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REAL ANALYTIC FUNCTIONS WITH ISOLATED ZEROES ON DOMAINS OF TOPOLOGICAL VECTOR SPACES

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Abstract. Here we show that for every open subset U of every Fréchet space V without any continuous norm there is no real analytic function on U with an isolated zero. We prove the same result for a certain Banach space V. We also prove an extension theorem for real analytic functions.

1. Introduction

The function $\sum_{i=1}^{n} x_i^2$ is a real analytic function on $\mathbf{R}^n$ with an isolated zero. Here we investigate the existence of such functions on open subsets of infinite-dimensional real topological vector spaces. In Section 2 we will prove the following result.

THEOREM 1. *Let V be a real Fréchet space without any continuous norm and U an open subset of V. Then there is no real analytic function on U with an isolated zero.*

We also have an example in which V is a Banach space.

EXAMPLE 1. Fix an uncountable discrete set A and let $C_0(A,\mathbf{R})$ (resp. $C_0(A,\mathbf{C})$) be the the Banach space of all $\mathbf{R}$-valued (resp. $\mathbf{C}$-valued) functions f on A which vanish at infinity with the supremum norm, i.e. such that for every $\epsilon > 0$ there is a finite set $S \subset A$ such that $|f(i)| < \epsilon$ for every $i \in A \backslash S$. For any open subset U of $C_0(A,\mathbf{C})$ every holomorphic function on U depends only on a countable number of variables ([3] or [1], Prop. 8). Using the complexification we see that for every open subset B of $C_0(A,\mathbf{R})$ and every real analytic function f on B, f depends only on a countable subset of A. In particular no such f may have an isolated zero or vanish exactly on a non-empty finite-dimensional real

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analytic subvariety of B.

Of course, for many topological vector spaces V there are many real analytic functions on V with exactly one zero.

EXAMPLE 2. Fix an infinite set I and a real number $p > 0$. Let $\ell^p(I)$ be the set of all p-summable real-valued functions on I with the usual norm $\| \|_p$; if $0 < p < 1$, this is not a norm, but a p-norm. Let $z_i : \ell^p(I) \to \mathbf{R}$ be the function defined by $z_i(f) = f(i)$ for all $f \in \ell^p(I)$. Fix an even integer $d \geq p$. The function $h := \sum_{i \in I} z_i^d$ is a continuous homogeneous real-valued polynomial on $\ell^p(I)$ (and hence a real analytic function on $\ell^p(I)$) with an isolated zero at $\{0\}$. For $p = 2$ this example covers all infinite-dimensional Hilbert spaces. Let $U \subsetneq \ell^p(I)$ and fix $P \in \partial(U)$. Take a real analytic function h on $\ell^p(I)$ which vanishes only at P and set $f := 1/h$. Thus f is a real analytic function on U which diverges at P. Hence U is a domain of holomorphy with respect to the set of all real analytic functions on U.

Then we will consider another, but related problem. As in [2], Theorem 4, we will obtain the following result.

THEOREM 2. *Let V be an infinite-dimensional real topological vector space such that its dual V' separates V. Let V_σ be V equipped with the weak topology. Let $K \subset V_\sigma$ be a compact convex set and U an open neighborhood of K in V_σ. Then every real analytic function on $U \backslash K$ is the restriction to $U \backslash K$ of a real analytic function on U.*

2. The proofs

LEMMA 1. *Let V be an infinite-dimensional real (resp. complex) topological vector space such that its dual V' separates V. Let V_σ be V equipped with the weak topology. Let U be an open subset of V_σ and f a real analytic (resp. holomorphic) function on U with respect to the sigma-topology. Then for every $P \in U$ there is an open neighborhood Ω of P in U, finitely many $f_i \in V'$, say f_i for $1 \leq i \leq s$, a neighborhood A of $(f_1(P), \ldots, f_s(P))$ in $\mathbf{R}^s$ (resp. $\mathbf{C}^s$) and a real analytic (resp. holomorphic) function g on A such that $f|\Omega = g \circ \tau|\Omega$, where $\tau = (f_1, \ldots, f_s)$. Furthermore, the minimal such integer s is a continuous function of P.*

Proof. Using the complexification we are reduced to prove the statement in the complex case. By the definition of weak topology there are finitely many $f_i \in V'$, say f_i for $1 \leq i \leq s$, and $\epsilon > 0$ such that $B := \{x \in V_\sigma : |f_i(x)| < \epsilon \text{ for all } i\}$ is contained in U and $f|B$ is bounded. Since every bounded holomorphic function on $\mathbf{C}$ is constant, the restriction of f to each fiber of τ is constant, proving the first assertion. For the last assertion copy the proof of [2], Th. 2. □

Use Lemma 1 to make the trivial modifications of the proof of [2], Th. 4, needed to obtain the following result.

THEOREM 3. *Let V be an infinite-dimensional real topological vector space such that its dual V' separates V. Let V_σ be V equipped with the weak topology. Let U be an open connected subset and L a closed subset of U such that for every $x \in L$ there is an open neighborhood Ω of x in U (for the weak topology) and finitely many $f_i \in V'$, say f_i for $1 \leq i \leq s$, such that for all integers $n \geq s$ and all $f_j \in V'$, $s+1 \leq j \leq n$, calling $\tau : V \to \mathbf{R}^n$ the map defined by $(f_1, \ldots, f_n)$, and every $z \in \Omega \cap L$ the set $\tau^{-1}(\tau(z)) \cap (U \backslash L)$ is connected and non-empty. Then every real analytic function on $U \backslash L$ is the restriction to $U \backslash L$ of a real analytic function on U.*

Proof of Theorem 2. Fix an integer $n \geq 2$ and $f_i \in V'$, $1 \leq i \leq n$. Call $\tau : V_\sigma \to \mathbf{R}^n$ the map defined by $(f_1, \ldots, f_n)$. Since $\tau(K)$ is compact and convex, $\mathbf{R}^n \backslash \tau(K)$ is connected and non-empty. Fix $P \in K$. Since K is convex and the fiber $\tau^{-1}(\tau(P))$ is an infinite-dimensional affine space, $\tau^{-1}(\tau(P)) \backslash \tau^{-1}(\tau(P)) \cap K$ is connected and non-empty. Hence we may apply Theorem 3. □

Proof of Theorem 1. Up to translation we may assume $0 \in U$. By [4], Th. 2.6.13, V has a subspace E isomorphic to $\mathbf{R}^{\mathbf{N}}$. Set $D := U \cap E$. It is sufficient to prove the same assertion for D. By [2], Cor. 1, every real analytic function on D depends locally only on finitely many variables and hence it cannot have any isolated zero. □

REMARK 1. The proof of Theorem 1 shows that we may take as V an arbitrary real topological vector space (even a non-locally convex one) containing a subspace isomorphic to $\mathbf{R}^{\mathbf{N}}$.

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