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THE CONSTRUCTION OF UNITS OF INFINITE ORDER IN THE CHARACTER RING OF A FINITE GROUP

By

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Abstract. The structure of the unit group consisting of units of finite order in the character ring of a finite group is well known (see [8]). We also have studied the unit group in the character ring of an alternating group A_n $(n \geq 5)$. In this article our objective is to construct units of infinite order in the character ring of a finite group concretely, by making use of units in $Z[\omega]$ where Z is the ring of rational integers, ω is a primitive p-th root of unity, and p (≥ 5) is a prime number.

1. Introduction

Throughout this article, G denotes always a finite group, Z the ring of rational integers, Q the rational field, and C the complex number field. For a finite set S, we denote by $|S|$ the number of elements in S.

Let $Irr(G) = \{\chi_1 = 1_G \text{ (the principal character)}, \ldots, \chi_h\}$ be the complete set of absolutely irreducible complex characters of G.

Let us set

$$R(G) = \left\{ \sum_{i=1}^{h} a_i \chi_i \;\middle|\; a_i \in Z \quad (i = 1, \ldots, h) \right\}.$$

That is, $R(G)$ is the set of generalized characters of G. It is well known that $R(G)$ forms a commutative ring with an identity element χ_1. We call $R(G)$ the character ring of a finite group G.

Let ζ be a primitive $|G|$-th root of unity and $K = Q(\zeta)$ be the smallest subfield of C containing Q and ζ. Then K is a splitting field for G.

We denote by A the ring of algebraic integers in K. In [8] we proved the following theorem and corollary concerning the units of finite order in the character

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ring of a finite group G.

THEOREM 1.1. *Any unit of finite order in $AR(G)$ has the form $\epsilon\chi$ for some linear character χ of G and some root ϵ of unity in A where $AR(G)$ is a A-algebra spanned by $Irr(G) = \{\chi_1, \dots, \chi_h\}$, that is, $AR(G) = \{\sum_{i=1}^{h} a_i\chi_i \mid a_i \in A\,(i = 1, \dots, h)\}$.*

COROLLARY 1.2. *Any unit of finite order in $R(G)$ has the form $\pm\chi$ for some linear character χ of G.*

Next we state some results, which have been obtained so far, concerning the units of infinite order in the character ring of a finite group G.

Here we fix the following notation;

Let R be a commutative ring with an identity element.

$U(R)$: $=$ the unit group of R,

$U_f(R)$: $=$ the subgroup of $U(R)$ which consists of units of finite order in R,

S_n, A_n: $=$ a symmetric group and an alternating group on n symbols for a natural number n, respectively.

We assume that $n \geq 5$ for a natural number n.

Since A_n $(n \geq 5)$ is a simple group, $A_n = D(A_n)$ holds where $D(A_n)$ is the commutator subgroup of A_n, and so A_n has only one linear character χ_1 (that is, the principal character of A_n). By Corollary 1.2 we have $U_f(R(A_n)) = \{\pm\chi_1\}$ and so we use a notation "$U(R(A_n))/\{\pm 1\}$" in place of "$U(R(A_n))/U_f(R(A_n))$" for simplicity, by identifying $\{\pm 1\}$ with $\{\pm\chi_1\}$. Let $c(n)$ be the number of self-associated frames of real type $[m_1, \dots, m_r], m_1 + \cdots + m_r = n$. (See Definition 2.4 in [9] about $c(n)$.)

In [9] we proved the following theorem.

THEOREM 1.3. *rank of $U(R(A_n))/\{\pm 1\} = c(n)$.*

In [10] we constructed $c(n)$ units of infinite order $\psi_1, \dots, \psi_{c(n)}$ in $R(A_n)$ concretely which are free generators, and showed that $U^2(R(A_n)) \subseteq \langle\psi_1, \dots, \psi_{c(n)}\rangle$ where $U^2(R(A_n)) = \{\psi^2 \mid \psi \in U(R(A_n))\}$ and $\langle\psi_1, \dots, \psi_{c(n)}\rangle$ is the abelian subgroup of $U(R(A_n))$ generated by $\psi_1, \dots, \psi_{c(n)}$. (See Theorem 3.4 in [10].)

We note that Theorem 1.3 is a direct consequence of Theorem 3.4 in [10]. These results concerning units in $R(A_n)$ were obtained by making use of the character table of A_n.

In this article we intend to construct units of infinite order in $R(G)$ for a finite group G, by making use of units in $Z[\omega]$ where ω is a primitive p-th root of unity and p (≥ 5) is a prime number.

2. Preliminaries

We shall keep the notation in Section 1.

Let ζ be a primitive $|G|$-th root of unity for a finite group G and $K = Q(\zeta)$ be the smallest subfield of C containing Q and ζ. We denote by A the ring of algebraic integers in K. Let B be a subring of A such that B contains all the values of characters of G. Here we note that $B \ni 1$ and so $B \supseteq Z$, because for any element $x \in G$, $\chi_1(x) = 1 \in B$ by assumption where χ_1 is the principal character of G.

In the above situation we have:

LEMMA 2.1. *If $U(B) = U_f(B)$ holds, then we have $U(R(G)) = U_f(R(G))$.*

Proof. For any element $u \in U(R(G))$, there exists $u' \in R(G)$ such that $uu' = \chi_1$ (the principal character of G). Hence, for any element $x \in G$ we have $(uu')(x) = u(x)u'(x) = \chi_1(x) = 1$. Since $u(x), u'(x) \in B$ by assumption, $u(x)$ is a unit in B. Hence $u(x)$ is a unit of finite order for any element x in G, because $U(B) = U_f(B)$. Let $\{x_1, \dots, x_g\}$ be all the elements of G, and l be the least common multiple of orders of elements $u(x_i)$ in B $(i = 1, \dots, g)$. Then we have $u^l = \chi_1$. That is, u is a unit of finite order in $R(G)$. Thus the result follows. ■

From now on we assume that p (≥ 5) is a prime number and ω is a primitive p-th root of unity. Let $Z[\omega]$ be the smallest subring of C containing Z and ω.

Let i and j be rational integers such that $1 \leq i, j < p, i \neq j$. Then there is a rational integer k, which is uniquely determined by i and j, such that $ik \equiv j \pmod p$, $1 \leq k < p$. For these rational integers i, j, and k, we define several functions of one variable as follows.

$$g_{ij}(x) = \frac{x^j + 1}{x^i + 1}, \; g_k(x) = \frac{x^k + 1}{x + 1},$$
$$f_k(x) = x^{p+k-1} - x^{p+k-2} + \cdots + (-1)^{m-1} x^{p+k-m} + \cdots + 1, \text{ if } k \text{ is even},$$
$$f_k(x) = x^{k-1} - x^{k-2} + \cdots + (-1)^{m-1} x^{k-m} + \cdots + 1, \text{ if } k \text{is odd},$$
$$f_{ij}(x) = f_k(x^i).$$

Then we have:

THEOREM 2.2. *In the above situation let ϵ be any p-th root of unity. Then we have:*

(i) $\dfrac{\epsilon^j + 1}{\epsilon^i + 1}$ *is a unit in* $Z[\omega]$

(ii) $g_{ij}(\epsilon) = g_k(\epsilon^i) = f_k(\epsilon^i) = f_{ij}(\epsilon)$

Proof. It is obvious that (i) and (ii) hold for $\epsilon = 1$, and so we may assume that

$\epsilon \neq 1$, that is, ϵ is a primitive p-th root of unity, in order to prove (i) and (ii).

(i) $\dfrac{1}{\epsilon+1} = \dfrac{\epsilon-1}{\epsilon^2-1} = \dfrac{\epsilon^{p+1}-1}{\epsilon^2-1} = \dfrac{(\epsilon^2)^{\frac{p+1}{2}}-1}{\epsilon^2-1} = (\epsilon^2)^{\frac{p-1}{2}} + (\epsilon^2)^{\frac{p-3}{2}} + \cdots + \epsilon^2 + 1.$

Hence $\dfrac{1}{\epsilon+1} \in Z[\epsilon]$. Since ϵ is a primitive p-th root of unity, $Z[\epsilon] = Z[\omega]$ holds. Hence $\dfrac{1}{\epsilon+1} \in Z[\omega]$, and so $\epsilon+1$ is a unit in $Z[\omega]$.

Since ϵ^i and ϵ^j $(1 \leq i,j < p)$ are primitive p-th roots of unity, ϵ^i+1 and ϵ^j+1 are units in $Z[\omega]$, and so $\dfrac{\epsilon^j+1}{\epsilon^i+1}$ is a unit in $Z[\omega]$.

(ii) Since $ik \equiv j \pmod p$ holds, $(\epsilon^i)^k = \epsilon^j$ holds. Therefore we have an equation $\dfrac{\epsilon^j+1}{\epsilon^i+1} = \dfrac{(\epsilon^i)^k+1}{\epsilon^i+1}$, and so $g_{ij}(\epsilon) = g_k(\epsilon^i)$ holds.

First we assume that k is even. Then $p+k$ is odd. Since $(\epsilon^i)^k = (\epsilon^i)^{p+k}$, $g_k(\epsilon^i) = \dfrac{(\epsilon^i)^{p+k}+1}{\epsilon^i+1} = (\epsilon^i)^{p+k-1} - (\epsilon^i)^{p+k-2} + \cdots + (-1)^{m-1}(\epsilon^i)^{p+k-m} + \cdots + 1 = f_k(\epsilon^i)$.

Next we assume that k is odd. Then we have

$$g_k(\epsilon^i) = \frac{(\epsilon^i)^k+1}{\epsilon^i+1} = (\epsilon^i)^{k-1} - (\epsilon^i)^{k-2} + \cdots + (-1)^{m-1}(\epsilon^i)^{k-m} + \cdots + 1 = f_k(\epsilon^i)$$

Therefore we have

$$g_{ij}(\epsilon) = g_k(\epsilon^i) = f_k(\epsilon^i) = f_{ij}(\epsilon)$$

Thus the result follows. ■

THEOREM 2.3. *Let $\epsilon = \pm 1$ and let $\eta, \theta,$ and λ be primitive p-th roots of unity. Suppose $\dfrac{\theta+1}{\eta+1} = \epsilon$ or $\epsilon\lambda$ holds. Then we have*

(i) $\epsilon = 1$

(ii) $\eta = \theta$ *or* $\eta\theta = 1$

Proof. We prove (i) and (ii) together. We note that not exceeding 4 $(\leq p-1)$ different p-th roots of unity are linearly independent over Q.

Suppose that $\dfrac{\theta+1}{\eta+1} = \epsilon$ holds. Then we have a formula $1+\theta-\epsilon\eta-\epsilon = 0$. If $\epsilon = -1$, then we get a formula $2 \cdot 1 + \theta + \eta = 0$, which is contrary to the above note, because $1, \theta$, and η are p-th roots of unity. Hence we have $\epsilon = 1$ and $\theta = \eta$.

Next suppose that $\dfrac{\theta+1}{\eta+1} = \epsilon\lambda$ holds. Then we have $1+\theta-\epsilon\lambda\eta-\epsilon\lambda = 0$. If $\epsilon = -1$, then we get a formula $1+\theta+\lambda\eta+\lambda = 0$, which is contrary to the above note, because $1, \theta, \lambda\eta$, and λ are p-th roots of unity. Hence it follows that $\epsilon = 1$ and $1+\theta-\lambda\eta-\lambda = 0$ hold. Now we assume that $\lambda\eta \neq 1$. Then we have $\theta \neq \lambda$ and $\lambda\eta \neq \theta$, because $\lambda \neq 1$. Since $\eta \neq 1$, we have $\lambda\eta \neq \lambda$. Hence $1, \theta, \lambda\eta,$

and λ are 4 different p-th roots of unity, and so are linearly independent over Q by the above note. This contradicts a formula $1+\theta-\lambda\eta-\lambda=0$. Therefore it follows that $\lambda\eta=1$ and $\theta=\lambda$ hold. Hence we have $\eta\theta=1$. This completes the proof of Theorem 2.3. ∎

3. Main theorems

We shall keep the notation of the preceding two sections.

In general, if $f(x)=a_nx^n+\cdots+a_1x+a_0\in Z[x]$ and ψ is a character of a finite group G, then we define a generalized character $f(\psi)$ of G as follows

$$f(\psi)=a_n\psi^n+\cdots+a_1\psi+a_0 1_G$$

where 1_G is the principal character of G.

From our earlier results [Theorems 2.2 and 2.3] we have:

THEOREM 3.1. *Let $\langle a\rangle$ be a cyclic group of order p and let ψ be a linear character of $\langle a\rangle$ such that $\psi(a)=\omega$ where p (≥ 5) is a prime number and ω is a primitive p-th root of unity.*

For rational integers i and j $(1\leq i,j<p, i\neq j)$, we assume that $i+j\neq p$. Then we have

(i) $\pm f_{ij}(\psi)$ *are not linear characters of* $\langle a\rangle$.
(ii) $\pm f_{ij}(\psi)$ *are units of infinite order in* $R(\langle a\rangle)$

where $f_{ij}(x)$ is the same polynomial over Z as the one given before Theorem 2.2.

Proof. (i) We assume that $\pm f_{ij}(\psi)$ are linear characters of $\langle a\rangle$. Since all the linear characters of $\langle a\rangle$ are given by ψ^m $(m=0,1,\cdots,p-1)$, we can write $\pm f_{ij}(\psi)=\psi^m$.

If we consider the values of these generalized characters at $a\in\langle a\rangle$ on both sides , then we have

$$\frac{\omega^j+1}{\omega^i+1}=\omega^m \quad \text{or} \quad \frac{\omega^j+1}{\omega^i+1}=-\omega^m,$$

because $f_{ij}(\psi)(a)=f_{ij}(\psi(a))=f_{ij}(\omega)=g_{ij}(\omega)$ by Theorem 2.2 (ii) where $g_{ij}(x)$ is the same function as the one given before Theorem 2.2.

By Theorem 2.3 (i), the second equation doesn't hold. Therefore we have an equation $\dfrac{\omega^j+1}{\omega^i+1}=\omega^m$. By Theorem 2.3 (ii) we have $\omega^i=\omega^j$ or $\omega^i\omega^j=\omega^{i+j}=1$. Hence $i=j$ or $i+j=p$, because $1\leq i,j<p$. These facts contradict our assumption that $i\neq j$ and $i+j\neq p$. Therefore $\pm f_{ij}(\psi)$ are not linear characters of $\langle a\rangle$. This completes the proof of (i).

(ii) For any element $a^l \in \langle a\rangle$ $(0 \leq l < p)$, by Theorem 2.2 (ii) we have

$$f_{ij}(\psi)(a^l) = f_{ij}(\psi(a^l)) = f_{ij}(\omega^l) = g_{ij}(\omega^l) = \frac{(\omega^l)^j + 1}{(\omega^l)^i + 1},$$

because $\psi(a) = \omega$ and ω^l is a p-th root of unity. Similarly we get $f_{ji}(\psi)(a^l) = \dfrac{(\omega^l)^i + 1}{(\omega^l)^j + 1}$ $(0 \leq l < p)$.

Therefore it follows that $(f_{ij}(\psi)f_{ji}(\psi))(x) = (f_{ij}(\psi)(x))(f_{ji}(\psi)(x)) = 1$ for any element $x \in \langle a\rangle$. This means $f_{ij}(\psi)f_{ji}(\psi) = 1_{\langle a\rangle}$ (the principal character of $\langle a\rangle$). Since $f_{ij}(\psi), f_{ji}(\psi) \in R(\langle a\rangle)$, $\pm f_{ij}(\psi)$ are units in $R(\langle a\rangle)$. By Corollary 1.2 and the statement (i) in this theorem, it follows that $\pm f_{ij}(\psi)$ are of infinite order. This completes the proof of (ii). ■

EXAMPLE 1. Let $\langle a\rangle$ be a cyclic group of order 5 and let ψ be a linear character given by $\psi(a) = \omega$ where ω is a primitive 5-th root of unity.

Since $1 \cdot 3 \equiv 3 \pmod 5$, for $i = 1$ and $j = 3$, we have $k = 3$. Hence we get $f_{13}(\psi) = f_3(\psi) = \psi^2 - \psi + 1_{\langle a\rangle}$ where $1_{\langle a\rangle}$ is the principal character of $\langle a\rangle$.

Since $3 \cdot 2 \equiv 1 \pmod 5$, for $i = 3$ and $j = 1$ we have $k = 2$ and $p + k = 7$. Hence we get

$$\begin{aligned} f_{31}(\psi) = f_2(\psi^3) &= (\psi^3)^6 - (\psi^3)^5 + (\psi^3)^4 - (\psi^3)^3 + (\psi^3)^2 - \psi^3 + 1_{\langle a\rangle} \\ &= -\psi^4 + \psi^2 + \psi, \end{aligned}$$

because $\psi^5 = 1_{\langle a\rangle}$ holds. Actually we can see easily that $f_{13}(\psi)f_{31}(\psi) = 1_{\langle a\rangle}$ holds, because $\psi^5 = 1_{\langle a\rangle}$ holds.

Thus $\pm(\psi^2 - \psi + 1_{\langle a\rangle})$ and $\pm(-\psi^4 + \psi^2 + \psi)$ are units of infinite order in $R(\langle a\rangle)$.

Hereafter when we consider $f_{ij}(\psi)$ where i and j are rational integers such that $1 \leq i, j < p, i \neq j, i + j \neq p$, and p (≥ 5) is a prime number, we assume that $f_{ij}(x)$ is the same polynomial over Z as the one given before Theorem 2.2.

Now we state some results about "tensor induction", which will be needed later.

Let H be a subgroup of G and choose a set T of representatives for the right cosets of H in G. Since G acts on the set of right cosets of H by $(Ht)g = Htg$ $(t \in T, g \in G)$. We write $t \cdot g \in T$ to denote the representative of the coset Htg, so that $(tg)(t \cdot g)^{-1} \in H$. Thus $\cdot$ defines an action of G on T.

Fix $g \in G$ and let $n(t)$ denote the size of the $\langle g\rangle$-orbit on T containing t. Then, by the same calculation as is used when developing the transfer map, we have $tg^{n(t)}t^{-1} \in H$ for $t \in T$. Let $T_\circ$ be a set of representatives for the $\langle g\rangle$-orbits on T.

Let φ be a class function of H. Then we define $\varphi^{\otimes G}$ on G, which is called a tensor induction of φ to G, by the formula

$$\varphi^{\otimes G}(g) = \prod_{t \in T_\circ} \varphi(tg^{n(t)}t^{-1}) \text{ for } g \in G \tag{$*$}$$

(see Definition 2.1 in [3]).

Expositions of tensor induction, including the details of the construction can be found in [2], [3], [5], and [6].

If φ is a generalized character of H, then $\varphi^{\otimes G}$ is also a generalized character of G by Theorem A in [3].

Let u be a unit in $R(H)$. Then we prove that $u^{\otimes G}$ is also a unit in $R(G)$. In fact, since u is a unit in $R(H)$, there is an element v in $R(H)$ such that $uv = 1_H$ (the principal character of H). By Theorem A in [3], $u^{\otimes G}$ and $v^{\otimes G}$ are generalized characters of G. For any element g in G, by a formula $(*)$ we have

$$\begin{aligned}(u^{\otimes G}v^{\otimes G})(g) &= (u^{\otimes G}(g))(v^{\otimes G}(g)) \\ &= \prod_{t \in T_\circ} u(tg^{n(t)}t^{-1}) \prod_{t \in T_\circ} v(tg^{n(t)}t^{-1}) \\ &= \prod_{t \in T_\circ} (uv)(tg^{n(t)}t^{-1}) = \prod_{t \in T_\circ} (1_H)(tg^{n(t)}t^{-1}) = 1,\end{aligned}$$

because $uv = 1_H$ and $tg^{n(t)}t^{-1} \in H$ for $t \in T_\circ$.

Hence $u^{\otimes G}v^{\otimes G}$ is the principal character of G. Therefore $u^{\otimes G}$ is a unit in $R(G)$.

By Theorem 3.1 we have at once:

THEOREM 3.2. *Let H be a cyclic subgroup with a generator a of order p in a finite group G where p (≥ 5) is a prime number. Let ψ be a linear character of H such that $\psi(a) = \omega$ where ω is a primitive p-th root of unity. Then $\pm f_{ij}(\psi)^{\otimes G}$ $(1 \leq i, j < p, i \neq j, i+j \neq p)$ are units in $R(G)$.*

Remark. In Theorem 3.2, $\pm f_{ij}(\psi)^{\otimes G}$ $(1 \leq i, j < p, i \neq j, i+j \neq p)$ are not always units of infinite order. We can give a counterexample. Let $G = S_n$ $(n \geq 5)$ be the symmetric group on n symbols and let χ be any character of G. Then $\chi(g) \in Z$ for any element $g \in G$. Hence any unit of $R(G)$ is of finite order by Lemma 2.1. Let H be a cyclic subgroup of G with a generator σ (a cyclic permutation of length p, $5 \leq p \leq n$, and p is a prime number). Then $H = \langle\sigma\rangle$ is a cyclic group of order p (≥ 5). Let ψ be a linear character of H such that $\psi(\sigma) = \omega$ where ω is a primitive p-th root of unity. Then by Theorem 3.1, $\pm f_{ij}(\psi)$ $(1 \leq i, j < p, i \neq j, i+j \neq p)$ are units of infinite order in $R(\langle\sigma\rangle)$, but $\pm f_{ij}(\psi)^{\otimes G}$ are units of finite order in $R(G)$. (See Example 2.)

Let G' be the commutator subgroup of G and $p \mid |G/G'|$ where p (≥ 5) is a prime number. Since G/G' is an abelian group, there is a normal subgroup H of G containing G' such that $G/H = \langle aH \rangle$ is a cyclic group of order p $(a \in G)$. Let ψ be a linear character of $\langle aH \rangle$ such that $\psi(aH) = \omega$ where ω is a primitive p-th root of unity. Then ψ can be viewed as a character of G.

In the above situation, by Theorem 3.1 we have:

THEOREM 3.3. *If $|G/G'|$ has a divisor p (≥ 5), then $\pm f_{ij}(\psi)$ $(1 \leq i, j < p, i \neq j, i+j \neq p)$ are units of infinite order in $R(G)$.*

THEOREM 3.4. *If G/G' is non-trivial (that is, $G \neq G'$) and $R(G)$ has no unit of infinite order for a finite group G where G' is the commutator subgroup of G, then G/G' is a $\{2,3\}$-group.*

Proof. Suppose that G/G' is not a $\{2,3\}$-group. Then $p \mid |G/G'|$ for some prime number p (≥ 5) because G/G' is non-trivial. By Theorem 3.3 $R(G)$ has units of infinite order. This contradicts that $R(G)$ has no unit of infinite order. Hence G/G' is a $\{2,3\}$-group. ■

EXAMPLE 2. Let $G = S_n$ $(n \geq 5)$ be a symmetric group on n symbols. Then $G' = A_n$ is an alternating group on n symbols and $|G/G'| = 2$ where G' is the commutator subgroup of G. Therefore S_n has two linear characters χ_1, χ_2 where χ_1 is the principal character of S_n and χ_2 is a linear character of S_n such that $\chi_2(\sigma) = 1$ if σ is an even permutation and $\chi_2(\sigma) = -1$ if σ is an odd permutation.

Since the values of characters of S_n are rational integers, by Lemma 2.1 the units in $R(S_n)$ are of finite order and so we have $U(R(S_n)) = U_f(R(S_n)) = \{\pm\chi_1, \pm\chi_2\}$ by Corollary 1.2.

EXAMPLE 3. Let $G = \langle a \rangle$ be a cyclic group of order m $(m = 3, 4, 6)$. Then $|G/G'| = m$, that is, G/G' is a $\{2,3\}$-group where G' is the commutator subgroup of G. Let A and B be the ring of algebraic integers in $Q(i)$ and the ring of algebraic integers in $Q(\sqrt{-3})$, respectively where $i = \sqrt{-1}$. Then $U(A) = U_f(A) = \{\pm 1, \pm i\}$ and $U(B) = U_f(B) = \{\pm 1, \pm\rho, \pm\rho^2\}$ hold where $\rho = \dfrac{-1+\sqrt{-3}}{2}$. Let ψ be a linear character of $\langle a \rangle$ such that $\psi(a) = \omega$ where ω is a primitive m-th root of unity. Since $\omega \in A$ or $\omega \in B$, by Lemma 2.1 and Corollary 1.2 we have $U(R(\langle a \rangle)) = U_f(R(\langle a \rangle)) = \{\pm\psi^i \mid i = 0, 1, \cdots, m-1\}$.

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