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NOTES ON STRICTLY ALMOST KÄHLER EINSTEIN MANIFOLDS OF DIMENSION FOUR

By

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Abstract. In [4], the authors defined some pair of 1-forms and a unitary frame field, called a special adapted pair, to study the integrability of almost complex structure of four-dimensional almost Kähler Einstein manifolds. In this paper, we define another kind of special adapted pair. These two pairs have similar properties, but the latter seems more useful in our discussion. Making use of the latter one, we show a result (§ 3, Proposition 3) on strictly almost Kähler Einstein manifolds of dimension four, and, as an application, we will give an another proof of the result by J. Armstrong [1].

1. Introduction

An almost Hermitian manifold $M = (M, J, g)$ is called an almost Kähler manifold if the corresponding Kähler form Ω defined by $\Omega(X, Y) = g(X, JY)$ is closed, or equivalently $\mathfrak{S}_{X,Y,Z}\, g((\nabla_X J)Y, Z) = 0$ for any vector fields X, Y, Z on M, where $\mathfrak{S}_{X,Y,Z}$ denotes the cyclic sum with respect to X, Y, Z. Therefore, a Kähler manifold ($\nabla J = 0$) is necessarily an almost Kähler manifold. It is well-known that an almost Kähler manifold with integrable almost complex structure is Kählerian. A non-Kähler almost Kähler manifold is called a strictly almost Kähler manifold. Examples of strictly almost Kähler manifold are constructed by many authors. In the framework of the study concerning the integrability of the complex structure of almost Kähler manifolds, the following conjecture by S.I Goldberg ([2]) is interesting.

CONJECTURE. A compact almost Kähler Einstein manifold is Kählerian.

K. Sekigawa ([6]) proved that the above conjecture is true if the scalar curvature τ of M is non-negative. Many progresses have been made under some additional conditions.

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2. Preliminaries

Let $M = (M, J, g)$ be a four-dimensional almost Hermitian manifold with the almost Hermitian structure (J, g). The Kähler form Ω of M is defined by $\Omega(X,Y) = g(X, JY)$ for X, $Y \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ is the Lie algebra of all smooth vector fields on M. We assume that M is oriented by the volume form $dV = \Omega^2/2$. We denote by ∇, R, ρ, and τ the Riemannian connection, the curvature tensor, the Ricci tensor and the scalar curvature of M, respectively. We assume that the curvature tensor is defined by $R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$ for X, Y, $Z \in \mathfrak{X}(M)$. We denote by ρ^* the Ricci $*$-tensor of M defined by

$$(2.1) \qquad \rho^*(x,y) = \frac{1}{2} \text{ trace of } (z \mapsto R(x, Jy)Jz)$$

for x, y, $z \in T_pM$, the tangent space of M at $p \in M$. The Ricci $*$-tensor satisfies

$$(2.2) \qquad \rho^*(x,y) = \rho^*(Jy, Jx)$$

for any x, $y \in T_pM$, $p \in M$. We note that if M is Kählerian, the Ricci tensor and the Ricci $*$-tensor coincide on M. The $*$-scalar curvature τ^* of M is the trace of the linear endomorphism Q^* defined by $g(Q^*x, y) = \rho^*(x,y)$ for x, $y \in T_pM$, $p \in M$. An almost Hermitian manifold M is called a weakly $*$-Einstein manifold if $\rho^* = \lambda^* g$ ($\lambda^* = \tau^*/4$), and a weakly $*$-Einstein manifold with constant $*$-scalar curvature is called a $*$-Einstein manifold. The following identity holds for any four-dimensional almost Hermitian manifold:

$$(2.3) \qquad \frac{1}{2}\{\rho(x,y) + \rho(Jx, Jy)\} - \frac{1}{2}\{\rho^*(x,y) + \rho^*(y,x)\} = \frac{\tau - \tau^*}{4} g(x,y)$$

for any x, $y \in T_pM$, $p \in M$.

Let $\wedge^2 M$ be the vector bundle of all 2-forms on M. The bundle $\wedge^2 M$ inherits a natural inner product g and we have an orthogonal decomposition

$$\wedge^2 M = \mathbb{R}\Omega \oplus LM \oplus \wedge_0^{1,1} M,$$

where LM (resp. $\wedge_0^{1,1}M$) is the bundle of J-skew-invariant (J-invariant) 2-forms on M perpendicular to Ω. We can identify the subbundle $\mathbb{R}\Omega \oplus LM$ (resp. $\wedge_0^{1,1}M$) with the bundle of self-dual (resp. anti-self-dual) 2-forms on M.

In the sequel, we assume that $M = (M, J, g)$ is a four-dimensional almost Kähler manifold. Then, we have

$$(2.4) \qquad \nabla\Omega = \alpha \otimes \Phi - J\alpha \otimes J\Phi,$$

where α is a local 1-form and $\{\Phi, J\Phi\}$ is a local orthonormal flame field of the bundle LM. Thus, we have

$$\|\alpha\|^2 = \frac{1}{2}\|\nabla\Omega\|^2 = \frac{1}{4}\|\nabla J\|^2 = \frac{\tau^* - \tau}{2}.$$

From this equality, we find that M is Kählerian if and only if $\tau^* - \tau = 0$ on M.

In this paper, for any orthonormal basis (resp. any local orthonormal frame field) $\{e_1, e_2, e_3, e_4\}$ of a point $p \in M$ (resp. on a neighborhood of p), we shall adopt the following notational convention:

$$\begin{aligned} J_{ij} &= g(Je_i, e_j), \quad \Gamma_{ijk} = g(\nabla_{e_i} e_j, e_k), \\ R_{ijkl} &= g(R(e_i, e_j)e_k, e_l), \ \ldots \ , R_{\bar{i}\bar{j}\bar{k}\bar{l}} = g(R(Je_i, Je_j)Je_k, Je_l), \\ \rho_{ij} &= \rho(e_i, e_j), \ \ldots \ , \rho_{\bar{i}\bar{j}} = \rho(Je_i, Je_j), \\ \rho^*_{ij} &= \rho^*(e_i, e_j), \ \ldots \ , \rho^*_{\bar{i}\bar{j}} = \rho^*(Je_i, Je_j), \\ \nabla_i J_{jk} &= g((\nabla_{e_i} J)e_j, e_k), \ \ldots \ , \nabla_{\bar{i}} J_{\bar{j}\bar{k}} = g((\nabla_{Je_i} J)Je_j, Je_k), \end{aligned} \tag{2.5}$$

and so on, where the latin indices run over the range 1, 2, 3, 4. We define a function G on M by

$$G = \sum_{i,j=1}^{4} (\rho^*_{ij} - \rho^*_{ji})^2.$$

Then, from (2.2), we have

$$G = 16\left\{(\rho^*_{13})^2 + (\rho^*_{14})^2\right\}. \tag{2.6}$$

3. Special adapted pair of the second kind

Let $M = (M, J, g)$ be a four-dimensional compact strictly almost Kähler Einstein manifold and put $M_0 = \{p \in M \mid \tau^* - \tau > 0 \text{ at } p\}$. Then, M_0 is a no-empty open submanifold of M.

Since M is a Einstein manifold, the Riemannian metric g is of class C^ω. Thus, we can choose a local C^ω-orthonormal frame field $\{e_i\}$ on a neighborhood of any point of M. Then, we may observe that the functions J_{ij} $(i, j = 1, \ldots, 2n)$ satisfy a certain system of elliptic partial differential equations of second order whose coefficients are C^ω-functions (we omit the details). Hence, from the well-known regularity theorem, we may assume that the almost Hermitian structure (J, g) is of class C^ω.

First, we recall the notion of a adapted pair defined in [4]. For the 1-form α of (2.4), we denote by $\mathcal{D}$ the 2-dimensional J-invariant distribution on M_0 spaned by $\{\alpha^*, J\alpha^*\}$ and by $\mathcal{D}^\perp$ the orthogonal complement of $\mathcal{D}$ in the tangent bundle of M, where α^* is the dual vector field of α. We choose a local unitary frame field $\{e_1, e_2 = Je_1, e_3, e_4 = Je_3\}$ on a neighborhood of any point of M_0 such that e_1, $e_2 \in \mathcal{D}$ and e_3, $e_4 \in \mathcal{D}^\perp$ and put $\{e^i\}$ the dual basis of $\{e_i\}$. Then, we have

$$\nabla\Omega = \alpha \otimes \frac{1}{\sqrt{2}}(e^1 \wedge e^3 - e^2 \wedge e^4) - J\alpha \otimes \frac{1}{\sqrt{2}}(e^1 \wedge e^4 + e^2 \wedge e^3).$$

We call the pair $\{\alpha, J\alpha, \{e_i\}\}$ an adapted pair to $\nabla\Omega$. If $\{\alpha, J\alpha, \{e_i\}\}$ is an adapted pair to $\nabla\Omega$, we have $\alpha_i = \alpha(e_i) = 0$ $(i = 3, 4)$ and hence

$$\|\alpha\|^2 = \alpha_1^2 + \alpha_2^2 = \frac{\tau^* - \tau}{2}. \tag{3.1}$$

REMARK 1. We remark that such a pair is not uniquely determined. In fact, for an adapted pair $\{\alpha, J\alpha, \{e_i\}\}$ and arbitrary local functions θ and φ, we define $\alpha(\theta,\varphi) = (\cos\theta)\alpha - (\sin\theta)J\alpha$, $J\alpha(\theta,\varphi) = (\sin\theta)\alpha + (\cos\theta)J\alpha$, $e_1(\theta,\varphi) = (\cos\varphi)e_1 - (\sin\varphi)e_2$, $e_2(\theta,\varphi) = (\sin\varphi)e_1 + (\cos\varphi)e_2$, $e_3(\theta,\varphi) = (\cos(\theta+\varphi))e_3 + (\sin(\theta+\varphi))e_4$, $e_4(\theta,\varphi) = -(\sin(\theta+\varphi))e_3 + (\cos(\theta+\varphi))e_4$. Then, $\{\alpha(\theta,\varphi), J\alpha(\theta,\varphi), \{e_i(\theta,\varphi)\}\}$ is again an adapted pair ([4]).

Now, let $\{\alpha, J\alpha, \{e_i\}\}$ be a local C^ω-adapted pair to $\nabla\Omega$. Then, from (2.38) and (2.43) of [4], we have

(3.2)
$$\begin{pmatrix} \Gamma_{132}\alpha_1 - \Gamma_{131}\alpha_2 & \Gamma_{131}\alpha_1 + \Gamma_{132}\alpha_2 \\ -(\Gamma_{131}\alpha_1 + \Gamma_{132}\alpha_2) & \Gamma_{132}\alpha_1 - \Gamma_{131}\alpha_2 - \frac{\sqrt{2}}{4}(\tau^* - \tau) \end{pmatrix} \begin{pmatrix} \rho^*_{13} \\ \rho^*_{14} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The determinant of the matrix in the left-hand side of (3.2) is

$$\begin{aligned} &(\Gamma_{132}\alpha_1 - \Gamma_{131}\alpha_2)\left\{\Gamma_{132}\alpha_1 - \Gamma_{131}\alpha_2 - \frac{\sqrt{2}}{4}(\tau^* - \tau)\right\} + (\Gamma_{131}\alpha_1 + \Gamma_{132}\alpha_2)^2 \\ &= (\Gamma_{131}^2 + \Gamma_{132}^2)(\alpha_1^2 + \alpha_2^2) - \frac{\sqrt{2}}{4}(\tau^* - \tau)(\Gamma_{132}\alpha_1 - \Gamma_{131}\alpha_2) \\ &= \frac{\tau^* - \tau}{2}\left\{(\Gamma_{131}^2 + \Gamma_{132}^2) - \frac{\sqrt{2}}{2}(\Gamma_{132}\alpha_1 - \Gamma_{131}\alpha_2)\right\} \end{aligned} \tag{3.3}$$

We assume that (3.3) does not vanish at some point $p_0 \in M_0$. Then, $G = 0$ on a neighborhood of p_0. Since G is real-analytic, $G = 0$ on M, and hence, M is weakly $*$-Einstein.

In the sequel, we assume that M is not weakly $*$-Einstein. Then, we may assume that

$$(\Gamma_{131}^2 + \Gamma_{132}^2) - \frac{\sqrt{2}}{2}(\Gamma_{132}\alpha_1 - \Gamma_{131}\alpha_2) = 0 \tag{3.4}$$

holds on M_0.

For a local C^ω-adapted pair $\{\alpha, J\alpha, \{e_i\}\}$, there exists a local C^ω-function ζ satisfying

$$\alpha = \|\alpha\|\{(\cos\zeta)e^1 - (\sin\zeta)e^2\}.$$

If we put

$$\begin{aligned}
&\alpha(0,\zeta) = \alpha, \quad J\alpha(0,\zeta) = J\alpha,\\
&e_1(0,\zeta) = (\cos\zeta)e_1 - (\sin\zeta)e_2, \quad e_2(0,\zeta) = Je_1(0,\zeta),\\
&e_3(0,\zeta) = (\cos\zeta)e_3 + (\sin\zeta)e_3, \quad e_4(0,\zeta) = Je_3(0,\zeta),
\end{aligned}$$

then $\{\alpha(0,\zeta), J\alpha(0,\zeta), \{e_i(0,\zeta)\}\}$ is an adapted pair satisfying

$$\alpha(0,\zeta) = \|\alpha\|e^1(0,\zeta), \quad J\alpha(0,\zeta) = \|\alpha\|e^2(0,\zeta). \tag{3.5}$$

So, let $\{\alpha, J\alpha, \{e_i\}\}$ be a local C^ω-adapted pair on an open set U in M_0 satisfying (3.5) and define a local C^ω-adapted pair $\{\alpha(\theta,\theta), J\alpha(\theta,\theta), \{e_i(\theta,\theta)\}\}$ by

$$\begin{aligned}
&\alpha(\theta,\theta) = (\cos\theta)\alpha - (\sin\theta)J\alpha, \quad J\alpha(\theta,\theta) = J\alpha(\theta,\theta),\\
&e_1(\theta,\theta) = (\cos\theta)e_1 - (\sin\theta)e_2, \quad e_2(\theta,\theta) = Je_1(\theta,\theta),\\
&e_3(\theta,\theta) = (\cos 2\theta)e_3 + (\sin 2\theta)e_4, \quad e_4(\theta,\theta) = Je_3(\theta,\theta),
\end{aligned} \tag{3.6}$$

where θ is defined by

$$\cos 2\theta = 1 - \frac{2\sqrt{2}}{\|\alpha\|}\Gamma_{132}, \quad \sin 2\theta = -\frac{2\sqrt{2}}{\|\alpha\|}\Gamma_{131}. \tag{3.7}$$

We may note that, from (3.4) and (3.5),

$$\begin{aligned}
&\left(1 - \frac{2\sqrt{2}}{\|\alpha\|}\Gamma_{132}\right)^2 + \left(-\frac{2\sqrt{2}}{\|\alpha\|}\Gamma_{131}\right)^2 = 1 - \frac{4\sqrt{2}}{\|\alpha\|}\Gamma_{132} + \frac{8}{\|\alpha\|^2}(\Gamma_{131}^2 + \Gamma_{132}^2)\\
&= 1 - \frac{4\sqrt{2}}{\|\alpha\|}\Gamma_{132} + \frac{8}{\|\alpha\|^2}\cdot\frac{\sqrt{2}}{2}\Gamma_{132}\|\alpha\| = 1.
\end{aligned}$$

Similarly, for an arbitrary local C^ω-adapted pair $\{\bar\alpha, J\bar\alpha, \{\bar e_i\}\}$ on $\bar U$ with $U \cap \bar U \neq \emptyset$ satisfying (3.5), we define an local C^ω-adapted pair $\{\bar\alpha(\bar\theta,\bar\theta), J\bar\alpha(\bar\theta,\bar\theta), \{\bar e_i(\bar\theta,\bar\theta)\}\}$ on $\bar U$ by

$$\begin{aligned}
&\bar\alpha(\bar\theta,\bar\theta) = (\cos\bar\theta)\bar\alpha - (\sin\bar\theta)J\bar\alpha, \quad J\alpha(\bar\theta,\bar\theta) = J\bar\alpha(\bar\theta,\bar\theta),\\
&\bar e_1(\bar\theta,\bar\theta) = (\cos\bar\theta)\bar e_1 - (\sin\bar\theta)\bar e_2, \quad \bar e_2(\bar\theta,\bar\theta) = J\bar e_1(\bar\theta,\bar\theta),\\
&\bar e_3(\bar\theta,\bar\theta) = (\cos 2\bar\theta)\bar e_3 + (\sin 2\bar\theta)\bar e_4, \quad \bar e_4(\bar\theta,\bar\theta) = J\bar e_3(\bar\theta,\bar\theta),
\end{aligned} \tag{3.8}$$

where $\bar{\theta}$ is defined by

$$\cos 2\bar{\theta} = 1 - \frac{2\sqrt{2}}{\|\alpha\|}\bar{\Gamma}_{132}, \quad \sin 2\bar{\theta} = -\frac{2\sqrt{2}}{\|\alpha\|}\bar{\Gamma}_{131}, \tag{3.9}$$

and $\bar{\Gamma}_{ijk} = g(\nabla_{\bar{e}_i}\bar{e}_j, \bar{e}_k)$. On $U \cap \bar{U}$, there exists a C^ω-function η on $U \cap \bar{U}$ such that

$$\begin{aligned} \bar{\alpha} &= (\cos\eta)\alpha - (\sin\eta)J\alpha, \\ \bar{e}_1 &= (\cos\eta)e_1 - (\sin\eta)e_2, \\ \bar{e}_3 &= (\cos 2\eta)e_3 + (\sin 2\eta)e_4. \end{aligned} \tag{3.10}$$

Thus, form (2.42) of [4], we have

$$\begin{aligned} \bar{\Gamma}_{131} &= \Gamma_{131}\cos 2\eta - \Gamma_{132}\sin 2\eta + \frac{\sqrt{2}}{2}\|\alpha\|\sin\eta\cos\eta \\ &= \left(\frac{\sqrt{2}}{4}\|\alpha\| - \Gamma_{132}\right)\sin 2\eta + \Gamma_{131}\cos 2\eta, \\ \bar{\Gamma}_{132} &= \Gamma_{131}\sin 2\eta + \Gamma_{132}\cos 2\eta + \frac{\sqrt{2}}{2}\|\alpha\|\sin^2\eta \\ &= \frac{\sqrt{2}}{4}\|\alpha\| + \Gamma_{131}\sin 2\eta - \left(\frac{\sqrt{2}}{4}\|\alpha\| - \Gamma_{132}\right)\cos 2\eta \end{aligned} \tag{3.11}$$

on $U \cap \bar{U}$. From (3. 7), (3. 9) and (3. 11), we have

$$\begin{aligned} \cos 2\bar{\theta} &= 1 - \frac{2\sqrt{2}}{\|\alpha\|}\left\{\frac{\sqrt{2}}{4}\|\alpha\| + \Gamma_{131}\sin 2\eta - \left(\frac{\sqrt{2}}{4}\|\alpha\| - \Gamma_{132}\right)\cos 2\eta\right\} \\ &= -\frac{2\sqrt{2}}{\|\alpha\|}(\Gamma_{131}\sin 2\eta + \Gamma_{132}\cos 2\eta) + \cos 2\eta \\ &= \sin 2\theta\sin 2\eta + \cos 2\theta\cos 2\eta \\ &= \cos 2(\theta - \eta), \\ \sin 2\bar{\theta} &= -\frac{2\sqrt{2}}{\|\alpha\|}\left\{\left(\frac{\sqrt{2}}{4}\|\alpha\| - \Gamma_{132}\right)\sin 2\eta + \Gamma_{131}\cos 2\eta\right\} \\ &= -\sin 2\eta + \frac{2\sqrt{2}}{\|\alpha\|}(\Gamma_{132}\sin 2\eta - \Gamma_{131}\cos 2\eta) \\ &= -\cos 2\theta\sin 2\eta + \sin 2\theta\cos 2\eta \\ &= \sin 2(\theta - \eta), \end{aligned}$$

and hence,

$$\bar{\theta} = \theta - \eta + m\pi \tag{3.12}$$

on $U \cap \bar{U}$ for some integer m. Therefore, we obtain

$$(3.13)\quad \begin{aligned} \bar{\alpha}(\bar{\theta},\bar{\theta}) &= (-1)^m\{\cos(\theta-\eta)((\cos\eta)\alpha-(\sin\eta)J\alpha) \\ &\qquad +\sin(\theta-\eta)((\cos\eta)J\alpha+(\sin\eta)\alpha)\} \\ &= (-1)^m\{(\cos\theta)\alpha+(\sin\theta)J\alpha\} = (-1)^m\alpha(\theta,\theta) \\ \bar{e}_1(\bar{\theta},\bar{\theta}) &= (-1)^m e_1(\theta,\theta), \\ \bar{e}_3(\bar{\theta},\bar{\theta}) &= \cos 2(\theta-\eta)\{(\cos 2\eta)e_3+(\sin 2\eta)e_4\} \\ &\qquad +\sin 2(\theta-\eta)\{(\cos 2\eta)e_4-(\sin 2\eta)e_3\} \\ &= (\cos 2\theta)e_3+(\sin 2\theta)e_4, = e_3(\theta,\theta) \end{aligned}$$

on $U \cap \bar{U}$. So, we have the following.

LEMMA 1. *On $U \cap \bar{U}(\neq \emptyset)$, either*

$$\bar{\alpha}(\bar{\theta},\bar{\theta}) = \alpha(\theta,\theta), \quad \bar{e}_1(\bar{\theta},\bar{\theta}) = e_1(\theta,\theta), \quad \bar{e}_3(\bar{\theta},\bar{\theta}) = e_3(\theta,\theta),$$

or

$$\bar{\alpha}(\bar{\theta},\bar{\theta}) = -\alpha(\theta,\theta), \quad \bar{e}_1(\bar{\theta},\bar{\theta}) = -e_1(\theta,\theta), \quad \bar{e}_3(\bar{\theta},\bar{\theta}) = e_3(\theta,\theta),$$

is holds.

REMARK 2. By choosing $\{-\bar{\alpha},-\bar{e}_1,\bar{e}_3\}$ instead of $\{\bar{\alpha},\bar{e}_1,\bar{e}_3\}$ on $\bar{U}$ if necessary, we may assume that the first case of the Lemma 1 always holds on $U \cap \bar{U}$. Thus, the universal covering of M_0 admits C^ω-absolute parallelism.

We will call the above $\{\alpha(\theta,\theta), J\alpha(\theta,\theta), \{e_i(\theta,\theta)\}\}$ *a special adapted pair of the second kind* and, for the brevity, put

$$\alpha(\theta) = \alpha(\theta,\theta), \quad J\alpha(\theta) = J\alpha(\theta,\theta), \quad e_i(\theta) = e_i(\theta,\theta), \quad i = 1,2,3,4.$$

We will call a special adapted pair defined in [4] *a one of the first kind.*

LEMMA 2. *With respect to a special adapted pair of the second kind $\{\alpha(\theta), J\alpha(\theta), \{e_i(\theta)\}\}$, we have*

$$\Gamma_{131}(\theta) = 0, \quad \Gamma_{132}(\theta) = 0, \quad \rho^*_{14}(\theta) = 0.$$

Proof. From (3.4), (3.7) and (2.42) of [4], we have

$$\Gamma_{131}(\theta) = \Gamma_{131}\cos 2\theta + \left(\frac{\sqrt{2}}{4}\|\alpha\| - \Gamma_{132}\right)\sin 2\theta = 0,$$

$$\begin{aligned} \Gamma_{132}(\theta) &= \Gamma_{131}\sin 2\theta + \Gamma_{132}\cos 2\theta + \frac{\sqrt{2}}{4}\|\alpha\|(1-\cos 2\theta) \\ &= -\frac{2\sqrt{2}}{\|\alpha\|}(\Gamma^2_{131}+\Gamma^2_{132}) + 2\Gamma_{132} = 0. \end{aligned}$$

From these equalities and (2.43) of [4], we have $\rho^*_{14}(\theta) = 0$.

From Lemma 2, for each point p of M_0, we can choose local C^ω-special adapted pair of the second kind $\{\alpha, J\alpha, \{e_i\}\}$ defined on a neighborhood of p satisfying

$$\begin{gathered}\alpha(e_1) = \|\alpha\| = \sqrt{\frac{\tau^* - \tau}{2}}, \quad \alpha(e_2) = 0, \\ \Gamma_{131} = \Gamma_{132} = 0, \quad \rho^*_{13} = \frac{\sqrt{G}}{4}(\geq 0), \quad \rho^*_{14} = 0.\end{gathered} \tag{3.14}$$

Therefore, similar equalities for a special adapted pair of the first kind, obtained by using merely the equalities (3.14), such as (3.14)~(3.17) in [4], are valid for a one of the second kind.

We remark that a special adapted pair of the first kind is not defined at the point (in M_0) where the function G vanishes (cf. [4]). But a special adapted pair of the second kind is defined on a neighborhood of each point of M_0, and we can prove the following.

PROPOSITION 3. *Let $M = (M, J, g)$ be a four-dimensional strictly almost Kähler Einstein manifold which is not weakly $*$-Einstein. Then, $G > 0$ at which $\tau^* - \tau > 0$.*

Proof. We suppose that G vanishes at $p_0 \in M_0$ and let $\{\alpha, J\alpha, \{e_i\}\}$ be a special adapted pair of the second kind on a neighborhood of p_0 which satisfies (3.14). Since $G(p_0) = 0$ is the minimum value of G, we have $e_i\rho^*_{13} = 0$ $(i = 1, 2, 3, 4)$ and $\rho^*_{13} = 0$ at p_0. Then, from the second equality of (3.16) in [4], we have $(\tau^* - \tau)\alpha_1 = (\tau^* - \tau)\|\alpha\| = 0$ at p_0. But this is a contradiction.

From this proposition, we immediately obtain the following.

THEOREM 4 ([1]). *Let $M = (M, J, g)$ be a compact four-dimensional almost Kähler Einstein manifold. Then, $\tau^* - \tau = 0$ holds at some point of M.*

Proof. First, we note that G must vanish somewhere on M ([5], Theorem 1). If $\tau^* - \tau > 0$ holds on M, then $M_0 = M$. Hence, from Proposition 3, $G > 0$ on M. But this is a contradiction.

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