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HOW TO FIND STABILITY IN A PURELY SEMISTABLE CONTEXT

By

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Dedicated to our teacher Professor Wilfried Hazod on the occasion of his 60th birthday

Abstract. By definition any stable distribution is semistable. For the converse relation we will show that certain logarithmic mixtures of semistable laws belong to the domain of normal attraction of a stable law. The mixtures themselves appear as limits for normalized sums of certain random numbers of random variables belonging to the domain of normal attraction of a semistable law. Combining the corresponding limit theorems, we observe a stable limit when starting in the domain of attraction of a semistable distribution. The results are given in a multivariate setting with operator normings and extend to the corresponding semi-selfsimilar respectively selfsimilar Lévy processes.

1. Introduction

This work is motivated by the following three observations concerning self-similar and semi-selfsimilar processes. Partly deviating from the original statements, we rather prefer to give a multivariate formulation with operator instead of scalar normings.

Let $\{X_t\}_{t\geq 0}$ be a stochastic process on $\mathbb{R}^d$, further let $c>1$ and Q be a linear operator on $\mathbb{R}^d$. We say that $\{X_t\}_{t\geq 0}$ is (c^Q,c)-*semi-selfsimilar* if for some drift-function $d_c:[0,\infty)\to\mathbb{R}^d$ the process obeys the space-time scaling

$$\{c^Q X_t + d_c(t)\}_{t\geq 0} \stackrel{f.d.}{=} \{X_{ct}\}_{t\geq 0}, \tag{1.1}$$

where $c^Q = e^{Q\log c} = \sum_{k=0}^{\infty}(k!)^{-1}(\log c)^k Q^k$ and $\stackrel{f.d.}{=}$ denotes equality of all finite dimensional marginal distributions. In case $d_c\equiv 0$ we call $\{X_t\}_{t\geq 0}$ *strictly* (c^Q,c)-semi-selfsimilar. Moreover, if (1.1) even holds for every $c>0$ we say that $\{X_t\}_{t\geq 0}$ is (strictly) *operator-selfsimilar* with *exponent* Q. For further details on (semi)-selfsimilar processes and existence of exponents we refer to [9], [13], [14], [15], and [18].

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We call the process $\{Y_t\}_{t\in\mathbb{R}}$ defined by $Y_t = e^{-tQ}X_{e^t}$, as introduced by Lamperti [11], the *Lamperti transform* of $\{X_t\}_{t\geq 0}$.

1.) As shown by Maejima and Sato [13], Theorem 13, the Lamperti transform of the (c^Q, c)-semi-selfsimilar process $\{X_t\}_{t\geq 0}$ is *periodically stationary* with *period* $\log c > 0$, i.e.

$$\{Y_{t+\log c}\}_{t\in\mathbb{R}} \stackrel{f.d.}{=} \{Y_t\}_{t\in\mathbb{R}}.$$

2.) Hurd [10], Theorem 1, provides a method to stationarize the periodically stationary process $\{Y_t\}_{t\in\mathbb{R}}$ by random time-shifting, as follows. Let θ be a random variable which is independent of $\{Y_t\}_{t\in\mathbb{R}}$ and uniformly distributed on the period interval $[0, \log c]$. Further define $Z_t(\omega) = Y_{t+\theta(\omega)}(\omega)$, which in case of Borel measurability defines a *stationary* process $\{Z_t\}_{t\in\mathbb{R}}$ in the sense that

$$\{Z_{t+s}\}_{t\in\mathbb{R}} \stackrel{f.d.}{=} \{Z_t\}_{t\in\mathbb{R}} \quad \text{for any } s > 0.$$

Hurd assumes joint measurability in t and ω to ensure that $\{Z_t\}_{t\in\mathbb{R}}$ is a well-defined stochastic process. For $t_1 < \cdots < t_m$ the finite dimensional marginal distributions are then given by

$$P_{(Z_{t_i}:1\leq i\leq m)} = \frac{1}{\log c}\int_0^{\log c} P_{(Y_{t_i+s}:1\leq i\leq m)}\,ds.$$

3.) Inverting the Lamperti transform, the stationary process $\{Z_t\}_{t\in\mathbb{R}}$ turns into an operator-selfsimilar process $\{U_t\}_{t\geq 0}$ by $U_0 = 0$ and $U_t = t^Q Z_{\log t}$ for $t > 0$, as has already been observed by Lamperti [11]; see also [5]. It is even possible to arbitrarily change the exponent in this step, but this will be of no further interest for us.

Combining these three results, presupposed we have measurability in the second step, the strictly (c^Q, c)-semi-selfsimilar process $\{X_t\}_{t\geq 0}$ can be transformed into a strictly operator-selfsimilar process $\{U_t\}_{t\geq 0}$ with exponent Q. The finite dimensional marginal distributions of the operator-selfsimilar process $\{U_t\}_{t\geq 0}$ are then given by

$$\begin{aligned}
P_{(U_{t_i}:1\leq i\leq m)} &= P_{(t_i^Q Z_{\log t_i}:1\leq i\leq m)} \\
&= \frac{1}{\log c}\int_0^{\log c} P_{(t_i^Q Y_{s+\log t_i}:1\leq i\leq m)}\,ds \\
&= \frac{1}{\log c}\int_0^{\log c} P_{(e^{-sQ} X_{e^s t_i}:1\leq i\leq m)}\,ds \\
&= \frac{1}{\log c}\int_1^{c} P_{(r^{-Q} X_{r t_i}:1\leq i\leq m)}\,\frac{dr}{r}.
\end{aligned}$$

Equivalently, for some random variable Θ which is independent of $\{X_t\}_{t\geq 0}$ and logarithmically distributed with probability density $x \mapsto (x \log c)^{-1} 1_{[1,c]}(x)$ we can write

$$U_t(\omega) = \Theta(\omega)^{-Q} X_{\Theta(\omega)t}(\omega) \quad \text{for any } t > 0 \tag{1.2}$$

such that $\{U_t\}_{t\geq 0} \stackrel{f.d.}{=} \{\Theta^{-Q} X_{\Theta t}\}_{t\geq 0}$. Thus taking logarithmic mixtures as in (1.2) equals out differences in scaling that might occur between two successive integer powers of c according to (1.1) such that the weaker semi-selfsimilarity turns into the stronger selfsimilarity property. It is easy to see that if the semi-selfsimilar process $\{X_t\}_{t\geq 0}$ has stationary increments, then so has the resulting selfsimilar process $\{U_t\}_{t\geq 0}$. Unfortunately, this is in general not true for independent increments, which will be our focus now. The aim of the paper is to show that independence of the increments is preserved by considering an approprite limit theorem.

From now on we will assume that the strictly (c^Q, c)-semi-selfsimilar process $\{X_t\}_{t\geq 0}$ has stationary and independent increments and is *continuous in law*, meaning that $t \mapsto P_{X_t}$ is continuous with respect to weak topology. Hence $\{X_t\}_{t\geq 0}$ is a Lévy process and thus there exists a version with càdlàg paths (continuous from the right with left limits) that guarantees measurability in the second step. We will prove in section 2 that an appropriately normalized and centered partial sum process of independent copies of $\{U_t\}_{t\geq 0}$ defined by (1.2) converges to an operator-selfsimilar Lévy process with exponent Q in the sense of convergence of all finite dimensional marginal distributions. Since $\{U_t\}_{t\geq 0}$ itself appears as a limit process of a normalized partial sum process with random sample size, we will further combine these two limiting procedures in section 3. As a consequence for the one-dimensional marginals of the processes, we observe an operator-stable limit when starting in the domain of normal attraction of an arbitrary strictly operator-semistable law.

2. Stabilizing Semistability

It is well known that the Fourier transform of an infinitely divisible probability measure ρ on $\mathbb{R}^d$ has the form $\exp(l(y))$ for $y \in \mathbb{R}^d$ with

$$l(y) = i\langle a, y\rangle - \tfrac{1}{2}\, q(y) + \int_{\mathbb{R}^d\setminus\{0\}} \Big(e^{i\langle y,x\rangle} - 1 - \frac{i\langle y, x\rangle}{1+\|x\|^2}\Big)\, d\phi(x),$$

where $a \in \mathbb{R}^d$, q is a nonnegative definite quadratic form on $\mathbb{R}^d$, $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product on $\mathbb{R}^d$ and ϕ is a σ-finite Borel measure on $\mathbb{R}^d \setminus \{0\}$

called the *Lévy measure* such that

$$\int_{\mathbb{R}^d\setminus\{0\}} \min\left(1, \|x\|^2\right) d\phi(x) < \infty.$$

The unique triple $[a, q, \phi]$ is called the *Lévy representation* and for short we will write $\rho \sim [a, q, \phi]$; see, e.g., Theorem 3.1.11 in [16].

Let $c > 1$ and Q be a linear operator on $\mathbb{R}^d$. An infinitely divisible probability measure ν is called *operator-semistable* or more precisely (c^Q, c)*-semistable* if

$$c^Q \nu = \nu^c * \varepsilon_{-d} \tag{2.1}$$

for some $d \in \mathbb{R}^d$, where ν^c denotes the c-fold convolution power of ν and ε_x denotes Dirac measure in $x \in \mathbb{R}^d$. In case $d = 0$ we call ν *strictly* (c^Q, c)-semistable. A probability measure η on $\mathbb{R}^d$ is said to belong to the *domain of normal attraction* of ν if

$$c^{-nQ} \eta^{\lfloor c^n \rfloor} * \varepsilon_{-d_n} \to \nu \quad \text{weakly as } n \to \infty \tag{2.2}$$

for some $d_n \in \mathbb{R}^d$. In case $d_n = 0$ for all $n \in \mathbb{N}$ we say that η belongs to the *strict* domain of normal attraction of ν.
An infinitely divisible probability measure ρ is called *operator-stable* with *exponent* Q if

$$t^Q \rho = \rho^t * \varepsilon_{-d(t)} \quad \text{for all } t > 0 \tag{2.3}$$

and some drift-function $d : (0, \infty) \to \mathbb{R}^d$. In case $d \equiv 0$ we call ρ *strictly* operator-stable. A probability measure μ on $\mathbb{R}^d$ is said to belong to the *domain of normal attraction* of ρ if

$$n^{-Q} \mu^n * \varepsilon_{-a_n} \to \rho \quad \text{weakly as } n \to \infty \tag{2.4}$$

for some $a_n \in \mathbb{R}^d$. In case $a_n = 0$ for all $n \in \mathbb{N}$ we say that μ belongs to the *strict* domain of normal attraction of ρ. For further details on operator-(semi-)stable distributions and their domains of attraction we refer to [16] and the literature cited therein.

We start with recalling the structure of a strictly (c^Q, c)-semi-selfsimilar Lévy process. Let $\{X_t\}_{t\geq 0}$ be strictly (c^Q, c)-semi-selfsimilar as in (1.1) (with $d_c \equiv 0$) for some $c > 1$ and some linear operator Q on $\mathbb{R}^d$, and let $\{X_t\}_{t\geq 0}$ have stationary and independent increments. Further assume that $\{X_t\}_{t\geq 0}$ is *proper* (meaning that for $t > 0$ the distribution of X_t is *full*, i.e. not concentrated on any lower dimensional hyperplane of $\mathbb{R}^d$) and continuous in law. Then, following the arguments given by Hudson and Mason [9], Theorem 7, (1.1) with $d_c \equiv 0$ is equivalent to the fact that $\nu = P_{X_1}$ is strictly (c^Q, c)-semistable. Especially ν is full and necessarily any eigenvalue of Q belongs to the halfplane $\{\mathrm{Re}(z) \geq 1/2\}$.

Now let ν be a full strictly (c^Q, c)-semistable law for some $c > 1$ and some linear operator $Q \in \mathrm{GL}(\mathbb{R}^d)$ and assume that ν is not operator-stable in which case we call ν *purely* semistable. Since the (multivariate) normal distribution is operator-stable, due to Theorem 7.1.10 of [16] we may and will assume throughout this section, that ν has no Gaussian component and hence has Lévy representation $\nu \sim [b, 0, \phi]$ for some $b \in \mathbb{R}^d$ and some Lévy measure ϕ and further the real part of any eigenvalue of the exponent Q exceeds $1/2$. Then (2.1) implies $c^Q\phi = c \cdot \phi$; see Lemma 7.1.6 of [16]. Note that $c \cdot \phi$ denotes multiplication, whereas $c^Q\phi$ denotes the image measure. Fix $0 = t_0 < t_1 < \cdots < t_m$ and consider

$$\xi_m = \big(X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_m} - X_{t_{m-1}}\big).$$

Since $\{X_t\}_{t\geq 0}$ has stationary and independent increments and the Lévy measure of $P_{X_t} = \nu^t$ is given by $t \cdot \phi$, the $(\mathbb{R}^d)^m$-valued random vector ξ_m has a full strictly (c^{Q_m}, c)-semistable distribution on $(\mathbb{R}^d)^m$ with exponent $Q_m = \mathrm{diag}(Q, \ldots, Q)$ and Lévy measure

$$\widetilde{\Phi}_{t_1,\ldots,t_m} = \sum_{i=1}^{m} (t_i - t_{i-1}) \cdot \phi_i,$$

where $\phi_i = \varepsilon_0 \otimes \cdots \varepsilon_0 \otimes \phi \otimes \varepsilon_0 \otimes \cdots \otimes \varepsilon_0$ is the product measure with ϕ in the i-th component and Dirac measure ε_0 (at the origin $0 \in \mathbb{R}^d$) in all other components. Especially we have

$$c^{Q_m} \widetilde{\Phi}_{t_1,\ldots,t_m} = c \cdot \widetilde{\Phi}_{t_1,\ldots,t_m}. \tag{2.5}$$

Further let $T_m : (\mathbb{R}^d)^m \to (\mathbb{R}^d)^m$ be defined by

$$T_m(x_1, \ldots, x_m) = (x_1, x_1 + x_2, \ldots, x_1 + \cdots + x_m).$$

Then T_m is an isomorphism of $(\mathbb{R}^d)^m$ and $T_m \circ t^{Q_m} = t^{Q_m} \circ T_m$ for all $t > 0$. Hence the finite dimensional marginals of $\{X_t\}_{t\geq 0}$ can be written as

$$\big(X_{t_1}, X_{t_2}, \ldots, X_{t_m}\big) = T_m(\xi_m),$$

which has a full strictly (c^{Q_m}, c)-semistable distribution with Lévy measure

$$\Phi_{t_1,\ldots,t_m} = T_m(\widetilde{\Phi}_{t_1,\ldots,t_m}) = \sum_{i=1}^{m} (t_i - t_{i-1}) \cdot T_m(\phi_i). \tag{2.6}$$

Note that for any $s > 0$ we have

$$\Phi_{s\cdot t_1,\dots,s\cdot t_m} = s \cdot \Phi_{t_1,\dots,t_m}. \tag{2.7}$$

Now let $\{X_t^{(n)}\}_{t\geq 0}$, $n \in \mathbb{N}$, be independent copies of $\{X_t\}_{t\geq 0}$. Then it follows from above, that whenever $0 < t_1^{(n)} < \cdots < t_m^{(n)}$ with $t_j^{(n)} \to t_j$ as $n \to \infty$ we have

$$c^{-nQ_m} \sum_{i=1}^{\lfloor c^n \rfloor} \left(X^{(i)}_{t_j^{(n)}} : 1 \leq j \leq m\right) \stackrel{d}{=} c^{-nQ_m} \left(X_{\lfloor c^n \rfloor t_j^{(n)}} : 1 \leq j \leq m\right)$$
$$\Rightarrow \left(X_{t_j} : 1 \leq j \leq m\right),$$

where $\Rightarrow$ denotes convergence in distribution. Hence by Corollary 8.2.11 of [16] we have

$$c^n \cdot P_{(c^{-nQ} X_{t_j} : 1 \leq j \leq m)} \to \Phi_{t_1,\dots,t_m} \tag{2.8}$$

uniformly on compact subsets of $\{0 < t_1 < \cdots < t_m\}$, where convergence to a Lévy measure ϕ is understood to hold for any ϕ-continuity set bounded away from the origin.

Further let Θ_n, $n \in \mathbb{N}$, be i.i.d. as Θ, logarithmically distributed with probability density $x \mapsto (x \log c)^{-1} 1_{[1,c]}(x)$ on $\mathbb{R}$, and assume that $(\{X_t^{(n)}\}_{t\geq 0}, \Theta_n : n \in \mathbb{N})$ are independent.

THEOREM 2.1. *With the above assumptions and notations there exists a sequence of functions $a_n : [0, \infty) \to \mathbb{R}^d$ with $a_n(0) = 0$ such that*

$$\left\{n^{-Q} \sum_{i=1}^{n} \Theta_i^{-Q} X^{(i)}_{\Theta_i t} - a_n(t)\right\}_{t\geq 0} \stackrel{f.d.}{\Longrightarrow} \{R_t\}_{t\geq 0}, \tag{2.9}$$

where $\stackrel{f.d.}{\Longrightarrow}$ denotes convergence of all finite dimensional marginal distributions and $\{R_t\}_{t\geq 0}$ is a Lévy process generated by a full operator-stable distribution $P_{R_1} = \rho$ with exponent Q and Lévy representation $\rho \sim [a, 0, \psi]$ for some $a \in \mathbb{R}^d$ and Lévy measure

$$\psi = \frac{1}{\log c} \int_1^c s^{-Q} \phi \, ds. \tag{2.10}$$

Proof. Fix any $0 = t_0 < t_1 < \cdots < t_m$ and note that, since ρ is full by Proposition 3.1.20 of [16], $(R_{t_1}, \dots, R_{t_m})$ is full on $(\mathbb{R}^d)^m$ and has Lévy measure

$$\Psi_{t_1,\dots,t_m} = \sum_{i=1}^{m} (t_i - t_{i-1}) \cdot T_m(\psi_i), \tag{2.11}$$

where $\psi_i = \varepsilon_0 \otimes \cdots \otimes \varepsilon_0 \otimes \psi \otimes \varepsilon_0 \otimes \cdots \otimes \varepsilon_0$ as above.
Now write $n = c^{m_n} r_n$ with $m_n \in \mathbb{N}_0$ and $r_n \in [1, c)$ and let $r \in [1, c]$ be an arbitrary limit point of (r_n) along some subsequence (n'). Using (2.8) and (2.7) we obtain along the subsequence (n')

$$
\begin{aligned}
& n \cdot \left(n^{-Q_m} P_{(\Theta^{-Q} X_{\Theta t_j} \,:\, 1 \le j \le m)}\right) \\
&= \frac{1}{\log c} \int_1^c c^{m_n} r_n \cdot P_{(c^{-m_n Q}(r_n s)^{-Q} X_{s \cdot t_j} \,:\, 1 \le j \le m)} \frac{ds}{s} \\
&= \frac{1}{\log c} \int_1^c c^{m_n} r_n \cdot \left((r_n s)^{-Q_m} P_{(c^{-m_n Q} X_{s \cdot t_j} \,:\, 1 \le j \le m)}\right) \frac{ds}{s} \\
&\to \frac{1}{\log c} \int_1^c r \cdot \left((rs)^{-Q_m} \Phi_{s \cdot t_1, \dots, s \cdot t_m}\right) \frac{ds}{s} \\
&= \frac{1}{\log c} \int_1^c (rs) \cdot \left((rs)^{-Q_m} \Phi_{t_1, \dots, t_m}\right) \frac{ds}{s} \\
&= \frac{1}{\log c} \int_r^{rc} s^{-Q_m} \Phi_{t_1, \dots, t_m} \, ds \\
&= \frac{1}{\log c} \int_r^{c} s^{-Q_m} \Phi_{t_1, \dots, t_m} \, ds + \frac{1}{\log c} \int_1^{r} c \cdot \left((cs)^{-Q_m} \Phi_{t_1, \dots, t_m}\right) ds \\
&= \frac{1}{\log c} \int_1^{c} s^{-Q_m} \Phi_{t_1, \dots, t_m} \, ds,
\end{aligned}
$$

where the last identity holds in view of $c \cdot (c^{-Q_m} \Phi_{t_1, \dots, t_m}) = \Phi_{t_1, \dots, t_m}$, which easily follows from (2.5) and (2.6). Moreover, the limit does not depend on r and hence we have convergence as $n \to \infty$ to the Lévy measure

$$
\begin{aligned}
\frac{1}{\log c} \int_1^c s^{-Q_m} \Phi_{t_1, \dots, t_m} \, ds &= \sum_{i=1}^m (t_i - t_{i-1}) \cdot \frac{1}{\log c} \int_1^c s^{-Q_m} T_m(\phi_i) \, ds \\
&= \sum_{i=1}^m (t_i - t_{i-1}) \cdot T_m\Big(\frac{1}{\log c} \int_1^c s^{-Q_m} \phi_i \, ds\Big) \\
&= \sum_{i=1}^m (t_i - t_{i-1}) \cdot T_m(\psi_i) = \Psi_{t_1, \dots, t_m}.
\end{aligned}
$$

Hence by Corollary 8.2.11 of [16] there exist $a_n^{(m)}(t_1, \dots, t_m) \in (\mathbb{R}^d)^m$ such that

(2.12)
$$
n^{-Q_m} \sum_{i=1}^n \left(\Theta_i^{-Q} X^{(i)}_{\Theta_i t_j} \,:\, 1 \le j \le m\right) - a_n^{(m)}(t_1, \dots, t_m) \Rightarrow (R_{t_1}, \dots, R_{t_m}).
$$

For $m = 1$ write $a_n^{(1)}(t) = a_n(t)$, then projecting (2.12) onto the j-th $\mathbb{R}^d$-component via $\pi_j : (\mathbb{R}^d)^m \to \mathbb{R}^d$, $\pi_j(x_1, \dots, x_m) = x_j$ shows that we can choose

$\pi_j\big(a_n^{(m)}(t_1,\dots,t_m)\big)$ as $a_n(t_j)$ so that $a_n^{(m)}(t_1,\dots,t_m) = \big(a_n(t_1),\dots,a_n(t_m)\big)$ completing the proof. □

Remark 2.2. The limiting process $\{R_t\}_{t\geq 0}$ in Theorem 2.1 is operator-selfsimilar with exponent Q and has stationary and independent increments. It is frequently called an *operator Lévy motion* and defines an *operator-stable process* with exponent Q in the sense of Maejima [12], i.e. for any $0 < t_1 < \cdots < t_m$ the distribution of the random vector $(R_{t_1},\dots,R_{t_m})$ is operator-stable in $(\mathbb{R}^d)^m$ with exponent $Q_m = \operatorname{diag}(Q,\dots,Q)$.

Remark 2.3. Let $\mathbb{R}^d = W_1 \oplus W_2 \oplus W_3$ be a direct sum decomposition of $\mathbb{R}^d$ into Q-invariant subspaces (possibly empty) such that the real part of any eigenvalue of the exponent Q is less than 1 on W_1, is equal to 1 on W_2 and exceeds 1 on W_3. Then by Corollary 8.2.15 of [16] the expectation $\mathbb{E}\langle\Theta^{-Q}X_{\Theta t}, w_1\rangle$ exists for all $w_1 \in W_1$ so that we can center to zero expectation on W_1. Further $\mathbb{E}|\langle\Theta^{-Q}X_{\Theta t}, w_3\rangle| = \infty$ for all $w_3 \in W_3$ and by Theorem 8.2.16 in [16] no centering is required on W_3. If we further assume that $\langle X_1, w_2\rangle$ is symmetric for all $w_2 \in W_2$, meaning that $P_{\langle X_1,w_2\rangle} = P_{\langle -X_1,w_2\rangle}$, we can choose $a_n(t) = 0$ for all $n \in \mathbb{N}$ and $t \geq 0$ in (2.9), which remains true for the stronger symmetry condition $P_{X_1} = P_{-X_1}$ in which case $a = 0$ in the Lévy representation of $\rho = P_{R_1} \sim [0,0,\psi]$.

Especially for the one-dimensional marginal distribution μ of U_1 given in (1.2), by Theorem 2.1 we obtain:

COROLLARY 2.4. *Let $\nu = P_{X_1}$ be as above. Then the distribution*

$$\mu = \frac{1}{\log c}\int_1^c r^{-Q}\nu^r\,\frac{dr}{r} \tag{2.13}$$

belongs to the domain of normal attraction of a full operator-stable law ρ with exponent Q and Lévy representation $\rho \sim [a,0,\psi]$ for some $a \in \mathbb{R}^d$ and Lévy measure ψ given by (2.10).

Remark 2.5. Corollary 2.4 shows that transforming the purely (c^Q,c)-semistable law ν into the logarithmic mixture μ in (2.13) is smoothing the tails in the sense that the R–O varying measure $\nu \in \mathrm{ROV}_\infty(Q,c)$ is transformed into a regularly varying measure $\mu \in \mathrm{RVM}_\infty(Q)$; see chapter 6 of [16] for notation and details.

3. Transitivity

A special logarithmic mixture (2.13) first appeared as an almost sure limit in [3] and in its general one-dimensional form has been obtained the same way in [4]. Furthermore, the logarithmic mixture μ in (2.13) also appears as a limit distribution for normalized sums of certain random numbers of i.i.d. random vectors belonging to the strict domain of normal attraction of the semistable law ν. Numerous choices of sequences of random numbers Θ_n converging in some sense to a logarithmically distributed Θ are possible to establish this result (see Remark 3.2 below). We will concentrate on the simple choice $\Theta_n = \lfloor n\Theta \rfloor$ that easily allows to combine this limit theorem with Theorem 2.1 in the sense of transitivity of Gnedenko; see [7] or, more generally, Theorem 9 in [6] for the semistable situation. Finally, we will observe a limit theorem with a stable limit when starting in the domain of normal attraction of a purely semistable law. Note that this is no contradiction, since we consider normalized sums of a random number of random variables. Again, we will give more general versions of the above mentioned results in the sense of convergence of all finite dimensional marginal distributions of the corresponding processes.

Now let $Y_1, Y_2, \ldots$ be i.i.d. random vectors on $\mathbb{R}^d$ with distribution $\eta = P_{Y_1}$ belonging to the strict domain of normal attraction of the strictly (c^Q, c)-semistable law ν and write

$$\{S_t = \sum_{k=1}^{\lfloor t \rfloor} Y_k\}_{t\geq 0}$$

for the corresponding partial sum process. Further let Θ be independent of $(Y_n)_{n\in\mathbb{N}}$ with logarithmic distribution as above.

LEMMA 3.1. *With the above assumptions and notations we have as $n \to \infty$*

$$\{(n\Theta)^{-Q} S_{n\Theta t}\}_{t\geq 0} \overset{f.d.}{\Longrightarrow} \{\Theta^{-Q} X_{\Theta t}\}_{t\geq 0} = \{U_t\}_{t\geq 0}.$$

Proof. Fix $0 < t_1 < \ldots < t_m$. Since by (2.2) we have $c^{-nQ} S_{c^n t} \Rightarrow X_t$ uniformly on compact subsets of $\{t > 0\}$, it follows by a standard argument considering independent increments that

$$(c^{-nQ} S_{c^n t_j} \ : 1 \leq j \leq m) \Rightarrow (X_{t_j} \ : 1 \leq j \leq m) \tag{3.1}$$

uniformly on compact subsets of $\{0 < t_1 < \ldots < t_m\}$. Now write $n = c^{m_n} r_n$ with $m_n \in \mathbb{N}_0$ and $r_n \in [1, c)$ and let $r \in [1, c]$ be an arbitrary limit point of (r_n) along some subsequence (n'). Using (3.1) and (1.1) we obtain along the subsequence (n')

$$
\begin{aligned}
&P_{((n\Theta)^{-Q}S_{n\Theta t_j}:1\le j\le m)}\\
&=\frac{1}{\log c}\int_1^c P_{((nx)^{-Q}S_{nxt_j}:1\le j\le m)}\,\frac{dx}{x}\\
&=\frac{1}{\log c}\int_1^c P_{((r_nx)^{-Q}c^{-m_nQ}S_{c^{m_n}r_nxt_j}:1\le j\le m)}\,\frac{dx}{x}\\
&\Rightarrow\frac{1}{\log c}\int_1^c P_{((rx)^{-Q}X_{rxt_j}:1\le j\le m)}\,\frac{dx}{x}\\
&=\frac{1}{\log c}\int_r^{rc} P_{(s^{-Q}X_{st_j}:1\le j\le m)}\,\frac{ds}{s}\\
&=\frac{1}{\log c}\int_r^{c} P_{(s^{-Q}X_{st_j}:1\le j\le m)}\,\frac{ds}{s}+\frac{1}{\log c}\int_1^{r} P_{((cs)^{-Q}X_{cst_j}:1\le j\le m)}\,\frac{ds}{s}\\
&=\frac{1}{\log c}\int_1^{c} P_{(s^{-Q}X_{st_j}:1\le j\le m)}\,\frac{ds}{s}\\
&=P_{(\Theta^{-Q}X_{\Theta t_j}:1\le j\le m)}.
\end{aligned}
$$

Since the limit does not depend on r, we have convergence as $n\to\infty$ which completes the proof. □

Remark 3.2. Alternatively, Lemma 3.1 can also be proven as an application to one of the following limit theorems. It can be shown that the distribution of mantissas $\mathcal{M}_c(\lfloor n\Theta\rfloor)$ converges weakly to the logarithmic distribution, where the mantissa to base $c>1$ is defined by $\mathcal{M}_c(x)=c^{\log_c x-\lfloor\log_c x\rfloor}\in[1,c)$ for $x>0$. In other words, if we uniquely write $x=c^m r$ with $m\in\mathbb{N}_0$ and $r\in[1,c)$ then $\mathcal{M}_c(x)=r$. Note that we implicitely used mantissas of the positive integers in the proofs of Theorem 2.1 and Lemma 3.1. With this knowledge, Lemma 3.1 follows from Theorem 2.4 in [2] considering independent increments. Moreover, since $\lfloor n\Theta\rfloor/n$ converges almost surely to Θ, Lemma 3.1 can be seen as an application to Gnedenko's transfer theorem [8]. Note that in general the transfer theorem for semistable laws only shows that the distributions of $\left((n\Theta)^{-Q}S_{n\Theta t}\right)_{n\in\mathbb{N}}$ are relatively compact, but due to the logarithmic distribution of Θ all distributional limit points coincide as is the case in the proofs of Theorem 2.1 and Lemma 3.1. This shows that the logarithmic distribution is essential for our purposes. Note further that it is also possible to drop the assumption of independence of Θ and $(Y_n)_{n\in\mathbb{N}}$ in Lemma 3.1 by means of results given in [1].

We will now combine Theorem 2.1 and Lemma 3.1 to a single limit theorem. Recall that $Y_1,Y_2,\dots$ is an i.i.d. sequence of random vectors on $\mathbb{R}^d$ with distribution $\eta=P_{Y_1}$ belonging to the strict domain of normal attraction of the strictly (c^Q,c)-semistable law ν. For $i\in\mathbb{N}$ let $(Y_n^{(i)})_{n\in\mathbb{N}}$ be i.i.d. as $(Y_n)_{n\in\mathbb{N}}$ and write

$$\{S_t^{(i)} = \sum_{k=1}^{\lfloor t \rfloor} Y_k^{(i)}\}_{t\geq 0} \quad \text{respectively} \quad \{S_t = \sum_{k=1}^{\lfloor t \rfloor} Y_k\}_{t\geq 0}$$

for the corresponding partial sum processes. Further let Θ_i, $i \in \mathbb{N}$, be i.i.d. as Θ, logarithmically distributed with probability density $x \mapsto (x \log c)^{-1} 1_{[1,c]}(x)$ on $\mathbb{R}$, and assume that $((Y_n^{(i)})_{n\in\mathbb{N}}, \Theta_i : i \in \mathbb{N})$ are independent.

THEOREM 3.3. *With the above assumptions and notations there exists a sequence of functions $b_n : [0,\infty) \to \mathbb{R}^d$ with $b_n(0) = 0$ such that*

$$\{n^{-Q} \sum_{i=1}^{n} (n\Theta_i)^{-Q} S^{(i)}_{n\Theta_i t} - b_n(t)\}_{t\geq 0} \overset{f.d.}{\Longrightarrow} \{R_t\}_{t\geq 0}, \tag{3.2}$$

where $\{R_t\}_{t\geq 0}$ is the operator-selfsimilar Lévy process appearing in Theorem 2.1

Proof. Let $0 = t_0 < t_1 < \cdots < t_m$ be arbitrary and recall that the distribution of $(R_{t_j} : 1 \leq j \leq m)$ has Lévy measure $\Psi_{t_1,\dots,t_m}$ given by (2.11). Further, as before let $Q_m = \operatorname{diag}(Q,\dots,Q)$, then $X_{n,i}^{(m)} = n^{-Q_m}((n\Theta_i)^{-Q} S^{(i)}_{n\Theta_i t_j} : 1 \leq j \leq m)$, $1 \leq i \leq n$, defines an infinitesimal array of rowwise i.i.d. random vectors on $(\mathbb{R}^d)^m$ by Lemma 3.1 and the fact that $n^{-Q_m} \to 0$ as $n \to \infty$, since the real part of any eigenvalue of Q_m exceeds 1/2. Now let $\eta_n^{(m)} = P_{X_{n,1}^{(m)}}$, then we will first prove that

$$n \cdot \eta_n^{(m)} \to \Psi_{t_1,\dots,t_m} \tag{3.3}$$

and

$$\lim_{\varepsilon\downarrow 0} \limsup_{n\to\infty} n \cdot \Big[\int_{\{\|y\|<\varepsilon\}} \langle y, z\rangle^2 \, d\eta_n^{(m)}(y) - \Big(\int_{\{\|y\|<\varepsilon\}} \langle y, z\rangle \, d\eta_n^{(m)}(y)\Big)^2\Big] = 0 \tag{3.4}$$

for any $z \in (\mathbb{R}^d)^m$, where $\langle\cdot,\cdot\rangle$ denotes some inner product on $(\mathbb{R}^d)^m$.

Now write $n = c^{m_n} r_n$ with $m_n \in \mathbb{N}_0$ and $r_n \in [1,c)$ and let $r \in [1,c]$ be an arbitrary limit point of (r_n) along some subsequence (n'). Then if $x_n \to x > 0$ by (3.1) we get along the subsequence (n')

$$\begin{aligned} &n^{-Q_m} \sum_{i=1}^{n} \big((nx_n)^{-Q} S^{(i)}_{nx_n t_j} : 1 \leq j \leq m\big) \\ &\overset{d}{=} n^{-Q_m} \big((nx_n)^{-Q} S_{n^2 x_n t_j} : 1 \leq j \leq m\big) \end{aligned}$$

$$= \left((r_n^2 x_n)^{-Q} c^{-2m_n Q} S_{c^{2m_n} r_n^2 x_n t_j} \;:\; 1 \le j \le m\right)$$
$$\Rightarrow \left((r^2 x)^{-Q} X_{r^2 x t_j} \;:\; 1 \le j \le m\right).$$

The limit distribution has no normal component and Lévy measure

$$(r^2 x)^{-Q_m} \Phi_{r^2 x t_1, \dots, r^2 x t_m} = (r^2 x) \cdot \left((r^2 x)^{-Q_m} \Phi_{t_1, \dots, t_m}\right),$$

where $\Phi_{t_1,\dots,t_m}$ is as in (2.6) and the above equality holds by (2.7). Hence, by convergence criteria for infinitesimal triangular arrays of random vectors due to Rvačeva [17] (see also Theorem 3.2.2 in [16]), we get along the subsequence (n')

$$n \cdot \left(n^{-Q_m} P_{((nx_n)^{-Q} S_{nx_n t_j} \,:\, 1 \le j \le m)}\right) \to (r^2 x) \cdot \left((r^2 x)^{-Q_m} \Phi_{t_1, \dots, t_m}\right) \tag{3.5}$$

and

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \limsup_{(n')} n \cdot \Big[&\int_{\{\|y\| < \varepsilon\}} \langle y, z\rangle^2 \, dn^{-Q_m} P_{((nx_n)^{-Q} S_{nx_n t_j} \,:\, 1 \le j \le m)}(y) \\ &- \Big(\int_{\{\|y\| < \varepsilon\}} \langle y, z\rangle \, dn^{-Q_m} P_{((nx_n)^{-Q} S_{nx_n t_j} \,:\, 1 \le j \le m)}(y) \Big)^2 \Big] = 0 \end{aligned} \tag{3.6}$$

for all $z \in (\mathbb{R}^d)^m$. Then by (3.5) we obtain along the subsequence (n')

$$\begin{aligned} n \cdot \eta_n^{(m)} &= \frac{1}{\log c} \int_1^c n \cdot \left(n^{-Q_m} P_{((nx)^{-Q} S_{nx t_j} \,:\, 1 \le j \le m)}\right) \frac{dx}{x} \\ &\to \frac{1}{\log c} \int_1^c (r^2 x) \cdot \left((r^2 x)^{-Q_m} \Phi_{t_1, \dots, t_m}\right) \frac{dx}{x} = \Psi_{t_1, \dots, t_m}, \end{aligned}$$

where the last identity follows as in the proof of Theorem 2.1. Since the limit does not depend on r we get (3.3). Moreover, since we have

$$\begin{aligned} n \cdot &\Big[\int_{\{\|y\| < \varepsilon\}} \langle y, z\rangle^2 \, d\eta_n^{(m)}(y) - \Big(\int_{\{\|y\| < \varepsilon\}} \langle y, z\rangle \, d\eta_n^{(m)}(y) \Big)^2 \Big] \\ &= \frac{1}{\log c} \int_1^c n \cdot \Big[\int_{\{\|y\| < \varepsilon\}} \langle y, z\rangle^2 \, dn^{-Q_m} P_{((nx)^{-Q} S_{nx t_j} \,:\, 1 \le j \le m)}(y) \\ &\qquad - \Big(\int_{\{\|y\| < \varepsilon\}} \langle y, z\rangle \, dn^{-Q_m} P_{((nx)^{-Q} S_{nx t_j} \,:\, 1 \le j \le m)}(y) \Big)^2 \Big] \frac{dx}{x}, \end{aligned}$$

(3.4) follows directly from (3.6).
Again, by convergence criteria in [17], (3.3) and (3.4) are equivalent to

$$n^{-Q_m} \sum_{i=1}^n \left((n\Theta_i)^{-Q} S^{(i)}_{n\Theta_i t_j} \;:\; 1 \le j \le m\right) - b_n^{(m)}(t_1, \dots, t_m) \Rightarrow (R_{t_1}, \dots, R_{t_m})$$

for some $b_n^{(m)}(t_1, \dots, t_m) \in (\mathbb{R}^d)^m$. As in the proof of Theorem 2.1, we can choose $b_n^{(m)}(t_1, \dots, t_m) = \big(b_n(t_1), \dots, b_n(t_m)\big)$ for some sequence of functions $b_n : [0, \infty) \to \mathbb{R}^d$, which completes the proof. □

Remark 3.4. It is known by [17] (see also Theorem 3.2.2 in [16]) that the centerings in Theorem 3.3 can be chosen as truncated moments

$$b_n(t) = n \cdot \mathbb{E}\big((n^2\Theta)^{-Q} S_{n\Theta t} \cdot 1_{\{\|(n^2\Theta)^{-Q}S_{n\Theta t}\|<R\}}\big)$$

for some $R > 0$ such that $\psi\{\|x\| = R\} = 0$ in which case $a = 0$ in the Lévy representation of $\rho = P_{R_1} \sim [0, 0, \psi]$. Especially, if we assume that Y_1 is symmetric, i.e. $\eta = P_{Y_1} = P_{-Y_1}$, no centering is needed in (3.2).

To summarize our procedure, we start with a partial sum process $\{S_t\}_{t\geq 0}$ of i.i.d. random vectors in some domain of attraction of a semistable law, which naturally has stationary and independent increments. Then the subordinated processes $\{(n\Theta)^{-Q}S_{n\Theta t}\}_{t\geq 0}$ for independent logarithmically distributed Θ have stationary increments that are no longer independent. By Lemma 3.1, their process limit as $n \to \infty$ is an operator-selfsimilar process $\{\Theta^{-Q}X_{\Theta t}\}_{t\geq 0}$ again with stationary but not independent increments. But taking independent copies of either of these processes leads via (2.9) respectively (3.2) to an operator-selfsimilar process with stationary and independent increments.

Especially, considering (3.2) for the one-dimensional marginal distributions of the processes involved in Theorem 3.3, we obtain an answer to the present title:

COROLLARY 3.5. *Let η be a probability measure on $\mathbb{R}^d$ belonging to the strict domain of normal attraction of a full strictly (c^Q, c)-semistable law $\nu \sim [b, 0, \phi]$. Then the normalized convolution powers of logarithmic mixtures*

$$n^{-Q}\Big(\frac{1}{\log c}\int_1^c (ns)^{-Q}\eta^{\lfloor ns\rfloor}\,\frac{ds}{s}\Big)^n \tag{3.7}$$

appropriately centered, converge weakly to a full operator-stable law $\rho \sim [a, 0, \psi]$ with exponent Q and Lévy measure ψ given by (2.10).

Note that since ν belongs to its own strict domain of normal attraction and since the logarithmic mixtures for $\eta = \nu$ in (3.7) coincide as in the proof of Lemma 3.1, we recover the result of Corollary 2.4.

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