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ON THE LOCAL SYMMETRY OF KAEHLER HYPERSURFACES

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Abstract. The purpose of the present paper is to prove that a Kaehler hypersurface with recurrent Ricci tensor is locally symmetric

1. Introduction

Let $\tilde{M}_{n+1}(\tilde{c})$ be a complex $(n+1)$-dimensional complex space form of constant holomorphic sectional curvature $\tilde{c}$ (i.e. complete, simply connected Kaehler manifold with constant holomorphic sectional curvature, say, $\tilde{c}$). For each real number $\tilde{c}$, there is (up to holomorphic isometry) exactly one complex space form in every dimension with holomorphic sectional curvature $\tilde{c}$. The complex space forms of holomorphic sectional curvature $\tilde{c}$ are denoted by $P_{n+1}(C)$, C_{n+1} and D_{n+1} depending on whether $\tilde{c}$ is positive, zero or negative, respectively. $P_{n+1}(C)$ is the complex projective space with Fubini-Study metric of constant holomorphic sectional curvature $\tilde{c}$. C_{n+1} is the complex Euclidean space. D_{n+1} is the open unit ball in C_{n+1} endowed with Bergman metric of constant holomorphic sectional curvature $\tilde{c}$.

Let M_n be a complex hypersurface in a complex space form $\tilde{M}_{n+1}(\tilde{c})$. From now on we call such a hypersurface M_n a Kaehler hypersurface. Let ∇ and S be the covariant differentiation on M_n and the Ricci tensor of M_n, respectively. K. Nomizu and B. Smyth [3] classified these Kaehler hypersurfaces with regard to the parallel Ricci tensor, i.e., $\nabla S = 0$. They proved that if the Ricci tensor S of M_n is parallel, then M_n is locally symmetric, that is, $\nabla R = 0$ and either M_n is totally geodesic in $\tilde{M}_{n+1}(\tilde{c})$ or M_n is locally the complex quadric, the latter case arising only when $\tilde{c} > 0$, where R denotes the curvature tensor of M_n (see Theorem C).

The Ricci tensor S is called the *recurrent Ricci tensor* if there exists a 1-form α such that $(\nabla_X S)Y = \alpha(X)SY$ for any X and Y tangent to M_n. And the

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Ricci tensor S is called the *birecurrent Ricci tensor* if there exists a covariant tensor field α of order 2 such that $(\nabla_X \nabla_Y S - \nabla_{\nabla_X Y} S)Z = \alpha(X,Y)SZ$ for any X, Y and Z tangent to M_n (See [6]).

The purpose of this paper is to classify Kaehler hypersurfaces with recurrent Ricci tensor in a complex space form. We note that this condition is weaker than $\nabla S = 0$. We prove the following theorem:

THEOREM. *Let M_n be a Kaehler hypersurface of complex dimension $n \geq 2$ with recurrent Ricci tensor in a complex space form $\tilde{M}_{n+1}(\tilde{c})$. Then the Ricci tensor of M_n is parallel.*

Remark. If M_n is a Kaehler hypersurface with recurrent curvature tensor, then M_n is called the recurrent hypersurface (See [6]). The above theorem shows that the recurrent hypersurface is locally symmetric.

2. Preliminaries

Let M_n be a Kaehler hypersurface of complex dimension n in a complex space form $\tilde{M}_{n+1}(\tilde{c})$ of constant holomorphic sectional curvature $\tilde{c}$. For each point $x_0 \in M_n$, we choose an unit normal vector field ξ defined in a neighborhood $U(x_0)$ of x_0. Denoting the complex structure on $\tilde{M}_{n+1}(\tilde{c})$ by J, $J\xi$ is also a normal vector field on $U(x_0)$. Let $\tilde{\nabla}$ (resp. ∇) be the covariant differentiation on $\tilde{M}_{n+1}(\tilde{c})$ (resp. M_n). Then, for any vector fields X, Y tangent to M_n on $U(x_0)$, we have

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)\xi + g(JAX, Y)J\xi, \tag{1}$$

$$\tilde{\nabla}_X \xi = -AX + s(X)J\xi, \tag{2}$$

where g, s and A are the induced Kaehler metric on M_n, the tensor field of type (0, 1) and the (1, 1)-type symmetric tensor field called the *second fundamental form*, respectively. It is easy to show that $AJ = -JA$.

Let R be the curvature tensor of M_n. Then, for any vector fields X, Y and Z on $U(x_0)$, we have the following (see [1], [3] and [5]):

$$\begin{aligned} R(X,Y)Z = \tilde{R}(X,Y)Z &+ g(AY,Z)AX - g(AX,Z)AY \\ &+ g(JAY,Z)JAX - g(JAX,Z)JAY, \end{aligned} \tag{3}$$

— Gauss equation

$$(\nabla_X A)Y - s(X)JAY = (\nabla_Y A)X - s(Y)JAX, \tag{4}$$

— Codazzi equation

where $\tilde{R}$ is the curvature tensor of $\tilde{M}_{n+1}(\tilde{c})$. Since $\tilde{M}_{n+1}(\tilde{c})$ is of constant holomorphic sectional curvature $\tilde{c}$, $\tilde{R}(X,Y)Z$ can be written as

$$\tilde{R}(X,Y)Z = \frac{\tilde{c}}{4}\{g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY + 2g(X,JY)JZ\}. \tag{5}$$

In particular, if Codazzi equation (4) satisfies

$$(\nabla_X A)Y = s(X)JAY \tag{6}$$

on a neighborhood of every point in M_n, then we say that *Codazzi equation reduces* (See [2] and [5]).

Next, we also denote the (1, 1)-type Ricci tensor of M_n by S. For any point x of $U(x_0)$, S is defined by

$$SX = \sum_{i=1}^{n} R(X,e_i)e_i + \sum_{i=1}^{n} R(X,Je_i)Je_i, \tag{7}$$

where $\{e_1,\ldots,e_n,Je_1,\ldots,Je_n\}$ is an orthonormal basis of the tangent space T_xM_n. Using Gauss equation (3) and the equation (5), we obtain

$$SX = \frac{n+1}{2}\tilde{c}X - 2A^2X \tag{8}$$

for any X tangent to M_n on $U(x_0)$.

We here recall the definitions of the recurrent Ricci tensor, the birecurrent Ricci tensor and the parallel Ricci tensor, again:

The Ricci tensor S is called the recurrent Ricci tensor if there exists a 1-form α such that $(\nabla_X S)Y = \alpha(X)SY$ for any X and Y tangent to M_n. And the Ricci tensor S is called the birecurrent Ricci tensor if there exists a covariant tensor field α of order 2 such that $(\nabla_X\nabla_Y S - \nabla_{\nabla_X Y}S)Z = \alpha(X,Y)SZ$ for any X, Y and Z tangent to M_n. If S satisfies $(\nabla_X S)Y = 0$ for any X and Y tangent to M_n, then the Ricci tensor S is said to be parallel.

Now, we prepare the following results without proof.

THEOREM A. (Ryan [4]). *Let M_n be a Kaehler hypersurface in a space of constant holomorphic sectional curvature $\tilde{c}$. Then $(R(X,Y)S)Z = 0$ on M_n if and only if one of the following is true:*

1. *$\tilde{c} \neq 0$ and A^2 is a multiple of I,*
2. *$\tilde{c} = 0$ and the nonzero eigenvalues of A^2 are equal,*

where I is the identity transformation on TM_n.

THEOREM B. (Nomizu and Smyth [3]). *If M_n is a Kaehler hypersurface in a complex space form $\tilde{M}_{n+1}(\tilde{c})$, then the following conditions are equivalent on M_n:*

1. *Codazzi equation reduces,*
2. *The Ricci tensor of M_n is parallel, that is, $\nabla S = 0$,*
3. *M_n is locally symmetric.*

THEOREM C. (Nomizu and Smyth [3]). *Let M_n be a Kaehler hypersurface of complex dimension $n \geq 1$ in a complex space form $\tilde{M}_{n+1}(\tilde{c})$ of constant holomorphic sectional curvature $\tilde{c}$. If the Ricci tensor of M_n is parallel, then M_n is locally symmetric and either M_n is of constant holomorphic sectional curvature $\tilde{c}$ and totally geodesic in $\tilde{M}_{n+1}(\tilde{c})$ or M_n is locally holomorphically isometric to the complex quadric Q_n in the complex projective space $P_{n+1}(C)$, the latter case arising only when $\tilde{c} > 0$.*

3. Lemmas

In this section, we show that if M_n is a Kaehler hypersurface with recurrent Ricci tensor S in $\tilde{M}_{n+1}(\tilde{c})$, then S satisfies $(R(X,Y)S)Z = 0$ (i.e. $R(X,Y)(SZ) - SR(X,Y)Z = 0$ for any X, Y and Z tangent to M_n). Consequently, we can get some information on the second fundamental form A of M_n (see Preliminaries, TheoremA).

LEMMA 1. *If M_n is a Kaehler hypersurface with recurrent Ricci tensor in $\tilde{M}_{n+1}(\tilde{c})$, then M_n has the birecurrent Ricci tensor.*

Proof. Suppose that M_n has the recurrent Ricci tensor. For any X, Y and Z tangent to M_n, we have

$$\begin{aligned}&(\nabla_X(\nabla_Y S))Z - (\nabla_{\nabla_X Y} S)Z \\ &\qquad = \nabla_X((\nabla_Y S)Z) - (\nabla_Y S)\nabla_X Z - (\nabla_{\nabla_X Y} S)Z.\end{aligned}$$

We use the assumption that S is the recurrent Ricci tensor and we obtain

$$\begin{aligned}&(\nabla_X(\nabla_Y S))Z - (\nabla_{\nabla_X Y} S)Z \\ &\qquad = \nabla_X(\alpha(Y)SZ) - \alpha(Y)S(\nabla_X Z) - \alpha(\nabla_X Y)SZ \\ &\qquad = X(\alpha(Y))SZ + \alpha(Y)(\nabla_X S)Z + \alpha(Y)S(\nabla_X Z) \\ &\qquad\quad - \alpha(Y)S(\nabla_X Z) - \alpha(\nabla_X Y)SZ \\ &\qquad = X(\alpha(Y))SZ + \alpha(Y)\alpha(X)SZ - \alpha(\nabla_X Y)SZ \\ &\qquad = (X(\alpha(Y)) + \alpha(Y)\alpha(X) - \alpha(\nabla_X Y))SZ..\end{aligned}$$

This equation implies that S is the birecurrent Ricci tensor.

LEMMA 2. *If M_n is a Kaehler hypersurface with birecurrent Ricci tensor S in $\tilde{M}_{n+1}(\tilde{c})$, then S satisfies $(R(X,Y)S)Z = 0$.*

Proof. We suppose that M_n is a Kaehler hypersurface with birecurrent Ricci tensor in $\tilde{M}_{n+1}(\tilde{c})$. In order to prove this lemma, we use the equation

$$\nabla_Y S^2 = (\nabla_Y S)S + S(\nabla_Y S) \tag{9}$$

and consider the equation

$$\begin{aligned}&(\nabla_X(\nabla_Y S^2))Z - (\nabla_{\nabla_X Y} S^2)Z \\ &\quad = \nabla_X((\nabla_Y S^2)Z) - (\nabla_Y S^2)\nabla_X Z - (\nabla_{\nabla_X Y} S^2)Z\end{aligned}$$

for any X, Y and Z tangent to M_n. From the equation (9), the above equation becomes

$$\begin{aligned}&(\nabla_X(\nabla_Y S^2))Z - (\nabla_{\nabla_X Y} S^2)Z \\ &\quad = \nabla_X((\nabla_Y S)SZ + S(\nabla_Y S)Z) - ((\nabla_Y S)S)\nabla_X Z - (S\nabla_Y S)\nabla_X Z \\ &\qquad - (\nabla_{\nabla_X Y} S)SZ - S(\nabla_{\nabla_X Y} S)Z \\ &\quad = (\nabla_X \nabla_Y S)SZ + (\nabla_Y S)(\nabla_X S)Z + (\nabla_Y S)S\nabla_X Z \\ &\qquad + (\nabla_X S)(\nabla_Y S)Z + S(\nabla_X \nabla_Y S)Z + S(\nabla_Y S)\nabla_X Z \\ &\qquad - (\nabla_Y S)S\nabla_X Z - (S\nabla_Y S)\nabla_X Z - (\nabla_{\nabla_X Y} S)SZ - S(\nabla_{\nabla_X Y} S)Z \\ &\quad = (\nabla_X \nabla_Y S - \nabla_{\nabla_X Y} S)SZ + S(\nabla_X \nabla_Y S - \nabla_{\nabla_X Y} S)Z \\ &\qquad + (\nabla_Y S)(\nabla_X S)Z + (\nabla_X S)(\nabla_Y S)Z.\end{aligned}$$

We now use the assumption that S is the birecurrent Ricci tensor, and it follows that

$$\begin{aligned}&(\nabla_X(\nabla_Y S^2))Z - (\nabla_{\nabla_X Y} S^2)Z \\ &\quad = (\alpha(X,Y)S)SZ + S(\alpha(X,Y)S)Z + (\nabla_Y S)(\nabla_X S)Z + (\nabla_X S)(\nabla_Y S)Z \\ &\quad = 2\alpha(X,Y)S^2 Z + (\nabla_Y S)(\nabla_X S)Z + (\nabla_X S)(\nabla_Y S)Z.\end{aligned}$$

A similar calculation shows that

$$\begin{aligned}&(\nabla_Y(\nabla_X S^2))Z - (\nabla_{\nabla_Y X} S^2)Z \\ &\qquad\qquad = 2\alpha(Y,X)S^2 Z + (\nabla_X S)(\nabla_Y S)Z + (\nabla_Y S)(\nabla_X S)Z.\end{aligned}$$

Hence we obtain

$$\begin{aligned}&(\nabla_X(\nabla_Y S^2) - \nabla_{\nabla_X Y} S^2)Z - (\nabla_Y(\nabla_X S^2) - \nabla_{\nabla_Y X} S^2)Z \\ &\qquad = (\nabla_X \nabla_Y S^2 - \nabla_Y \nabla_X S^2 - \nabla_{[X,Y]} S^2)Z \\ &\qquad = 2(\alpha(X,Y) - \alpha(Y,X))S^2 Z.\end{aligned}$$

From this equation and the commutativity of the trace and the derivation, we have

$$(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}) \operatorname{trace} S^2 = 2(\alpha(X,Y) - \alpha(Y,X)) \operatorname{trace} S^2.$$

Since trace S^2 is a differentiable function on M_n, the left side of this equation equals to zero. Therefore we get $\alpha(X,Y) = \alpha(Y,X)$ or trace $S^2 = 0$. If trace $S^2 = 0$, then we deduce that $S = 0$. Hence we can see that $(R(X,Y)S)Z = 0$. If $\alpha(X,Y) = \alpha(Y,X)$, then we have

$$\begin{aligned}(R(X,Y)S)Z &= (\nabla_X \nabla_Y S - \nabla_Y \nabla_X S - \nabla_{[X,Y]} S)Z \\ &= (\nabla_X \nabla_Y S - \nabla_Y \nabla_X S - \nabla_{\nabla_X Y} S + \nabla_{\nabla_Y X} S)Z \\ &= (\alpha(X,Y) - \alpha(Y,X))SZ \\ &= 0.\end{aligned}$$

Therefore we conclude that $(R(X,Y)S)Z = 0$ for any vector fields X, Y and Z tangent to M_n.

4. Proof of Theorem

Now, we prove the following theorem:

THEOREM. *Let M_n be a Kaehler hypersurface of complex dimension $n \geq 2$ with recurrent Ricci tensor in a complex space form $\tilde{M}_{n+1}(\tilde{c})$. Then the Ricci tensor of M_n is parallel.*

Proof. For each point x of $U(x_0)$, we choose an orthonormal basis of $T_x M_n$ $\{e_1, \ldots, e_n, Je_1, \ldots, Je_n\}$ for which the matrix of A is of the form

$$\begin{bmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_n & & & \\ & & & -\lambda_1 & & \\ & & & & \ddots & \\ & & & & & -\lambda_n \end{bmatrix},$$

i.e.,

$$Ae_i = \lambda_i e_i, \ AJe_i = -\lambda_i Je_i \ \text{ and } \ \lambda_i \geq 0 \ \text{ for } \ 1 \leq i \leq n \, (\text{see [5], lemma 1}).$$

Since M_n has the recurrent Ricci tensor S, it follows that $(R(X,Y)S)Z = 0$ for any X, Y and Z on $U(x_0)$ from Lemma 1 and Lemma 2. Therefore, by using (3), (5) and (8), we have

$$\begin{aligned}(R(e_i,e_j)S)e_j &= R(e_i,e_j)(Se_j) - S(R(e_i,e_j)e_j)\\ &= \Big(\frac{\tilde c}{4}+\lambda_i\lambda_j\Big)\Big(\frac{n+1}{2}\tilde c - 2\lambda_j^2\Big)e_i - \Big(\frac{\tilde c}{4}+\lambda_i\lambda_j\Big)\Big(\frac{n+1}{2}\tilde c - 2\lambda_i^2\Big)e_i\\ &= \Big(\frac{\tilde c}{4}+\lambda_i\lambda_j\Big)(2\lambda_i^2 - 2\lambda_j^2)e_i\\ &= 0\end{aligned}$$

for $i \neq j$.
Similarly, we get

$$\begin{aligned}(R(e_i,Je_j)S)e_j &= R(e_i,Je_j)(Se_j) - S(R(e_i,Je_j)e_j)\\ &= -\Big(\frac{\tilde c}{4}-\lambda_i\lambda_j\Big)\Big(\frac{n+1}{2}\tilde c - 2\lambda_j^2\Big)Je_i\\ &\quad + \Big(\frac{\tilde c}{4}-\lambda_i\lambda_j\Big)\Big(\frac{n+1}{2}\tilde c - 2\lambda_i^2\Big)Je_i\\ &= \Big(\frac{\tilde c}{4}-\lambda_i\lambda_j\Big)(2\lambda_j^2 - 2\lambda_i^2)Je_i\\ &= 0.\end{aligned}$$

Hence it follows that

$$\Big(\frac{\tilde c}{4}+\lambda_i\lambda_j\Big)(\lambda_i^2-\lambda_j^2) = 0 \quad \text{and} \quad \Big(\frac{\tilde c}{4}-\lambda_i\lambda_j\Big)(\lambda_i^2-\lambda_j^2) = 0 \tag{10}$$

for $1 \leq i, j \leq n$ (for details, see [4]).

We first consider the case of $\tilde c \neq 0$. It is easy to see that (10) is equivalent to $\lambda_i^2 - \lambda_j^2 = 0$. Hence we have $\lambda_i = \lambda_j = \lambda$. Then by the assumption of $n \geq 2$ we can choose linearly independent vector fields X, Y on $U(x_0)$ such that $AX = \lambda X$ and $AY = \lambda Y$. From Codazzi equation

$$(\nabla_X A)Y - s(X)JAY = (\nabla_Y A)X - s(Y)JAX,$$

we have

$$\begin{aligned}&(X\lambda)Y + (\lambda I - A)\nabla_X Y - \lambda s(X)JY\\ &= (Y\lambda)X + (\lambda I - A)\nabla_Y X - \lambda s(Y)JX.\end{aligned}$$

Therefore we know that $\lambda =$ constant on $U(x_0)$. From the equation (8), we obtain

$$S = \Big(\frac{n+1}{2}\tilde c - 2\lambda^2\Big) I.$$

Hence we find that $\nabla S = 0$.

Next, we consider the case of $\tilde{c} = 0$. From the equation (10), we see that $\lambda_i \lambda_j (\lambda_i^2 - \lambda_j^2) = 0$. Thus we have two subcases of $\lambda_i = \lambda_j = \lambda$ at $x \in U(x_0)$ and $\lambda_i = \lambda \neq \lambda_j = 0$ at $x \in U(x_0)$. The set of points such that $\lambda_i = \lambda_j = \lambda$ is an open set in $U(x_0)$, since $\lambda =$ constant as above. Since it is obviously a closed set and we may assume that $U(x_0)$ is connected set, either $U(x_0)$ satisfies $\lambda_i = \lambda_j = \lambda$ or $U(x_0)$ so $\lambda_i = \lambda \neq \lambda_j = 0$. We show that the second case cannot occur. In order to proceed with the argument, we consider three distributions on $U(x_0)$ defined by

$$\begin{aligned} T_\lambda(x) &= \{X \in T_x M_n | AX = \lambda X\}, \\ T_{-\lambda}(x) &= \{X \in T_x M_n | AX = -\lambda X\}, \\ T_0(x) &= \{X \in T_x M_n | AX = 0\}. \end{aligned}$$

The tangent space $T_x M_n$ satisfies $T_x M_n = T_\lambda(x) \oplus T_{-\lambda}(x) \oplus T_0(x)$. We here suppose that $\dim T_\lambda(x) \geq 2$. Then we can choose a neighborhood U of $x \in U(x_0)$ such that $\dim T_\lambda \geq 2$ on U. Since S is the recurrent Ricci tensor, it follows that $(\nabla_X S)Y = \alpha(X)SY$ for any X and Y tangent to M_n. From (8) this equation becomes

$$(\nabla_X A)AY + A(\nabla_X A)Y = \alpha(X)A^2 Y. \tag{11}$$

First, we take vector fields $X_1, X_2 \in T_\lambda$ so that X_1 and X_2 are linearly independent. From the equation (11), we have

$$(\lambda I + A)(\nabla_{X_1} A)X_2 = \lambda^2 \alpha(X_1)X_2.$$

Taking the T_0-component of the above equation, we get

$$\lambda((\nabla_{X_1} A)X_2)_0 = 0,$$

i.e.,

$$((\nabla_{X_1} A)X_2)_0 = 0.$$

This means that $(\nabla_{X_1} A)X_2 \in T_\lambda \oplus T_{-\lambda}$. On the other hand, the equation

$$\begin{aligned} (\nabla_{X_1} A)X_2 &= \nabla_{X_1}(AX_2) - A\nabla_{X_1} X_2 \\ &= (X_1\lambda)X_2 + (\lambda I - A)\nabla_{X_1} X_2 \end{aligned}$$

and Codazzi equation (4) lead us to

$$\begin{aligned} &(X_1\lambda)X_2 + (\lambda I - A)\nabla_{X_1} X_2 - \lambda s(X_1)JX_2 \\ &= (X_2\lambda)X_1 + (\lambda I - A)\nabla_{X_2} X_1 - \lambda s(X_2)JX_1. \end{aligned}$$

We take the T_λ-component of this equation. Then we obtain $X_1\lambda = X_2\lambda = 0$. This implies that $X\lambda = 0$ for any $X \in T_\lambda$. Similarly, if Y_1 and Y_2 are vector fields in $T_{-\lambda}$, then we can see that $Y\lambda = 0$ for any $Y \in T_{-\lambda}$.

Next, we consider $X \in T_\lambda$ and $Y \in T_{-\lambda}$. Using (11), we have

$$(-\lambda I + A)(\nabla_X A)Y = \lambda^2\alpha(X)Y.$$

We take the T_0-component of this equation. Hence we get

$$-\lambda((\nabla_X A)Y)_0 = 0,$$

i.e.,

$$((\nabla_X A)Y)_0 = 0.$$

Therefore we have $(\nabla_X A)Y \in T_\lambda \oplus T_{-\lambda}$. On the other hand, since $X\lambda = 0$ for $X \in T_\lambda$, $(\nabla_X A)Y$ is written as

$$\begin{aligned}(\nabla_X A)Y &= \nabla_X(AY) - A\nabla_X Y\\ &= -(X\lambda)Y + (-\lambda I - A)\nabla_X Y\\ &= (-\lambda I - A)\nabla_X Y.\end{aligned}$$

Hence we have $(\nabla_X A)Y \in T_\lambda$. Similarly, from (11) we can consider the equation

$$(\lambda I + A)(\nabla_Y A)X = \lambda^2\alpha(Y)X.$$

Taking the T_0-component of above equation, we have

$$\lambda((\nabla_Y A)X)_0 = 0,$$

i.e.,

$$((\nabla_Y A)X)_0 = 0.$$

Hence we obtain $(\nabla_Y A)X \in T_\lambda \oplus T_{-\lambda}$. Using the fact that $Y\lambda = 0$ for $Y \in T_{-\lambda}$, we have

$$\begin{aligned}(\nabla_Y A)X &= \nabla_Y(AX) - A\nabla_Y X\\ &= (Y\lambda)X + (\lambda I - A)\nabla_Y X\\ &= (\lambda I - A)\nabla_Y X.\end{aligned}$$

Thus we get $(\nabla_Y A)X \in T_{-\lambda}$. Then Codazzi equation (4) can be written as

$$(-\lambda I - A)\nabla_X Y + \lambda s(X)JY = (\lambda I - A)\nabla_Y X - \lambda s(Y)JX.$$

Since the left side of this equation is in T_λ and the right side is in $T_{-\lambda}$, we conclude that $(\nabla_X A)Y = s(X)JAY$ and $(\nabla_Y A)X = s(Y)JAX$ for $X \in T_\lambda$, $Y \in T_{-\lambda}$. Thus if $X_1, X_2 \in T_\lambda$, then we find

$$\begin{aligned}(\nabla_{X_1}A)X_2 &= -(\nabla_{X_1}A)JJX_2\\ &= J(\nabla_{X_1}A)JX_2\\ &= Js(X_1)JAJX_2\\ &= s(X_1)JAX_2.\end{aligned}$$

Similarly, we have

$$\begin{aligned}(\nabla_{Y_1}A)Y_2 &= -(\nabla_{Y_1}A)JJY_2\\ &= J(\nabla_{Y_1}A)JY_2\\ &= Js(Y_1)JAJY_2\\ &= s(Y_1)JAY_2\end{aligned}$$

for any $Y_1, Y_2 \in T_{-\lambda}$.

Finally, if X is any tangent vector field on U and $Z \in T_0$, then we obtain $A(\nabla_X A)Z = 0$ from (11). Therefore we see that $(\nabla_X A)Z \in T_0$. By using the equation

$$\begin{aligned}(\nabla_X A)Z &= \nabla_X(AZ) - A\nabla_X Z\\ &= -A\nabla_X Z\end{aligned}$$

and the fact that $(\nabla_X A)Z \in T_0$ and the T_0-component of $A\nabla_X Z$ is zero, we have $(\nabla_X A)Z = 0$. On the other hand, it is easy to see that $s(X)JAZ = 0$. Hence we get $(\nabla_X A)Z = s(X)JAZ$. Then Codazzi equation (4) gives $(\nabla_Z A)X = s(Z)JAX$. We conclude that $(\nabla_X A)Y = s(X)JAY$ for any X and Y on U, that is, Codazzi equation reduces. Therefore, the Ricci tensor is parallel from Theorem B. Hence U must satisfies $\lambda_i = \lambda_j = \lambda$, which is a contradiction.

It remains that the case of dim T_λ = dim $T_{-\lambda} = 1$ on $U(x_0)$. For $X \in T_\lambda$ and $Y \in T_0$ from (11) we have

$$g(\nabla_X Y, X) = g(\nabla_X Y, JX) = g(\nabla_{JX} Y, X) = g(\nabla_{JX} Y, JX) = 0.$$

Moreover, for $X \in T_\lambda$ from (11)

$$2\lambda(X\lambda)JX + (-\lambda I - A)\nabla_X JX = \lambda^2\alpha(X)JX.$$

Hence we get

$$g(\nabla_X JX, X) = 0.$$

Similarly, we have

$$g(\nabla_{JX}JX, X) = g(\nabla_{JX}X, JX) = g(\nabla_X X, JX) = 0.$$

Therefore for a unit vector X we obatin

$$\nabla_X X = \nabla_X JX = \nabla_{JX}X = \nabla_{JX}JX = 0.$$

Then we get

$$\begin{aligned} &g(R(X, JX)JX, X) \\ &= g(\nabla_X\nabla_{JX}JX - \nabla_{JX}\nabla_X JX - \nabla_{[X,JX]}JX, X) \\ &= 0. \end{aligned}$$

On the other hand, from (3) we have

$$g(R(X, JX)JX, X) = -\lambda^2,$$

which is a contradiction. This completes the proof of Theorem.

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