space $T_g\mathcal{R}_o^{vol}$ then decomposes into the subspaces $\Phi \oplus C_o^\infty(M)g$, where

$$\Phi = \{h \mid \mathrm{tr}_g h = 0,\ \delta_g h = 0\,\}$$

and

$$C_o^\infty(M)g = \{h = fg \mid f \in C^\infty(M),\ \int_M f dv_g = 0\,\},$$

the tangent space to the conformal deformations $\{e^f g \in \mathcal{R}^{vol} \mid f \in C^\infty(M)\}$ of the metric g.

The first variation formula of $S : \mathcal{R} \longrightarrow \mathbf{R}$ is

$$-S'_g(h) = \int_M \left(r_g - \frac{s_g}{2} g, h\right)_g dv_g,$$

where r_g is the Ricci tensor and $g(t) : (-\epsilon, \epsilon) \longrightarrow \mathcal{R}$ is a curve with $g(0) = g$ and $\frac{d}{dt} g(t)|_{t=0} = h$. This formula is derived from the following formulae([10], [1]);

$$\left(\frac{d}{dt} s_{g(t)}\right)|_{t=0} = \Delta_g(\mathrm{tr}_g h) + \delta_g(\delta_g h) - (r_g, h)_g$$
$$\left(\frac{d}{dt} dv_{g(t)}\right)_{|t=0} = \frac{1}{2}\mathrm{tr}_g h\, dv_g$$

Therefore, we have

PROPOSITION 2.1. *Let g be a smooth metric of negative constant scalar curvature s_g. Then, the following are equivalent.*

(i) *g is Einstein,*
(ii) *g is critical for the total scalar curvature functional $S : \Sigma \longrightarrow \mathbf{R}$.*

This proposition, a well known result of D. Hilbert, appears in [1] as Proposition 4.47. The proof of this proposition is easy so that we omit it.

3. The quadratic form Q

Let g be an Einstein metric on a compact manifold M with $s_g < 0$.

PROPOSITION 3.1 (Koiso [9], Theorems 2.4, 2.5 and Besse [1], Theorem 4.60). *The second variation formula of $S : \Sigma \cap \mathcal{R}_o^{vol} \longrightarrow \mathbf{R}$ at g is given by*

$$-2S_g''(h,h) = \int_M \{|Dh|^2 - 2(\hat{R}(h), h)\} dv_g, \tag{2}$$

where $\hat{R} : S_0^2(M) \longrightarrow S_0^2(M)$, $S_0^2(M) = \{h \in S^2(M) \,|\, \mathrm{tr}_g h = 0\}$, is an endomorphism defined by

$$\hat{R}(h)_{ij} = \sum_{k,\ell} R_{ikj\ell} h_{k\ell}$$

in terms of an orthonormal basis. Here $R = (R_{ikj\ell})$ is the Riemannian curvature tensor of (M, g).

We remark here that the tangent space at g to $\Sigma \cap \mathcal{R}_o^{vol}$ is given, as noticed in p.131, [1], by

$$T_g(\Sigma \cap \mathcal{R}_o^{vol}) = \{h \in \Gamma(M; S^2 M) \,|\, \mathrm{tr}_g h = 0,\ \delta_g h = 0\}.$$

In fact, take a one-parameter family $g(t)$ in $\Sigma \cap \mathcal{R}_o^{vol}$ with $g(0) = g$. Then $h = \frac{d}{dt} g(t)_{|t=0}$ satisfies $\delta_g h = 0$ and $\int_M \mathrm{tr}_g h dv_g = 0$. Also from the scalar curvature constant condition we have $\Delta_{g(t)} s_{g(t)} = 0$ and by differentiating this by t

$$\Delta_g(\Delta_g(\mathrm{tr}_g h) + \delta_g(\delta_g h) - \frac{s_g}{n} \mathrm{tr}_g h) = 0.$$

Since $\delta_g h = 0$, $\Delta_g(\mathrm{tr}_g h) - \frac{s_g}{n}\mathrm{tr}_g h$ must be constant. However, this constant must be further zero, because $\int_M \mathrm{tr}_g h dv_g = 0$. Then $\mathrm{tr}_g h$ is an eigenfunction of negative eigenvalue $\frac{s_g}{n}$. This implies that h is trace free.

Now we will investigate the second variation formula (2). We define the quadratic form Q associated with this formula.

The rough Laplacian D^*D for $h \in \Gamma(M; S_0^2(M))$ can be written as

$$D^*Dh = (\delta_g^D d_g^D + d_g^D \delta_g^D)h - h \circ r + \hat{R}(h), \tag{3}$$

where

$$(h \circ r)_{ij} = \sum_k h_{ik} r_{kj}.$$

We will follow the argument given by [2].

Take an arbitrary $\alpha \in (0,1)$. Then, since $\delta_g h = 0$, it holds from the above formula that

$$\begin{aligned} -2S_g''(h,h) = \alpha \int_M |Dh|^2 dv_g + (1-\alpha) \int_M |d_g^D h|^2 dv_g \\ - 2\alpha \int_M (\hat{R}(h), h) dv_g + (1-\alpha) \int_M Q(h,h) dv_g. \end{aligned} \tag{4}$$

Here the quadratic form $Q = Q_x$ at $x \in M$ is defined by

$$Q_x(h,h) = -\{(h \circ r, h)_g(x) + (\hat{R}(h), h)_g(x)\} \text{ for } h \in S_0^2(M). \tag{5}$$

The positive definiteness of Q gives a sufficient criterion to the variational stability of Einstein metrics in the following lemma.

LEMMA 3.1 (Besson, Courtois and Gallot [2]). *If there exists a positive constant λ such that for any $x \in M$ and for any h of* $\mathrm{tr}_g h = 0$

$$Q_x(h,h) \geq \lambda |h|^2,$$

then there exists also $\lambda' > 0$ such that

$$-2S_g''(h,h) \geq \lambda' \int_M |h|_g^2 dv_g. \tag{6}$$

for all h.

If, in fact, we take $\alpha > 0$ sufficiently small, then we obtain (6) from (4).

We will verify our theorems and corollaries stated in Introduction. Let g be an Einstein metric of $s_g < 0$. Then, since $r = \frac{s_g}{n} g$, the Riemannian curvature tensor R_g is written as

$$R_g = W - \frac{s_g}{n(n-1)} g \owedge g,$$

where W is the Weyl curvature tensor, so that

$$\hat{R}(h) = \hat{W}(h) - \frac{s_g}{n(n-1)}\, h, \quad h \in S_0^2(M).$$

The quadratic form $Q_x(h,h)$ is therefore

$$Q_x(h,h) = -\left\{ \frac{n-2}{n(n-1)} s_g |h|^2 + (\hat{W}(h), h) \right\}.$$

Diagonalize h as

$$h(e_i, e_i) = h_{ii}, \quad h(e_i, e_j) = h_{ij} = 0, \quad i \neq j.$$

We have then $|h|^2 = \sum_{i=1}^{n} h_{ii}^2$ and

$$\hat{W}(h)_{ij} = \sum_{k,\ell} W_{ikj\ell} h_{k\ell} = \sum_{k=1}^{n} W_{ikjk} h_{kk}$$

so that

$$\begin{aligned}(\hat{W}(h), h) &= \sum_{i,j} (\hat{W}(h))_{ij} h_{ij} = \sum_{i,k} (W_{ikik} h_{kk}) h_{ii} \\ &= \sum_{i \neq k} W_{ikik} h_{ii} h_{kk}\end{aligned}$$

which turns out to be

$$\frac{1}{2} \sum_{i \neq k} W_{ikik} (h_{ii} + h_{kk})^2 = \sum_{i<k} W_{ikik} (h_{ii} + h_{kk})^2,$$

since $\sum_i W_{ikij} = 0$. We have

$$W_{ikik} = (W(\theta^i \wedge \theta^k), \theta^i \wedge \theta^k) \leq w |\theta^i \wedge \theta^k|^2 = w,$$

for each i, k, $i \neq k$, where $w = w(x)$ is the maximum eigenvalue of the self-adjoint operator $W : \Lambda^2(M) \longrightarrow \Lambda^2(M)$ at $x \in M$. Then

$$\sum_{i<k} W_{ikik} (h_{ii} + h_{kk})^2 \leq w(x) \sum_{i<k} (h_{ii} + h_{kk})^2 = w \left\{ (n-1) \sum_i h_{ii}^2 + 2 \sum_{i<k} h_{ii} h_{kk} \right\}.$$

Further we have from $\mathrm{tr}_g h = 0$

$$2 \sum_{i<k} h_{ii} h_{kk} = -\sum_i h_{ii}^2 = -|h|^2$$

from which it follows

$$(\hat{W}(h), h) \ \leq (n-2)\ w(x)\ |h|^2, \ \ h \in S_0^2(M)$$

and then

$$\frac{n-2}{n(n-1)} s_g |h|^2 + (\hat{W}(h), h) \ \leq \ (n-2)\ (w(x) + \frac{1}{n(n-1)} s_g)|h|^2$$

at each x.

Since, by the assumption,

$$\sup_{x \in M} w(x) + \frac{1}{n(n-1)} s_g < 0,$$

there exists a positive constant λ such that

$$Q_x(h, h) \geq \lambda |h|^2, \ \ h \in S_0^2(M)$$

therefore for sufficiently small α we can use Lemma 3.1 to have

$$-2S''(h, h) \geq \lambda' ||h||^2$$

for any $h \in \Gamma(M; S_0^2(M))$ satisfying $\delta_g h = 0$. This verifies Theorem 1.

Proof of Theorem 2. Since $\dim M = 4$, we have $h_{ii} + h_{jj} = -(h_{kk} + h_{\ell\ell})$ for distinct i, j, k, ℓ so that

$$\sum_{i<k} W_{ikik}(h_{ii} + h_{kk})^2 = (W_{1212} + W_{3434})(h_{11} + h_{22})^2$$
$$+ (W_{1313} + W_{2424})(h_{11} + h_{33})^2 + (W_{1414} + W_{2323})(h_{11} + h_{44})^2.$$

We have

$$W_{1212} + W_{3434} = (W(\theta^1 \wedge \theta^2), \theta^1 \wedge \theta^2) + (W(\theta^3 \wedge \theta^4), \theta^3 \wedge \theta^4)$$

and the similar formulae for $W_{1313} + W_{2424}$ and $W_{1414} + W_{2323}$.

Since the space of 2-forms $\Lambda^2(M)$ splits into $\Lambda_+^2 \oplus \Lambda_-^2$ equipped with the orthogonal basis

$$f_1^{\pm} = \frac{1}{2}(\theta^1 \wedge \theta^2 \pm \theta^3 \wedge \theta^4),$$
$$f_2^{\pm} = \frac{1}{2}(\theta^1 \wedge \theta^3 \pm \theta^4 \wedge \theta^2),$$
$$f_3^{\pm} = \frac{1}{2}(\theta^1 \wedge \theta^4 \pm \theta^2 \wedge \theta^3),$$

respectively, we have

$$\theta^1 \wedge \theta^2 = f_1^+ + f_1^-, \quad \theta^3 \wedge \theta^4 = f_1^+ - f_1^-$$

so that

$$W_{1212} = (W(f_1^+ + f_1^-), f_1^+ + f_1^-) = (W^+(f_1^+), f_1^+) + (W^-(f_1^-), f_1^-)$$

which coincides with W_{3434} so that

$$W_{1212} + W_{3434} = 2\{(W^+(f_1^+), f_1^+) + (W^-(f_1^-), f_1^-)\}.$$

Here we used the four dimensional Weyl curvature property, namely, $W : \Lambda^2(M) \longrightarrow \Lambda^2(M)$ decomposes into $W^\pm : \Lambda^2_\pm \longrightarrow \Lambda^2_\pm$.

Let w^+ and w^- be the maximum eigenvalue of $W^\pm$, respectively. Then, we have

$$W_{1212} + W_{3434} \leq 2\ (\ w^+|f_1^+|^2 + w^-|f_1^-|^2\ )\ =\ w^+ + w^-$$

(here $|f_i^\pm|^2 = \frac{1}{2}$, $i = 1, 2, 3$) and hence

$$\sum_{i<k} W_{ikik}(h_{ii} + h_{kk})^2 \leq (w^+ + w^-)\{(h_{11} + h_{22})^2 + (h_{11} + h_{33})^2 + (h_{11} + h_{44})^2\}$$
$$= (w^+ + w^-)|h|^2.$$

From the assumption of Theorem 2, $\sup_{x\in M}\{w^+(x) + w^-(x)\} + \frac{1}{6}s_g < 0$. Then, it follows that for some positive constant λ

$$Q_x(h, h) \geq \lambda|h|^2, \quad h \in S^2_0(M)$$

at each point x of M. This implies from Lemma 3.1 the variational stability of our Einstein metric g.

4. Bochner curvature and Kähler-Einstein metrics

The Bochner curvature tensor $B = B_{ikj\ell}$ is defined on a Kähler manifold (M, J, g), quite similarly to the Weyl conformal curvature tensor on a real Riemannian manifold. The following stability theorem is a generalization of Corollary 2 in §1, because in four dimension B coincides with W^-.

THEOREM 4.1. *Let (M, g, J) be a compact Kähler-Einstein manifold of real dimension n. Suppose $s_g < 0$. If the Bochner curvature tensor $B = B_{ikj\ell}$ satisfies*

$$\sup{}_x \beta^+(x) + \frac{1}{n+2}s_g < 0,$$

then g is variationally stable. Here β^+ denotes the largest eigenvalue of the self-adjoint endomorphism B over the space $\{h \in S^2_0(M)\,|\, h$ is anti-J-invariant$\}$.

Proof. As explained in the proof of Lemma 12.94, [1], the operator $D^*D - 2\hat{R}$ preserves J-invariant and anti-J-invariant symmetric tensors. Therefore, for $h = h_1 + h_2$, where h_1 is J-invariant and h_2 is anti-J-invariant, we have

$$((D^*D - 2\hat{R})h, h)_g = ((D^*D - 2\hat{R})h_1, h_1)_g + ((D^*D - 2\hat{R})h_2, h_2)_g.$$

For J-invariant h_1, set a 2-form ψ as $\psi(X, Y) = h_1(X, JY)$. Then by arranging the Weitzenböck formula 12.92' in [1] as

$$((D^*D - 2\hat{R})h_1)(X, Y) = -2\frac{s_g}{n}h_1(X, Y) - (\Delta\psi)(X, JY),$$

we obtain

$$((D^*D - 2\hat{R})h_1, h_1)_g = -2\frac{s_g}{n}|h_1|^2 + 2(\Delta\psi, \psi)_g$$

and by integrating

$$-2S''(h_1, h_1) = \int_M ((D^*D - 2\hat{R})h_1, h_1)_g \, dv_g \geq -2\frac{s_g}{n}\int_M |h_1|_g^2 \, dv_g,$$

since $\int_M (\Delta\psi, \psi) dv_g \geq 0$.

On the other hand, for an anti-J-invariant h_2 we make use of the Bochner curvature tensor and apply Lemma 3.1. The Bochner curvature tensor for a Kähler-Einstein metric g is

$$B_{ijk\ell} = R_{ijk\ell} - \frac{s_g}{n(n+2)}\{(g_{ik}g_{j\ell} - g_{i\ell}g_{jk}) + (J_{ik}J_{j\ell} - J_{jk}J_{i\ell} + 2J_{ij}J_{k\ell})\}$$

(see for example [6] and [16]) so that a complex space form metric is a Kähler metric whose Bochner curvature tensor vanishes. So, similarly to the proof of Theorem 1 we have

$$(\hat{R}(h_2), h_2)_g = (\hat{B}(h_2), h_2)_g - 2\frac{s_g}{n(n+2)}|h_2|_g^2$$

and the quadratic form Q is then

$$\begin{aligned} Q_x(h_2, h_2) &= -\{\frac{s_g}{n}|h_2|^2 + (\hat{B}(h_2), h_2) - 2\frac{s_g}{n(n+2)}|h_2|^2\} \\ &= -\{\frac{s_g}{n+2}|h_2|^2 + (\hat{B}(h_2), h_2)\}. \end{aligned}$$

Therefore, from the assumption of the theorem there exists $\lambda > 0$ such that $Q_x(h_2, h_2) \geq \lambda|h_2|^2$ for all h_2. By combining these two arguments we obtain the theorem.

5. Stability and deformations of complex structures

Let (M, g, J) be a compact Kähler-Einstein manifold and of $s_g < 0$.

For the proof of Theorem 3 it suffices to show that there is a $\lambda > 0$ such that

$$\int_M ((D^*D - 2\hat{R})(h), h)_g \, dv_g \geq \lambda \int_M |h|_g^2 dv_g$$

for all h satisfying $\mathrm{tr}_g h = 0$ and $\delta_g h = 0$.

As discussed in the above we divide the argument into the J-invariant case and the anti-J-invariant case. For the invariant case we have

$$\int_M ((D^*D - 2\hat{R})(h_1), h_1)_g \, dv_g \geq -2\frac{s_g}{n} \int_M |h_1|^2 dv_g.$$

On the other hand, for h_2 we set $I \in \Gamma(M; \mathrm{End}\ (TM))$ by $g(X, IY) = h_2(X, JY)$. Then I is symmetric, i.e., $g(IX, Y) = g(X, IY)$ and anti-commutes with J, i.e., $IJ + JI = 0$. Therefore we may regard I as a $T^{1,0}M$-valued $(0,1)$-form. By making use of the Weitzenböck formula

$$(D^*D - 2\hat{R})(h_2)(X, JY) = g(X, (\Delta'' I)(Y)),$$

where $\Delta'' = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ is the complex Laplacian associated with the Dolbeault complex of the holomorphic tangent bundle $T^{1,0}M$. See for this, formula 12.93', [1]. Therefore

$$(D^*D - 2\hat{R})(h_2)(X, Y) = -g(X, (\Delta'' I)JY)$$

which is equal to $g(X, J(\Delta'' I)Y)$, since $IJ + JI = 0$. Then

$$((D^*D - 2\hat{R})(h_2), h_2) = \sum_{i,j} g(e_i, J(\Delta'' I)(e_j))h_2(e_i, e_j) = (\Delta'' I, I)_g,$$

since $h_2(e_i, e_j) = g(e_i, JIe_j)$. Therefore

$$\begin{aligned}\int_M ((D^*D - 2\hat{R})(h_2), h_2)_g dv_g &= \int_M (\Delta'' I, I)_g dv_g \\ &\geq \lambda \int_M |I|^2 dv_g = \lambda \int_M |h_2|^2 \, dv_g,\end{aligned}$$

because the operator Δ'' is positive definite, self-adjoint elliptic and the assumption Ker $\Delta'' = H^1(M; \Theta) = 0$ ensures that the first eigenvalue of Δ'' is positive. Theorem 3 follows from these arguments.

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