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PERIODIC SOLUTIONS IN A CLASS OF PERIODIC DELAY PREDATOR-PREY SYSTEMS

By

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Abstract. In the present paper, we investigate the existence and the global attractivity of positive periodic solutions of a system of m-species prey and n-species predators.

1. Introduction

On the fields of mathematical biology and mathematical economics, Lotka-Volterra type equations have been extensively studied by many authors. In recent years, the theoretical analyses of the existence and the global stability of equilibria or periodic solutions for a multi-species population dynamics with time delays are especially vigorous subjects. Concerning the basic results, we refer the readers to Gopalsamy [5], Hofbauer & Sigmund [6], Kuang [7] and their lists of references.

Our purpose in this paper is to show the existence and the global attractivity of positive periodic solutions of the predator-prey system with periodic parameters and periodic delays of the form

(LV)

$$\begin{cases} \dot{u}_i(t) = u_i(t)\left(\alpha_i(t) - \sum_{j=1}^{m} a_{ij}(t)u_j(t-\sigma_{ij}(t)) - \sum_{k=1}^{n} b_{ik}(t)v_k(t-\tau_{ik}(t))\right), \\ \dot{v}_k(t) = v_k(t)\left(-\beta_k(t) + \sum_{i=1}^{m} c_{ki}(t)u_i(t-\mu_{ki}(t)) - \sum_{l=1}^{n} d_{kl}(t)v_l(t-\nu_{kl}(t))\right), \\ \qquad\qquad t \in \mathbb{R}, \quad i = 1,\ldots,m, \quad k = 1,\ldots,n, \end{cases}$$

where $\alpha_i, \beta_k \in C(\mathbb{R})$ and $a_{ij}, b_{ik}, c_{ki}, d_{kl}, \sigma_{ij}, \tau_{ik}, \mu_{ki}, \nu_{kl} \in C(\mathbb{R}; [0,\infty))$ are periodic functions of t with period $T > 0$ for all $i, j = 1,\ldots,m$ and all $k, l = 1,\ldots,n$. System (LV) is modeling a food web of m-species prey and n-species predators.

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Variables u_i and v_k represent the population densities of ith prey and kth predator respectively. Function α_i is the reproduction rate of ith prey and β_k is the natural mortality rate of kth predator, where these parameters may be either positive or negative according to the periodic environmental factors. Function a_{ij} is the competitive coefficient between ith and jth prey, d_{kl} kth and lth predator, then these parameters represent intra-specific competition for $i = j$ or $k = l$, inter-specific competition $i \neq j$ or $k \neq l$. One prey and one predator do not compete with each other for common resources. Function b_{ik} is the uptake rate of ith prey by kth predator. Function c_{ki} is the yield rate, which converts the uptake of ith prey into growth of kth predator. System (LV) has timelags σ_{ij} and ν_{kl} in the processes of competition, τ_{ik} uptake, μ_{ki} yield.

As for this type of nonautonomous systems, the sufficient conditions for periodic solutions to exist are often given by either way of using the supremum and the infimun or the average value of each periodic parameter. Our main result adopts the latter. In case that (LV) has no delays, applying Brouwer's fixed point theorem, Pinghua & Rui [11] have given an existence result in the former way. Fan et al [1] have examined the existence of periodic solutions in the latter way and the global stability for the periodic n-species competition system with periodic delays of the form

$$\dot{u}_i(t) = u_i(t)\left(\alpha_i(t) - \sum_{j=1}^{n} a_{ij}(t)u_j(t - \sigma_{ij}(t))\right), \quad t \in \mathbb{R}, \quad i = 1, \dots, n,$$

where $\alpha_i \in C(\mathbb{R})$ and $a_{ij}, \sigma_{ij} \in C(\mathbb{R}; [0, \infty))$ are periodic functions of t with period $T > 0$ for all $i, j = 1, \dots, n$. The results of [1] motivate the present paper. Gai et al [2] have established an existence result of periodic solutions for the periodic delay difference system corresponding to (LV), however, the conditions assumed in [2] includes an improper point from a biological perspective.

On the other hand, most of the authors, e.g. [1, 3, 11], have constructed a suitable Lyapunov function for the considering system in order to show that the solution guaranteed by the existence theorem is unique and globally attractive or that the system attains even the weaker concept of stability — permanence defined in Section 3.

Throughout this paper, we shall use the following notation. Let $\mathbb{R}_+ = (0, \infty)$, and $J_N = \{i \in \mathbb{Z} : 1 \leq i \leq N\}$. We denote an element of $\mathbb{R}^N$ by column vector $x = (x_1, x_2, \dots, x_N)^{\mathrm{T}}$. We also denote by $B_r(a)$ an open ball centered at a with radius $r > 0$ of a suitable space. For a given set Ω, its closure is written by $\overline{\Omega}$, its boundary $\partial\Omega$. For a given T-periodic function $f \in C(\mathbb{R})$, we put $\overline{f} = \frac{1}{T}\int_0^T f(s)\,ds$ and $\widetilde{f} = \frac{1}{T}\int_0^T |f(s)|\,ds$.

2. Existence of Positive Periodic Solutions

In this section, we establish an existence result for periodic solutions of (LV). In order to prove our result, we shall make use of the continuation theorem in coincidence degree theory due to Mawhin et al. See [4] or [10].

Let X and Z be Banach spaces. A linear operator $\mathcal{L} : D(\mathcal{L}) \subset X \to Z$ is called a Fredholm operator if its kernel $K(\mathcal{L}) = \{x \in D(\mathcal{L}) : \mathcal{L}x = 0\}$ has finite dimension and its range $R(\mathcal{L}) = \{\mathcal{L}x : x \in D(\mathcal{L})\}$ is closed and has finite codimension. The index of $\mathcal{L}$ is defined by the integer $\dim K(\mathcal{L}) - \mathrm{codim} R(\mathcal{L})$. If $\mathcal{L}$ is a Fredholm operator with index 0, then there exists continuous projections $\mathcal{P} : X \to X$ and $\mathcal{Q} : Z \to Z$ such that $R(\mathcal{P}) = K(\mathcal{L})$ and $K(\mathcal{Q}) = R(\mathcal{L})$. Then $\mathcal{L}|_{D(\mathcal{L})\cap K(\mathcal{P})} : D(\mathcal{L}) \cap K(\mathcal{P}) \to R(\mathcal{L})$ is bijective, and its inverse operator is denoted by $\mathcal{K}_{\mathcal{P}} : R(\mathcal{L}) \to D(\mathcal{L}) \cap K(\mathcal{P})$. Since $K(\mathcal{L})$ is isomorphic to $R(\mathcal{Q})$, there exists a bijection $\mathcal{J} : K(\mathcal{L}) \to R(\mathcal{Q})$. Let Ω be a bounded open subset of X and let $\mathcal{N} : X \to Z$ be a continuous operator. If $\mathcal{Q}\mathcal{N}(\overline{\Omega})$ is bounded and $\mathcal{K}_{\mathcal{P}}(\mathcal{I} - \mathcal{Q})\mathcal{N} : \overline{\Omega} \to X$ is compact, then $\mathcal{N}$ is called $\mathcal{L}$-compact on $\overline{\Omega}$, where $\mathcal{I}$ is the identity.

Let $\mathcal{L}$ be a Fredholm linear operator with index 0 and let $\mathcal{N}$ be a $\mathcal{L}$-compact mapping on $\overline{\Omega}$. Define mapping $\mathcal{F} : D(\mathcal{L}) \cap \overline{\Omega} \to Z$ by $\mathcal{F} = \mathcal{L} - \mathcal{N}$. If $\mathcal{L}x \neq \mathcal{N}x$ for all $x \in D(\mathcal{L}) \cap \partial\Omega$, then by using $\mathcal{P}, \mathcal{Q}, \mathcal{K}_{\mathcal{P}}, \mathcal{J}$ defined above, the coincidence degree of $\mathcal{F}$ in Ω with respect to $\mathcal{L}$ is defined by

$$D_{\mathcal{L}}(\mathcal{F}, \Omega) = \deg\left(\mathcal{I} - \mathcal{P} - \left(\mathcal{J}^{-1}\mathcal{Q} + \mathcal{K}_{\mathcal{P}}(\mathcal{I} - \mathcal{Q})\right)\mathcal{N}, \Omega, 0\right),$$

where $\deg(\mathcal{G}, D, p)$ is the Leray-Schauder degree of $\mathcal{G}$ at p relative to D.

And then the following lemma holds.

LEMMA 2.1 (Continuation theorem). *Let $\mathcal{L}$ be a Fredholm operator with index 0 and let $\mathcal{N}$ be $\mathcal{L}$-compact on $\overline{\Omega}$. Assume that the following conditions hold:*

(i) *$\mathcal{L}x \neq \lambda\mathcal{N}x$ for all $\lambda \in (0,1]$ and all $x \in D(\mathcal{L}) \cap \partial\Omega$,*
(ii) *$\mathcal{Q}\mathcal{N}x \neq 0$ for all $x \in K(\mathcal{L}) \cap \partial\Omega$.*

Then

$$D_{\mathcal{L}}(\mathcal{F}, \Omega) = \deg\left(-\mathcal{J}^{-1}\mathcal{Q}\mathcal{N}|_{K(\mathcal{L})}, K(\mathcal{L}) \cap \Omega, 0\right).$$

And if $D_{\mathcal{L}}(\mathcal{F}, \Omega) \neq 0$, then $\mathcal{L}x = \mathcal{N}x$ has at least one solution $x \in D(\mathcal{L}) \cap \Omega$.

To state our main result, we need some preliminaries. Throughout the present paper, the initial conditions for (LV) are arbitrarily given as follows:

(C1)

$$\begin{cases} \phi_i \in C([-\kappa, 0]; \mathbb{R}_+), \ u_i(s) = \phi_i(s) \text{ for all } s \in [-\kappa, 0] \text{ and each } i \in J_m, \\ \psi_k \in C([-\kappa, 0]; \mathbb{R}_+), \ v_k(s) = \psi_k(s) \text{ for all } s \in [-\kappa, 0] \text{ and each } k \in J_n, \end{cases}$$

where $\kappa = \max\{\sigma_{ij}(t), \tau_{ik}(t), \mu_{ki}(t), \nu_{kl}(t) : t \in [0,T],\ i,j \in J_m,\ k,l \in J_n\}$. Furthermore, we consider the case where the following condition is satisfied:

(C2) $\overline{\alpha}_i > 0$ for each $i \in J_m$ and $\overline{\beta}_k > 0$ for each $k \in J_n$.

REMARK 2.2. Condition (C2) means that the population density of one prey and one predator must increase and decrease exponentially in the long run respectively if any other species does not participate in the food web.

Our main result is as follows.

THEOREM 2.3. *Suppose that the following conditions* (2.1) *and* (2.2) *hold:*

$$\overline{a}_{ii} > 0 \quad \text{and} \quad \overline{\alpha}_i > \sum_{\substack{j=1\\j\neq i}}^{m} \overline{a}_{ij}\mathbb{C}_j + \sum_{k=1}^{n} \overline{b}_{ik}\mathbb{D}_k \quad \text{for each } i \in J_m, \tag{2.1}$$

$$\overline{d}_{kk} > 0 \quad \text{and} \quad \sum_{i=1}^{m} \overline{c}_{ki}\mathbb{E}_i > \overline{\beta}_k + \sum_{\substack{l=1\\l\neq k}}^{n} \overline{d}_{kl}\mathbb{D}_l \quad \text{for each } k \in J_n, \tag{2.2}$$

where

$$\mathbb{C}_i = \frac{\overline{\alpha}_i}{\overline{a}_{ii}} \exp\{(\overline{\alpha}_i + \widetilde{\alpha}_i)T\}, \tag{2.3}$$

$$\mathbb{D}_k = \frac{1}{\overline{d}_{kk}} \sum_{i=1}^{m} \overline{c}_{ki}\mathbb{C}_i \exp\left\{\left(\widetilde{\beta}_k - \overline{\beta}_k + 2\sum_{i=1}^{m} \overline{c}_{ki}\mathbb{C}_i\right) T\right\}, \tag{2.4}$$

$$\mathbb{E}_i = \frac{1}{\overline{a}_{ii}} \left(\overline{\alpha}_i - \sum_{\substack{j=1\\j\neq i}}^{m} \overline{a}_{ij}\mathbb{C}_j - \sum_{k=1}^{n} \overline{b}_{ik}\mathbb{D}_k\right) \exp\{-(\overline{\alpha}_i + \widetilde{\alpha}_i)T\}. \tag{2.5}$$

Then problem (LV) *has at least one positive T-periodic solution.*

REMARK 2.4. Conditions $\overline{a}_{ii} > 0$ of (2.1) and $\overline{d}_{kk} > 0$ of (2.2) indicate that any species has the intra-specific competition, which does not let its own population density explode. Roughly speaking, the others of (2.1) and (2.2) describe the conditions for all species of the food web to be survivable eternally. In fact, when we set $T = 0$, (2.1) and (2.2) give a sufficient condition for the autonomous system corresponding to (LV) replaced all of the periodic parameters and delays by nonnegative constants to have a positive equilibrium. We will mention this problem in Corollary 2.8.

The proof of Theorem 2.3 follows after some lemmata.

LEMMA 2.5. *Let* $(u(t), v(t))$ *be a solution of* (LV) *satisfying condition* (C1). *Then* $(u(t), v(t)) \in \mathbb{R}_+^{m+n}$ *for all* $t \in [0, \infty)$.

Proof. Integrating (LV) over $[0, t]$ for $t \geq 0$, we have

$$\int_0^t \frac{\dot{u}_i(s)}{u_i(s)} ds = \int_0^t \Big\{\alpha_i(s) - \sum_{j=1}^m a_{ij}(s) u_j(s - \sigma_{ij}(s)) - \sum_{k=1}^n b_{ik}(s) v_k(s - \tau_{ik}(s))\Big\} ds$$

for each $i \in J_m$. That is

$$u_i(t) = u_i(0) \exp\left\{\int_0^t \Big\{\alpha_i(s) - \sum_{j=1}^m a_{ij}(s) u_j(s - \sigma_{ij}(s)) - \sum_{k=1}^n b_{ik}(s) v_k(s - \tau_{ik}(s))\Big\} ds\right\}$$

for all $t \in [0, \infty)$ and each $i \in J_m$. Since $u_i(0) \in \mathbb{R}_+$ from (C1), the above equality implies $u_i(t) \in \mathbb{R}_+$ for all $t \in [0, \infty)$. We have the same estimate for $v_k(t)$ and each $k \in J_n$. Therefore, the assertion holds. □

Lemma 2.5 and (C1) enable us to change variables $u_i(t) = \exp\{x_i(t)\}$ and $v_k(t) = \exp\{y_k(t)\}$ for $t \in [-\kappa, \infty)$, each $i \in J_m$ and each $k \in J_n$ respectively. Then (LV) is equivalent to

(LV′)

$$\begin{cases} \dot{x}_i(t) = \alpha_i(t) - \sum_{j=1}^m a_{ij}(t) \exp\{x_j(t - \sigma_{ij}(t))\} - \sum_{k=1}^n b_{ik}(t) \exp\{y_k(t - \tau_{ik}(t))\}, \\ \dot{y}_k(t) = -\beta_k(t) + \sum_{i=1}^m c_{ki}(t) \exp\{x_i(t - \mu_{ki}(t))\} - \sum_{l=1}^n d_{kl}(t) \exp\{y_l(t - \nu_{kl}(t))\}, \\ \qquad t \in [0, \infty), \quad i \in J_m, \quad k \in J_n. \end{cases}$$

Here, we put

$$z(t) = (z_1(t), \ldots, z_{m+n}(t))^{\mathbb{T}} = (x_1(t), \ldots, x_m(t), y_1(t), \ldots, y_n(t))^{\mathbb{T}}.$$

We also put

$$X = \left\{z \in C\left(\mathbb{R}; \mathbb{R}^{m+n}\right) : z(t) = z(t + T), t \in \mathbb{R}\right\}.$$

X is a Banach space with the norm $\|\cdot\|$ defined by

$$\|z\| = \max_{t \in [0,T]} \left(\sum_{i=1}^{m+n} |z_i(t)|^2\right)^{1/2} \quad \text{for each } z \in X.$$

Let us define a linear operator $\mathcal{L} : D(\mathcal{L}) \subset X \to X$ such that, for $t \in [0, \infty)$ and each $z \in D(\mathcal{L})$,

$$z_i(t) \mapsto \dot{z}_i(t) \quad \text{for each } i \in J_{m+n},$$

where $D(\mathcal{L}) = \{z \in X : z \in C^1(\mathbb{R}; \mathbb{R}^{m+n})\}$. While, $\mathcal{N}_1$, $\mathcal{N}_2$ and $\mathcal{N}$ are continuous operators defined by as follows:
$\mathcal{N}_1 : X \to X$ such that for $t \in [0, \infty)$ and each $z \in X$,

$$\begin{cases} x_i(t) \mapsto \alpha_i(t) - a_{ii}(t) \exp\{x_i(t - \sigma_{ii}(t))\} & \text{for each } i \in J_m, \\ y_k(t) \mapsto -\beta_k(t) + \sum_{i=1}^{m} c_{ki}(t) \exp\{x_i(t - \mu_{ki}(t))\} & \\ \qquad\qquad -d_{kk}(t) \exp\{y_k(t - \nu_{kk}(t))\} & \text{for each } k \in J_n, \end{cases}$$

$\mathcal{N}_2 : X \to X$ such that for $t \in [0, \infty)$ and each $z \in X$,

$$\begin{cases} x_i(t) \mapsto -\sum_{\substack{j=1 \\ j \neq i}}^{m} a_{ij}(t) \exp\{x_j(t - \sigma_{ij}(t))\} & \\ \qquad\qquad -\sum_{k=1}^{n} b_{ik}(t) \exp\{y_k(t - \tau_{ik}(t))\} & \text{for each } i \in J_m, \\ y_k(t) \mapsto -\sum_{\substack{l=1 \\ l \neq k}}^{n} d_{kl}(t) \exp\{y_l(t - \nu_{kl}(t))\} & \text{for each } k \in J_n, \end{cases}$$

and

$$\mathcal{N} : X \to X \quad \text{such that} \quad \mathcal{N} = \mathcal{N}_1 + \mathcal{N}_2.$$

Then one can see that the existence problem of T-periodic solutions for (LV′) corresponds to the abstract equation $\mathcal{L}z = \mathcal{N}z$ on $D(\mathcal{L}) \cap X$.

LEMMA 2.6. *Suppose that* (2.1) *and* (2.2) *hold. Then the set*

$$S_1 = \{z \in D(\mathcal{L}) \cap X : \mathcal{L}z = \mathcal{N}_1 z + \xi \mathcal{N}_2 z \quad \textit{for some } \xi \in [0, 1]\}$$

is bounded in X.

Proof. Fix $\xi \in [0, 1]$. Let $z \in D(\mathcal{L}) \cap X$ be a possible solution of

$$\mathcal{L}z = \mathcal{N}_1 z + \xi \mathcal{N}_2 z. \tag{2.6}$$

Integrating (2. 6) over [0,T], we find that

$$\overline{\alpha}_i T - \int_0^T a_{ii}(s)\exp\{x_i(s-\sigma_{ii}(s))\}\,ds - \xi\int_0^T \sum_{\substack{j=1\\ j\neq i}}^m a_{ij}(s)\exp\{x_j(s-\sigma_{ij}(s))\}\,ds - \xi\int_0^T \sum_{k=1}^n b_{ik}(s)\exp\{y_k(s-\tau_{ik}(s))\}\,ds = 0 \tag{2.7}$$

for each $i \in J_m$ and

$$-\overline{\beta}_k T + \int_0^T \sum_{i=1}^m c_{ki}(s)\exp\{x_i(s-\mu_{ki}(s))\}\,ds - \int_0^T d_{kk}(s)\exp\{y_k(s-\nu_{kk}(s))\}\,ds - \xi\int_0^T \sum_{\substack{l=1\\ l\neq k}}^n d_{kl}(s)\exp\{y_l(s-\nu_{kl}(s))\}\,ds = 0 \tag{2.8}$$

for each $k \in J_n$. On the other hand, noting that $a_{ij}(t) \geq 0$, $b_{ik}(t) \geq 0$, $c_{ki}(t) \geq 0$ and $d_{kl}(t) \geq 0$ for all $t \in \mathbb{R}$, we find from (2. 7) and (2. 8) that

$$\begin{aligned}\int_0^T |\dot{x}_i(s)|\,ds &\leq \widetilde{\alpha}_i T + \int_0^T a_{ii}(s)\exp\{x_i(s-\sigma_{ii}(s))\}\,ds \\ &\quad + \xi\int_0^T \sum_{\substack{j=1\\ j\neq i}}^m a_{ij}(s)\exp\{x_j(s-\sigma_{ij}(s))\}\,ds \\ &\quad + \xi\int_0^T \sum_{k=1}^n b_{ik}(s)\exp\{y_k(s-\tau_{ik}(s))\}\,ds \\ &= (\widetilde{\alpha}_i + \overline{\alpha}_i)T\end{aligned} \tag{2.9}$$

for each $i \in J_m$ and

$$\begin{aligned}\int_0^T |\dot{y}_k(s)|\,ds &\leq \widetilde{\beta}_k T + \int_0^T \sum_{i=1}^m c_{ki}(s)\exp\{x_i(s-\mu_{ki}(s))\}\,ds \\ &\quad + \int_0^T d_{kk}(s)\exp\{y_k(s-\nu_{kk}(s))\}\,ds \\ &\quad + \xi\int_0^T \sum_{\substack{l=1\\ l\neq k}}^n d_{kl}(s)\exp\{y_l(s-\nu_{kl}(s))\}\,ds \\ &= (\widetilde{\beta}_k - \overline{\beta}_k)T + 2\int_0^T \sum_{i=1}^m c_{ki}(s)\exp\{x_i(s-\mu_{ki}(s))\}\,ds\end{aligned} \tag{2.10}$$

for each $k \in J_n$. It follows from the continuities and periodicities of $x_i(t)$ and $y_k(t)$ that there exists $\hat{\zeta}_i, \check{\zeta}_i \in [0,T]$ and $\hat{\eta}_k, \check{\eta}_k \in [0,T]$ such that

$$x_i(\hat{\zeta}_i) = \max_{t\in[0,T]} x_i(t), \quad x_i(\check{\zeta}_i) = \min_{t\in[0,T]} x_i(t) \quad \text{for each } i \in J_m, \tag{2.11}$$

$$y_k(\hat{\eta}_k) = \max_{t\in[0,T]} y_k(t), \quad y_k(\check{\eta}_k) = \min_{t\in[0,T]} y_k(t) \quad \text{for each } k \in J_n. \tag{2.12}$$

By standerd arguments, we have that for all $t \in [0,\infty)$, each $i \in J_m$ and each $k \in J_n$,

$$x_i(\hat{\zeta}_i) - \int_0^T |\dot{x}_i(s)|\,ds \le x_i(t) \le x_i(\check{\zeta}_i) + \int_0^T |\dot{x}_i(s)|\,ds, \tag{2.13}$$

$$y_k(\hat{\eta}_k) - \int_0^T |\dot{y}_k(s)|\,ds \le y_k(t) \le y_k(\check{\eta}_k) + \int_0^T |\dot{y}_k(s)|\,ds. \tag{2.14}$$

From (2.1), (2.7) and (2.11), we have

$$\overline{a}_{ii} T \exp\{x_i(\check{\zeta}_i)\} \le \overline{\alpha}_i T,$$

that is

$$x_i(\check{\zeta}_i) \le \log \frac{\overline{\alpha}_i}{\overline{a}_{ii}} \quad \text{for each } i \in J_m. \tag{2.15}$$

From (2.9), (2.13) and (2.15), we obtain

$$x_i(t) \le \log \frac{\overline{\alpha}_i}{\overline{a}_{ii}} + (\widetilde{\alpha}_i + \overline{\alpha}_i)T \quad \text{for all } t \in [0,\infty) \text{ and each } i \in J_m. \tag{2.16}$$

By using notation (2.3), $\exp\{x_i(t)\} \le \exp\{x_i(\hat{\zeta}_i)\} \le \mathbb{C}_i$ for all $t \in [0,\infty)$ and each $i \in J_m$. From (2.2), (2.8), (2.11), (2.12) and (2.16), we have

$$\overline{d}_{kk} T \exp\{y_k(\check{\eta}_k)\} \le \int_0^T \sum_{i=1}^m c_{ki}(s) \exp\{x_i(\hat{\zeta}_i)\}\,ds \le T \sum_{i=1}^m \overline{c}_{ki} \mathbb{C}_i,$$

that is

$$y_k(\check{\eta}_k) \le \log \left\{ \frac{1}{\overline{d}_{kk}} \sum_{i=1}^m \overline{c}_{ki} \mathbb{C}_i \right\} \quad \text{for each } k \in J_n. \tag{2.17}$$

From (2.10), (2.12), (2.14), (2.16) and (2.17), we obtain

$$\begin{aligned} y_k(t) &\le \log \left\{ \frac{1}{\overline{d}_{kk}} \sum_{i=1}^m \overline{c}_{ki} \mathbb{C}_i \right\} + (\widetilde{\beta}_k - \overline{\beta}_k)T \\ &\qquad + 2\int_0^T \sum_{i=1}^m c_{ki}(s) \exp\{x_i(\hat{\zeta}_i)\}\,ds \\ &\le \log \left\{ \frac{1}{\overline{d}_{kk}} \sum_{i=1}^m \overline{c}_{ki} \mathbb{C}_i \right\} + \left(\widetilde{\beta}_k - \overline{\beta}_k + 2\sum_{i=1}^m \overline{c}_{ki} \mathbb{C}_i \right) T \end{aligned} \tag{2.18}$$

for all $t \in [0,\infty)$ and each $k \in J_n$. By using notation (2.4), $\exp\{y_k(t)\} \leq \exp\{y_k(\hat{\eta}_k)\} \leq \mathbb{D}_k$ for all $t \in [0,\infty)$ and each $k \in J_n$. While, we have from (2.7), (2.11), (2.12), (2.16) and (2.18) that

$$\begin{aligned}\overline{a}_{ii}T\exp\{x_i(\hat{\zeta}_i)\} &\geq \overline{\alpha}_i T - \xi\int_0^T \sum_{\substack{j=1\\ j\neq i}}^{m} a_{ij}(s)\exp\{x_j(\hat{\zeta}_j)\}\,ds\\ &\qquad - \xi\int_0^T \sum_{k=1}^{n} b_{ik}(s)\exp\{y_k(\hat{\eta}_k)\}\,ds\\ &\geq \overline{\alpha}_i T - T\sum_{\substack{j=1\\ j\neq i}}^{m}\overline{a}_{ij}\mathbb{C}_j - T\sum_{k=1}^{n}\overline{b}_{ik}\mathbb{D}_k\end{aligned} \tag{2.19}$$

for each $i \in J_m$. By using (2.1), we obtain from (2.9), (2.13) and (2.19) that

$$x_i(t) \geq \log\left\{\frac{1}{\overline{a}_{ii}}\left(\overline{\alpha}_i - \sum_{\substack{j=1\\ j\neq i}}^{m}\overline{a}_{ij}\mathbb{C}_j - \sum_{k=1}^{n}\overline{b}_{ik}\mathbb{D}_k\right)\right\} - (\widetilde{\alpha}_i + \overline{\alpha}_i)T \tag{2.20}$$

for all $t \in [0,\infty)$ and each $i \in J_m$. By using notation (2.5), $\exp\{x_i(t)\} \geq \exp\{x_i(\check{\zeta}_i)\} \geq \mathbb{E}_i$ for all $t \in [0,\infty)$ and each $i \in J_m$. From (2.8), (2.11), (2.14), (2.18) and (2.20), we have

$$\begin{aligned}\overline{d}_{kk}T\exp\{y_k(\hat{\eta}_k)\} &\geq -\overline{\beta}_k T + \int_0^T \sum_{i=1}^{m} c_{ki}(s)\exp\{x_i(\check{\zeta}_i)\}\,ds\\ &\qquad - \xi\int_0^T \sum_{\substack{l=1\\ l\neq k}}^{n} d_{kl}(s)\exp\{y_l(\hat{\eta}_l)\}\,ds\\ &\geq -\overline{\beta}_k T + T\sum_{i=1}^{m}\overline{c}_{ki}\mathbb{E}_i - T\sum_{\substack{l=1\\ l\neq k}}^{n}\overline{d}_{kl}\mathbb{D}_l\end{aligned} \tag{2.21}$$

for each $k \in J_n$. By using (2.2), we obtain from (2.10), (2.14) and (2.21) that

$$\begin{aligned}y_k(t) &\geq \log\left\{\frac{1}{\overline{d}_{kk}}\left(-\overline{\beta}_k + \sum_{i=1}^{m}\overline{c}_{ki}\mathbb{E}_i - \sum_{\substack{l=1\\ l\neq k}}^{n}\overline{d}_{kl}\mathbb{D}_l\right)\right\}\\ &\qquad - (\widetilde{\beta}_k - \overline{\beta}_k)T - 2\int_0^T \sum_{i=1}^{m} c_{ki}(s)\exp\{x_i(\hat{\zeta}_i)\}\,ds\\ &\geq \log\left\{\frac{1}{\overline{d}_{kk}}\left(-\overline{\beta}_k + \sum_{i=1}^{m}\overline{c}_{ki}\mathbb{E}_i - \sum_{\substack{l=1\\ l\neq k}}^{n}\overline{d}_{kl}\mathbb{D}_l\right)\right\}\\ &\qquad - \left(\widetilde{\beta}_k - \overline{\beta}_k + 2\sum_{i=1}^{m}\overline{c}_{ki}\mathbb{C}_i\right)T\end{aligned} \tag{2.22}$$

for all $t \in [0,\infty)$ and each $k \in J_n$. Let us denote by $\mathbb{F}_k$,

$$\frac{1}{\overline{d}_{kk}}\left(-\overline{\beta}_k+\sum_{i=1}^{m}\overline{c}_{ki}\mathbb{E}_i-\sum_{\substack{l=1\\ l\neq k}}^{n}\overline{d}_{kl}\mathbb{D}_l\right)\exp\left\{-\left(\widetilde{\beta}_k-\overline{\beta}_k+2\sum_{i=1}^{m}\overline{c}_{ki}\mathbb{C}_i\right)T\right\}$$

for each $k \in J_n$, then $\exp\{y_k(t)\} \geq \exp\{y_k(\check{\eta}_k)\} \geq \mathbb{F}_k$ for all $t \in [0,\infty)$ and each $k \in J_n$. Consequently, we obtain that

$$\max_{t\in[0,T]}|x_i(t)| \leq \max\{|\log\mathbb{C}_i|, |\log\mathbb{E}_i|\} \quad \text{for each } i \in J_m,$$
$$\max_{t\in[0,T]}|y_k(t)| \leq \max\{|\log\mathbb{D}_k|, |\log\mathbb{F}_k|\} \quad \text{for each } k \in J_n,$$

it follows that there exists $r > 0$ independent of ξ such that

$$||z|| \leq r \quad \text{for all } z \in S_1.$$

This completes the proof. □

LEMMA 2.7. *Under conditions* (2.1) *and* (2.2), *the set*

$$S_2 = \{z \in D(\mathcal{L}) \cap X : \mathcal{L}z = \lambda\mathcal{N}_1 z \quad \text{for some } \lambda \in (0,1]\}$$

is bounded in X.

Proof. The proof is entirely similar to the proof of Lemma 2.6. Fix $\lambda \in (0,1]$ and let $z \in D(\mathcal{L}) \cap X$ be a possible solution of $\mathcal{L}z = \lambda\mathcal{N}_1 z$. Then we obtain that $||z|| \leq r$ independently of λ, where r is the identical constant adopted at the end of the proof of Lemma 2.6. □

Proof of Theorem 2.3. It is obvious from the definition of $\mathcal{L}$ that

$$K(\mathcal{L}) = \{z \in X : z(t) = c, t \in \mathbb{R}, c \in \mathbb{R}^{m+n}\},$$

and

$$R(\mathcal{L}) = \left\{z \in X : \int_0^T z(s)\,ds = 0\right\}.$$

Since $R(\mathcal{L})$ is a closed subspace of X and $\dim K(\mathcal{L}) = \operatorname{codim} R(\mathcal{L}) = m+n$, $\mathcal{L}$ is a Fredholm operator with index 0. Now, we define continuous projections $\mathcal{P}, \mathcal{Q} : X \to X$ as

$$\mathcal{P}z = \mathcal{Q}z = \frac{1}{T}\int_0^T z(s)\,ds \quad \text{for each } z \in X.$$

Then $K(\mathcal{L}) = R(\mathcal{Q})$, and we can choose identity $\mathcal{I}$ as a bijection $\mathcal{J} : K(\mathcal{L}) \to R(\mathcal{Q})$. We denote by $\mathcal{K}_{\mathcal{P}} : R(\mathcal{L}) \to D(\mathcal{L}) \cap K(\mathcal{P})$ the inverse operator of the restriction of $\mathcal{L}$ to $D(\mathcal{L}) \cap K(\mathcal{P})$, that is,

$$\mathcal{K}_{\mathcal{P}} z = \int_0^t z(s)\, ds - \frac{1}{T} \int_0^T \int_0^t z(s)\, ds\, dt \quad \text{for each } z \in R(\mathcal{L}).$$

Let $z = z^* \in K(\mathcal{L})$ be a solution of $\mathcal{Q}\mathcal{N}_1 z = 0$, that is, constant vector value function $z(t) = z^* = (x_1^*, \ldots, x_m^*, y_1^*, \ldots, y_n^*)^{\mathbb{T}} \in \mathbb{R}^{m+n}$ for all $t \in [0, \infty)$ satisfies

$$\overline{\alpha}_i - \overline{a}_{ii} \exp\{x_i^*\} = 0 \quad \text{for each } i \in J_m,$$
$$-\overline{\beta}_k + \sum_{i=1}^m \overline{c}_{ki} \exp\{x_i^*\} - \overline{d}_{kk} \exp\{y_k^*\} = 0 \quad \text{for each } k \in J_n.$$

Then, by using (2.1) and (2.2), we have

$$x_i^* = \log\left\{\frac{\overline{\alpha}_i}{\overline{a}_{ii}}\right\} \quad \text{for each } i \in J_m, \tag{2.23}$$

$$y_k^* = \log\left\{\frac{1}{\overline{d}_{kk}}\left(-\overline{\beta}_k + \sum_{i=1}^m \overline{c}_{ki}\frac{\overline{\alpha}_i}{\overline{a}_{ii}}\right)\right\} \quad \text{for each } k \in J_n. \tag{2.24}$$

It is obvious from (2.23) and (2.24) that z^* is the unique solution belonging to $K(\mathcal{L})$ of $\mathcal{Q}\mathcal{N}_1 z = 0$.

We define mappings $\mathcal{H}_\xi : X \to X$ by

$$\mathcal{H}_\xi = \mathcal{N}_1 + \xi \mathcal{N}_2 \quad \text{for each } \xi \in [0,1].$$

We note that $\mathcal{H}_0 = \mathcal{N}_1$ and $\mathcal{H}_1 = \mathcal{N}$. Then, for all $\xi \in [0,1]$, we can see that $\mathcal{Q}\mathcal{H}_\xi(\overline{\Omega})$ is bounded for any bounded open set $\Omega \subset X$ and mapping $\mathcal{K}_{\mathcal{P}}(\mathcal{I} - \mathcal{Q})\mathcal{H}_\xi$ is compact by Ascoli-Arzela's theorem. Therefore, $\mathcal{H}_\xi$ is a homotopy of $\mathcal{L}$-compact mappings on X. Here, we fix $r_0 > 0$ such that

$$r_0 > \max\left\{ \|z^*\|, \sup_{z \in S_1 \cup S_2} \|z\| \right\},$$

where S_1 and S_2 are defined in Lemma 2.6 and 2.7 respectively. We put $\Omega = B_{r_0}(0) \subset X$. We also define mapping $\mathcal{F} : (D(\mathcal{L}) \cap \overline{\Omega}) \times [0,1] \to X$ by

$$\mathcal{F}(z, \xi) = \mathcal{L}z - \mathcal{H}_\xi z \quad \text{for each } z \in D(\mathcal{L}) \cap \overline{\Omega} \text{ and each } \xi \in [0,1].$$

Then, by Lemma 2.6, we find that $\mathcal{L}z \neq \mathcal{H}_\xi z$ for all $z \in D(\mathcal{L}) \cap \partial\Omega$ and all $\xi \in [0,1]$, it follows from the homotopy invariance property of Leray-Schauder degree that

$$D_{\mathcal{L}}(\mathcal{F}(\,.\,,1), \Omega) = D_{\mathcal{L}}(\mathcal{F}(\,.\,,0), \Omega).$$

Furthermore, by Lemma 2.7, $\mathcal{N}_1$ and Ω satisfy conditions (i) and (ii) of Lemma 2.1. And then we find

$$D_{\mathcal{L}}\left(\mathcal{F}(\,.\,,0),\Omega\right) = \deg\left(-\mathcal{J}^{-1}\mathcal{Q}\mathcal{N}_1|_{K(\mathcal{L})}, K(\mathcal{L})\cap\Omega, 0\right).$$

We recall that $\mathcal{J} = \mathcal{I}$ and $K(\mathcal{L}) \simeq \mathbb{R}^{m+n}$. Mapping $\mathcal{J}^{-1}\mathcal{Q}\mathcal{N}_1|_{K(\mathcal{L})} : K(\mathcal{L}) \to K(\mathcal{L})$ is given by

$$\begin{pmatrix} x_1 \\ \vdots \\ x_m \\ y_1 \\ \vdots \\ y_n \end{pmatrix} \mapsto \begin{pmatrix} \bar{\alpha}_1 \\ \vdots \\ \bar{\alpha}_m \\ -\bar{\beta}_1 \\ \vdots \\ -\bar{\beta}_n \end{pmatrix} - \begin{pmatrix} \bar{a}_{11} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \bar{a}_{mm} & 0 & \cdots & 0 \\ -\bar{c}_{11} & \cdots & -\bar{c}_{1m} & \bar{d}_{11} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\bar{c}_{n1} & \cdots & -\bar{c}_{nm} & 0 & \cdots & \bar{d}_{nn} \end{pmatrix} \begin{pmatrix} \exp\{x_1\} \\ \vdots \\ \exp\{x_m\} \\ \exp\{y_1\} \\ \vdots \\ \exp\{y_n\} \end{pmatrix}$$

for each $z = (x_1, \ldots, x_m, y_1, \ldots, y_n)^{\mathrm{T}} \in K(\mathcal{L})$. While, we have already stated explicitly that z^* is the unique solution belonging to $K(\mathcal{L})\cap\Omega$ of $\mathcal{Q}\mathcal{N}_1 z = 0$ in the last paragraph. Thus we obtain

$$\begin{aligned} &\deg(-\mathcal{J}^{-1}\mathcal{Q}\mathcal{N}_1|_{K(\mathcal{L})}, K(\mathcal{L})\cap\Omega, 0) \\ &= \operatorname{sgn}\left\{(-1)^{m+n}\prod_{i=1}^{m}(-\bar{a}_{ii})\prod_{k=1}^{n}(-\bar{d}_{kk})\exp\left\{\sum_{i=1}^{m}x_i^* + \sum_{k=1}^{n}y_k^*\right\}\right\} \\ &= (-1)^{2m+2n} = 1. \end{aligned}$$

Hence it follows that $D_{\mathcal{L}}\left(\mathcal{F}(\,.\,.1),\Omega\right) = 1$, by using Lemma 2.1 again, this implies that $\mathcal{L}z = \mathcal{N}z$ has at least one solution $z \in D(\mathcal{L})\cap\Omega$. Therefore, problem (LV) has at least one positive T-periodic solution. □

COROLLARY 2.8. *Under assumptions* (2.1) *and* (2.2) *with* $T = 0$, *the autonomous system corresponding to* (LV) *replaced all of the periodic parameters and delays by nonnegative constants has a positive equilibrium.*

Proof. By the similar arguments to Lemma 2.6 and 2.7, we have that $\mathcal{Q}\mathcal{H}_\xi \neq 0$ for $\xi \in [0,1]$ and all $z \in K(\mathcal{L})\cap\partial\Omega$, where $\mathcal{H}_\xi$ and Ω are the same as those in the proof of Theorem 2.3. By using the homotopy invariance property of Brouwer degree, we obtain $\deg(\mathcal{Q}\mathcal{N}|_{K(\mathcal{L})}, K(\mathcal{L})\cap\Omega, 0) = \deg(\mathcal{Q}\mathcal{N}_1|_{K(\mathcal{L})}, K(\mathcal{L})\cap\Omega, 0) = (-1)^{m+n}$ from the conclusion at the end of the proof of Theorem 2.3. This implies that $\mathcal{Q}\mathcal{N}z = 0$ has at least one solution $z \in K(\mathcal{L})\cap\Omega$, which is coincident with a positive equilibrium of the autonomous system corresponding to (LV). We may choose 0 as T of (2.1) and (2.2) because the autonomous system is obviously independent of t. □

3. Global attractivity

Before stating our result of the present section, we define the following concept concerning the stability of a solution.

DEFINITION 3.1. Let $w^*(t) = (u_1^*(t), \ldots, u_m^*(t), v_1^*(t), \ldots, v_n^*(t))^{\mathbb{T}}$ be a positive T-periodic solution of (LV). If

$$\lim_{t\to\infty} \sum_{i=1}^{m} |u_i(t) - u_i^*(t)| = 0 \quad \text{and} \quad \lim_{t\to\infty} \sum_{k=1}^{n} |v_k(t) - v_k^*(t)| = 0$$

for any solution $w(t) = (u_1(t), \ldots, u_m(t), v_1(t), \ldots, v_n(t))^{\mathbb{T}}$ of (LV) satisfying initial condition (C1), then $w^*(t)$ is said to be globally attractive.

In the present section, we consider the special case where $\sigma_{ij}(t)$, $\tau_{ik}(t)$, $\mu_{ki}(t)$ and $\nu_{kl}(t)$ are constant functions, that is, for every $i, j \in J_m$ and every $k, l \in J_n$,

(C3) $\quad \sigma_{ij}(t) \equiv \sigma_{ij},\ \tau_{ik}(t) \equiv \tau_{ik},\ \mu_{ki}(t) \equiv \mu_{ki}$ and $\nu_{kl}(t) \equiv \nu_{kl}$ for all $t \in \mathbb{R}$,

and (LV) is free from the delays of intra-specific competition, namely,

(C4) $\quad \sigma_{ii} = 0$ for each $i \in J_m$ and $\nu_{kk} = 0$ for each $k \in J_n$.

Throughout the rest of this paper, we assume that (C3) and (C4) hold.

PROPOSITION 3.2. *In addition to the conditions of Theorem 2.3, suppose that the following conditions hold:*

$$\grave{a}_{ii} > \sum_{\substack{j=1\\ j\neq i}}^{m} \acute{a}_{ji} + \sum_{k=1}^{n} \acute{c}_{ki} \quad \textit{for each } i \in J_m, \tag{3.1}$$

$$\grave{d}_{kk} > \sum_{i=1}^{m} \acute{b}_{ik} + \sum_{\substack{l=1\\ l\neq k}}^{n} \acute{d}_{lk} \quad \textit{for each } k \in J_n, \tag{3.2}$$

where $\acute{f} = \max\{f(t) : t \in [0,T]\}$ *and* $\grave{f} = \min\{f(t) : t \in [0,T]\}$ *for a given T-periodic function* $f \in C(\mathbb{R}; [0,\infty))$. *Then the positive T-periodic solution of* (LV) *is globally attractive.*

Proof. Let $w(t)$ be a solution of (LV) with (C1) and let $w^*(t)$ be a positive T-periodic solution of (LV). We denote by D^+ the right-hand derivative. Then we find for $t \in [0,\infty)$,

$$\begin{aligned} D^+|\log u_i(t) - \log u_i^*(t)| \le -\grave{a}_{ii}|u_i(t) - u_i^*(t)| & \\ + \sum_{\substack{j=1\\ j\neq i}}^{m} \acute{a}_{ij}|u_j(t-\sigma_{ij}) - u_j^*(t-\sigma_{ij})| & \\ + \sum_{k=1}^{n} \acute{b}_{ik}|v_k(t-\tau_{ik}) - v_k^*(t-\tau_{ik})| & \end{aligned}$$

for each $i \in J_m$ and

$$D^+|\log v_k(t) - \log v_k^*(t)| \le -\grave{d}_{kk}|v_k(t) - v_k^*(t)| + \sum_{i=1}^{m} \acute{c}_{ki}|u_i(t-\mu_{ki}) - u_i^*(t-\mu_{ki})| + \sum_{\substack{l=1\\l\neq k}}^{n} \acute{d}_{kl}|v_l(t-\nu_{kl}) - v_l^*(t-\nu_{kl})|$$

for each $k \in J_n$. Define Lyapunov function $L(t)$ for $t \in [0,\infty)$ by

$$\begin{aligned} L(t) = &\sum_{i=1}^{m} |\log u_i(t) - \log u_i^*(t)| + \sum_{k=1}^{n} |\log v_k(t) - \log v_k^*(t)| \\ &+ \sum_{i=1}^{m}\left\{\sum_{\substack{j=1\\j\neq i}}^{m} \acute{a}_{ij}\int_{t-\sigma_{ij}}^{t} |u_j(s) - u_j^*(s)|\,ds + \sum_{k=1}^{n} \acute{b}_{ik}\int_{t-\tau_{ki}}^{t} |v_k(s) - v_k^*(s)|\,ds\right\} \\ &+ \sum_{k=1}^{n}\left\{\sum_{i=1}^{m} \acute{c}_{ki}\int_{t-\mu_{ki}}^{t} |u_i(s) - u_i^*(s)|\,ds + \sum_{\substack{l=1\\l\neq k}}^{n} \acute{d}_{kl}\int_{t-\nu_{kl}}^{t} |v_l(s) - v_l^*(s)|\,ds\right\}. \end{aligned}$$

Then we have for $t \in [0,\infty)$,

$$\begin{aligned} D^+L(t) \le &\sum_{i=1}^{m}\Bigg\{ -\grave{a}_{ii}|u_i(t) - u_i^*(t)| + \sum_{\substack{j=1\\j\neq i}}^{m} \acute{a}_{ij}|u_j(t-\sigma_{ij}) - u_j^*(t-\sigma_{ij})| \\ &\qquad + \sum_{k=1}^{n} \acute{b}_{ik}|v_k(t-\tau_{ik}) - v_k^*(t-\tau_{ik})| \\ &\qquad + \sum_{\substack{j=1\\j\neq i}}^{m} \acute{a}_{ij}\left[|u_j(t) - u_j^*(t)| - |u_j(t-\sigma_{ij}) - u_j^*(t-\sigma_{ij})|\right] \\ &\qquad + \sum_{k=1}^{n} \acute{c}_{ki}\left[|u_i(t) - u_i^*(t)| - |u_i(t-\mu_{ki}) - u_i^*(t-\mu_{ki})|\right]\Bigg\} \\ &+ \sum_{k=1}^{n}\Bigg\{ -\grave{d}_{kk}|v_k(t) - v_k^*(t)| + \sum_{i=1}^{m} \acute{c}_{ki}|u_i(t-\mu_{ki}) - u_i^*(t-\mu_{ki})| \\ &\qquad + \sum_{\substack{l=1\\l\neq k}}^{n} \acute{d}_{kl}|v_l(t-\nu_{kl}) - v_l^*(t-\nu_{kl})| \\ &\qquad + \sum_{i=1}^{m} \acute{b}_{ik}\left[|v_k(t) - v_k^*(t)| - |v_k(t-\tau_{ki}) - v_k^*(t-\tau_{ki})|\right] \\ &\qquad + \sum_{\substack{l=1\\l\neq k}}^{n} \acute{d}_{kl}\left[|v_l(t) - v_l^*(t)| - |v_l(t-\nu_{kl}) - v_l^*(t-\nu_{kl})|\right]\Bigg\} \end{aligned}$$

$$=\sum_{i=1}^{m}\left\{\left(-\grave{a}_{ii}+\sum_{\substack{j=1\\j\neq i}}^{m}\acute{a}_{ji}+\sum_{k=1}^{n}\acute{c}_{ki}\right)|u_i(t)-u_i^*(t)|\right\}$$
$$+\sum_{k=1}^{n}\left\{\left(-\grave{d}_{kk}+\sum_{i=1}^{m}\acute{b}_{ik}+\sum_{\substack{l=1\\l\neq k}}^{n}\acute{d}_{lk}\right)|v_k(t)-v_k^*(t)|\right\}.$$

By using (3. 1) and (3. 2), we have there exists $\varepsilon>0$ such that

(3. 3)
$$\varepsilon=\min\left\{\grave{a}_{ii}-\sum_{\substack{j=1\\j\neq i}}^{m}\acute{a}_{ji}-\sum_{k=1}^{n}\acute{c}_{ki},\ \grave{d}_{kk}-\sum_{i=1}^{m}\acute{b}_{ik}-\sum_{\substack{l=1\\l\neq k}}^{n}\acute{d}_{lk}: i\in J_m,\ k\in J_n\right\}.$$

And then

$$D^+L(t)\le-\varepsilon\left\{\sum_{i=1}^{m}|u_i(t)-u_i^*(t)|+\sum_{k=1}^{n}|v_k(t)-v_k^*(t)|\right\}\quad\text{for } t\in[0,\infty).$$

Integrating the above inequality over $[0,t]$, we find for $t\in[0,\infty)$,

$$L(t)+\varepsilon\int_0^t\left\{\sum_{i=1}^{m}|u_i(s)-u_i^*(s)|+\sum_{k=1}^{n}|v_k(s)-v_k^*(s)|\right\}ds\le L(0), \tag{3. 4}$$

it follows from $L(0)<\infty$ that

$$\limsup_{t\to\infty}\int_0^t\left\{\sum_{i=1}^{m}|u_i(s)-u_i^*(s)|+\sum_{k=1}^{n}|v_k(s)-v_k^*(s)|\right\}ds\le\frac{L(0)}{\varepsilon}<\infty.$$

This implies that $\sum_{i=1}^{m}|u_i(t)-u_i^*(t)|+\sum_{k=1}^{n}|v_k(t)-v_k^*(t)|$ is integrable on $[0,\infty)$.

While, we find from the definition of $L(t)$ and (3. 4) that for $t\in[0,\infty)$,

$$\sum_{i=1}^{m}|\log u_i(t)-\log u_i^*(t)|+\sum_{k=1}^{n}|\log v_k(t)-\log v_k^*(t)|\le L(t)\le L(0).$$

Clearly, we have for $t\in[0,\infty)$, $|\log u_i(t)-\log u_i^*(t)|\le L(0)$ for each $i\in J_m$ and $|\log v_k(t)-\log v_k^*(t)|\le L(0)$ for each $k\in J_n$. That is, for $t\in[0,\infty)$,

$$u_i^*(t)\exp\{-L(0)\}\le u_i(t)\le u_i^*(t)\exp\{L(0)\}\quad\text{for each } i\in J_m,$$
$$v_k^*(t)\exp\{-L(0)\}\le v_k(t)\le v_k^*(t)\exp\{L(0)\}\quad\text{for each } k\in J_n.$$

Since $w^*(t)$ is a positive T-periodic solution of (LV), using positive constants $\mathbb{C}_i$, $\mathbb{D}_k$, $\mathbb{E}_i$ and $\mathbb{F}_k$ of Section 2, we get from the inequalities above that for $t \in [0, \infty)$,

$$0 < \mathbb{E}_i \exp\{-L(0)\} \leq u_i(t) \leq \mathbb{C}_i \exp\{L(0)\} < \infty \quad \text{for each } i \in J_m,$$
$$0 < \mathbb{F}_k \exp\{-L(0)\} \leq v_k(t) \leq \mathbb{D}_k \exp\{L(0)\} < \infty \quad \text{for each } k \in J_n.$$

Now that $u_i(t)$ and $v_k(t)$ satisfy (LV), we obtain that $\dot{u}_i(t), \dot{u}_i^*(t)$ and $\dot{v}_k(t), \dot{v}_k^*(t)$ are uniformly bounded on $[0, \infty)$ for every $i \in J_m$ and every $k \in J_n$ respectively. Therefore, nonnegative function $\sum_{i=1}^m |u_i(t) - u_i^*(t)| + \sum_{k=1}^n |v_k(t) - v_k^*(t)|$ is integrable and uniformly continuous on $[0, \infty)$, it follows that

$$\lim_{t\to\infty} \sum_{i=1}^m |u_i(t) - u_i^*(t)| = 0 \quad \text{and} \quad \lim_{t\to\infty} \sum_{k=1}^n |v_k(t) - v_k^*(t)| = 0.$$

This completes the proof. □

REMARK 3.3. From a viewpoint of biology, if there exist $\underline{M} > 0$ and $\overline{M} > 0$ such that for any solution $w(t)$ of (LV) with (C1),

$$\underline{M} \leq \liminf_{t\to\infty} w_i(t) \leq \limsup_{t\to\infty} w_i(t) \leq \overline{M} \quad \text{for all } i \in J_{m+n},$$

then system (LV) is said to be permanent. Proposition 3.2 indicates the positive T-periodic solution of (LV) is unique and attracts all orbits with (C1). Thus, one can see that (LV) attains permanence under conditions (3. 1) and (3. 2). Actually, by the argument of Lemma 2.6, we may take $\underline{M} = \min\{\mathbb{E}_i, \mathbb{F}_k : i \in J_m, k \in J_n\}$ and $\overline{M} = \max\{\mathbb{C}_i, \mathbb{D}_k : i \in J_m, k \in J_n\}$.

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