YOKOHAMA MATHEMATICAL
JOURNAL VOL. 51, 2004

ON THE CONVERGENCE OF THE MODIFIED SZASZ-MIRAKYAN OPERATORS

By

ZBIGNIEW WALCZAK

(Received March 25, 2003; Revised November 4, 2003)

Abstract. In this paper we introduce certain modification of the Szasz-Mirakyan operators. We give theorems on the degree of approximation of functions from exponential weighted spaces by introduced operators, using norms of these spaces.

1. Introduction

1.1. Approximation properties of Szasz-Mirakyan operators

(1)

$$S_n(f;x) := e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right), \quad x\in R_0=[0,+\infty),\ n\in N:=\{1,2,\dots\},$$

in exponential weighted spaces C_q were examined in [1]. The space C_q, $q>0$, considered in [1] is associated with the weighted function

$$v_q(x) := e^{-qx}, \qquad x\in R_0, \tag{2}$$

and consists of all real-valued functions f continuous on R_0 for which $v_q f$ is uniformly continuous and bounded on R_0.

Let B_{qr}, $q>0$, $r\in R_2:=[2,+\infty)$, be the set of real-valued functions f defined on R_0 for which $\sup_{x\in R_0} v_{qr}(x)|f(x)|<\infty$. The norm on B_{qr} is defined by

$$\|f\|_{qr}\equiv\|f(\cdot)\|_{qr} := \sup_{x\in R_0} v_{qr}(x)|f(x)|. \tag{3}$$

In [1] was proved that S_n is a positive linear operator from the space C_q into C_p provided that $p>q>0$ and $n>n_0>q/\ln(p/q)$. For $f\in C_q$ was proved that

$$v_p(x)|S_n(f;x)-f(x)|\le M_1(q)\omega_2\left(f;C_q;\sqrt{\frac{x}{n}}\right), \qquad x\in R_0, \quad n>n_0,$$

2000 Mathematics Subject Classification: 41A36
Key words and phrases: Szasz-Mirakyan operator, exponential weighted space, approximation theorem

where $M_1(q) = const. > 0$ and $\omega_2(f; C_q; \cdot)$ is the modulus of smoothness of the order 2.

Thus it follows that if $f \in C_q^1 = \{f \in C_q : f' \in C_q\}$ and $p > q > 0$, then

$$v_p(x)|S_n(f;x) - f(x)| \leq O\left(1/\sqrt{n}\right) \tag{4}$$

for every fixed $x \in R_0$ and for $n > n_0$.

In this paper by $M_k(\alpha, \beta)$ we shall denote suitable positive constants depending only on indicated parameters α, β.

1.2. In this paper we modify the formula (1), i.e. we introduce the following class of operators in the space of all real-valued functions f defined on R_0 for which $\sup_{x \in R_0}(1 + x^2)^{-1} v_{qr}(x)|f(x)| < \infty$.

DEFINITION. Let $r \in R_2$ and $q, s > 0$ be fixed numbers. Define a class of operators $A_n(f; \cdot) \equiv A_n(f; q, r, s; \cdot)$ by the formula

$$A_n(f;q,r,s;x) := e^{-(n^s x+1)^r} \sum_{k=0}^{\infty} \frac{(n^s x + 1)^{rk}}{k!} f\left(\frac{k}{n^s(n^s x+1)^{r-1} + qr}\right), \tag{5}$$
$$x \in R_0, \quad n \in N.$$

In Section 2 we shall prove that $A_n(f; q, r, s)$, $n \in N$, is a positive linear operator from the space C_{qr} (the norm on C_{qr} is given by (3)) into B_{qr}. Moreover we shall prove that the order of approximation of function $f \in C_{qr}$ by $A_n(f; q, r, s)$ for $s > 1/2$ is better than (4).

We shall apply the modulus of continuity of $f \in C_{qr}$ defined by

$$\omega_1(f; C_{qr}; t) := \sup_{0 \leq h \leq t} \|\Delta_h f(\cdot)\|_{qr}, \qquad t \in R_0, \tag{6}$$

where $\Delta_h f(x) := f(x+h) - f(x)$ for $x, h \in R_0$.

From (6) it follows that

$$\lim_{t \to 0+} \omega_1(f; C_{qr}; t) = 0 \tag{7}$$

for every $f \in C_{qr}$, $q > 0$, $r \in R_2$. Moreover if $f \in C_{qr}^1$, then

$$\omega_1(f; C_{qr}; t) \leq M_2 \|f'\|_{qr} t, \qquad (0 < t \leq 1, \quad M_2 \text{ a positive constant}). \tag{8}$$

2. Main results

2.1. In this section we shall give some properties of the above operators, which we shall apply to the proofs of the main theorems.

By elementary calculations we obtain

LEMMA 1. *Let $r \in R_2$ and $q, s > 0$ be fixed numbers. Then for all $x \in R_0$ and $n \in N$ we have*

$$A_n(1; q, r, s; x) = 1, \tag{9}$$

$$\begin{aligned} A_n(t; q, r, s; x) &= \frac{(n^s x + 1)^r}{\alpha_n}, \\ A_n(t^2; q, r, s; x) &= \frac{(n^s x + 1)^{2r}}{\alpha_n^2}\left[1 + \frac{1}{(n^s x + 1)^r}\right], \\ A_n(e^{qrt}; q, r, s; x) &= e^{\beta_n (n^s x + 1)^r}, \\ A_n(te^{qrt}; q, r, s; x) &= \frac{(n^s x + 1)^r}{\alpha_n} e^{qr/\alpha_n} e^{\beta_n (n^s x + 1)^r}, \\ A_n(t^2 e^{qrt}; q, r, s; x) &= \left[\left(\frac{(n^s x + 1)^r}{\alpha_n} e^{qr/\alpha_n}\right)^2 + \frac{(n^s x + 1)^r}{\alpha_n^2} e^{qr/\alpha_n}\right] e^{\beta_n (n^s x + 1)^r}, \end{aligned} \tag{10}$$

where

$$\alpha_n = n^s (n^s x + 1)^{r-1} + qr, \qquad \beta_n = e^{qr/\alpha_n} - 1. \tag{11}$$

Moreover

$$\begin{aligned} A_n(t - x; q, r, s; x) &= \frac{(n^s x + 1)^r}{\alpha_n} - x, \\ A_n((t - x)^2; q, r, s; x) &= \left(\frac{(n^s x + 1)^r}{\alpha_n} - x\right)^2 + \frac{(n^s x + 1)^r}{\alpha_n^2}, \\ A_n((t - x)^2 e^{qrt}; q, r, s; x) & \\ = \left[\left(\frac{(n^s x + 1)^r}{\alpha_n} e^{qr/\alpha_n} - x\right)^2 \right. &+ \left. \frac{(n^s x + 1)^r}{\alpha_n^2} e^{qr/\alpha_n}\right] e^{\beta_n (n^s x + 1)^r} \end{aligned} \tag{12}$$

for $x \in R_0$ and $n \in N$.

Now we shall prove two fundamental lemmas.

LEMMA 2. *Let $q, s > 0$ and $r \in R_2$ be fixed numbers. Then*

$$\|A_n(1/v_{qr}(t); q, r, s; \cdot)\|_{qr} \le e^{qr}, \qquad n \in N. \tag{13}$$

Moreover for every function $f \in C_{qr}$ we have

$$\|A_n(f; q, r, s; \cdot)\|_{qr} \le e^{qr} \|f\|_{qr}, \qquad n \in N. \tag{14}$$

The formula (5) and the inequality (14) show that $A_n(f; q, r, s; \cdot)$, $n \in N$, is a positive linear operator from the space C_{qr} into B_{qr} .

Proof. From (2) and (10) we have

$$v_{qr}(x)A_n(1/v_{qr}(t);q,r,s;x) = e^{\beta_n(n^s x+1)^r - qrx}, \qquad x \in R_0,\ n \in N.$$

By (11) we get

$$\begin{aligned}\beta_n &= \sum_{k=1}^{\infty} \frac{1}{k!}\left(\frac{qr}{n^s(n^s x+1)^{r-1}+qr}\right)^k < \sum_{k=1}^{\infty}\left(\frac{qr}{n^s(n^s x+1)^{r-1}+qr}\right)^k \\ &= \frac{qr}{n^s(n^s x+1)^{r-1}}.\end{aligned}$$

Hence we can write

$$v_{qr}(x)A_n(1/v_{qr}(t);q,r,s;x) \le e^{qr}, \qquad x \in R_0,\ n \in N,$$

which implies (13). The formulas (5) and (3) yield

$$\|A_n(f(t);q,r,s;\cdot)\|_{qr} \le \|f\|_{qr}\|A_n(1/v_{qr}(t);q,r,s;\cdot)\|_{qr}, \qquad n \in N, \quad r \in R_2,$$

for every $f \in C_{qr}$. Applying (13), we obtain (14). This completes the proof of Lemma 2.

LEMMA 3. *For every fixed $q, s > 0$ and $r \in R_2$ there exists a positive constant $M_3(q,r)$ such that*

$$\sup_{x\in R_0} v_{qr}(x)A_n\left((t-x)^2/v_{qr}(t);q,r,s;x\right) \le \frac{M_3(q,r)}{n^{2s}} \qquad \textit{for all } n \in N. \tag{15}$$

Proof. From (10)–(12) we have

$$\begin{aligned} v_{qr}(x)A_n\left((t-x)^2/v_{qr}(t);q,r,s;x\right) &= v_{qr}(x)A_n\left(1/v_{qr}(t);q,r,s;x\right)\cdot \\ &\cdot\left[\left(\frac{(n^s x+1)^r}{\alpha_n}e^{qr/\alpha_n} - x\right)^2 + \frac{(n^s x+1)^r}{\alpha_n^2}e^{qr/\alpha_n}\right]\end{aligned}$$

for $x \in R_0$ and $n \in N$. Observe that

$$\begin{aligned}\left(\frac{(n^s x+1)^r}{\alpha_n}e^{qr/\alpha_n} - x\right)^2 &\le 2\left(\frac{(n^s x+1)^r}{\alpha_n}\left(e^{qr/\alpha_n}-1\right)\right)^2 \\ &+ 2\left(\frac{(n^s x+1)^r}{\alpha_n} - x\right)^2 := W_1 + W_2.\end{aligned}$$

Using (11) and the inequality $e^t - 1 \le te^t$ for $t \in R_0$, we obtain

$$W_1 \le 2e^2\frac{(rq)^2}{n^{4s}(n^s x+1)^{2r-4}} \le \frac{M_4(q,r)}{n^{4s}}, \qquad n \in N.$$

Moreover, for $r \geq 2$, we have

$$W_2 = 2\left(\frac{(n^s x+1)^r - n^s x(n^s x+1)^{r-1} - qrx}{\alpha_n}\right)^2$$
$$= \frac{2}{\alpha_n^2}\left((n^s x+1)^{r-1}\left(1 - \frac{qrx}{(n^s x+1)^{r-1}}\right)\right)^2$$
$$\leq \frac{2}{n^{2s}}\left(1 + \frac{qrx}{(n^s x+1)^{r-1}}\right)^2 \leq \frac{M_5(q,r)}{n^{2s}}$$

and

$$\frac{(n^s x+1)^r}{\alpha_n^2} e^{qr/\alpha_n} \leq \frac{e}{n^{2s}(n^s x+1)^{r-2}} \leq \frac{e}{n^{2s}}.$$

From this and in view of Lemma 2 we get

$$v_{qr}(x) A_n\left((t-x)^2/v_{qr}(t); q, r, s; x\right) \leq \frac{M_6(q,r)}{n^{2s}}$$

for $x \in R_0$ and $n \in N$. This ends the proof of (15).

2.2. Now we shall give approximation theorems for A_n.

THEOREM 1. *For every fixed $q, s > 0$ and $r \in R_2$ there exists a positive constant $M_7(q,r)$ such that for every $f \in C^1_{qr}$ we have*

$$\|A_n(f; q, r, s; \cdot) - f(\cdot)\|_{qr} \leq \frac{M_7(q,r)}{n^s}\|f'\|_{qr} \qquad \textit{for all } n \in N. \tag{16}$$

Proof. Fix $x \in R_0$. Since

$$f(t) - f(x) = \int_x^t f'(u)du, \qquad t \in R_0,$$

for $f \in C^1_{2q}$ and the fixed x, we have by (9) that

$$A_n(f(t); q, r, s; x) - f(x) = A_n\left(\int_x^t f'(u)du; q, r, s; x\right), \qquad n \in N.$$

But by (2) and (3) we have

$$\left|\int_x^t f'(u)du\right| \leq \|f'\|_{qr}\left(\frac{1}{v_{qr}(t)} + \frac{1}{v_{qr}(x)}\right)|t-x|, \qquad t \in R_0.$$

This implies that

$$\begin{aligned} &v_{qr}(x)|A_n(f; q, r, s; x) - f(x)| \\ &\leq \|f'\|_{qr}\left\{A_n(|t-x|; q, r, s; x) + v_{qr}(x) A_n(|t-x|/v_{qr}(t); q, r, s; x)\right\} \end{aligned} \tag{17}$$

for $n \in N$ and $x \in R_0$. By the Hölder inequality, (9) and Lemmas 1 - 3, we obtain

$$A_n\left(|t-x|;q,r,s;x\right) \le \left\{A_n\left((t-x)^2;q,r,s;x\right)A_n\left(1;q,r,s;x\right)\right\}^{1/2} \le \frac{M_8(q,r)}{n^s}$$

and

$$\begin{aligned}
&v_{qr}(x)A_n\left(|t-x|/v_{qr}(t);q,r,s;x\right)\\
&\le v_{qr}(x)\left\{A_n\left((t-x)^2/v_{qr}(t);q,r,s;x\right)\right\}^{1/2}\left\{A_n\left(1/v_{qr}(t);q,r,s;x\right)\right\}^{1/2}\\
&\le \frac{M_9(q,r)}{n^s}, \qquad n \in N.
\end{aligned}$$

From this and by (17) we immediately obtain (16).

THEOREM 2. *Suppose that $q, s > 0$, $r \in R_2$ are fixed numbers and $f \in C_{qr}$. Then there exists a positive constant $M_{10}(q,r)$ such that*

$$\|A_n(f;q,r,s;\cdot) - f(\cdot)\|_{qr} \le M_{10}(q,r)\omega_1(f;C_{qr};1/n^s), \qquad n \in N. \tag{18}$$

Proof. We use Steklov function f_h of $f \in C_{qr}$

$$f_h(x) := \frac{1}{h}\int_0^h f(x+t)dt, \qquad x \in R_0, \quad h > 0. \tag{19}$$

From (19) we get

$$f_h(x) - f(x) = \frac{1}{h}\int_0^h \Delta_t f(x)dt, \quad f_h'(x) = \frac{1}{h}\Delta_h f(x), \quad x \in R_0, \quad h > 0,$$

and

$$\|f_h - f\|_{qr} \le \omega_1\left(f;C_{qr};h\right), \tag{20}$$

$$\|f_h'\|_{qr} \le h^{-1}\omega_1\left(f;C_{qr};h\right), \tag{21}$$

for $h > 0$. From this we have that $f_h \in C^1_{qr}$ for $f \in C_{qr}$ and $h > 0$. Observe that

$$\begin{aligned}
v_{qr}(x)|A_n(f;q,r,s;x) - f(x)| &\le v_{qr}(x)\left[|A_n\left(f-f_h;q,r,s;x\right)|\right.\\
&\quad \left.+ |A_n\left(f_h;q,r,s;x\right) - f_h(x)| + |f_h(x) - f(x)|\right] := L_1(x) + L_2(x) + L_3(x)
\end{aligned}$$

for $x \in R_0$, $n \in N$, $r \in R_2$ and $h > 0$. From (14) and (20) we obtain

$$\begin{aligned}
&\|L_1\|_{qr} \le e^{qr}\|f_h - f\|_{qr} \le e^{qr}\omega_1\left(f;C_{qr};h\right),\\
&\|L_3\|_{qr} \le \omega_1\left(f;C_{qr};h\right).
\end{aligned}$$

Using Theorem 1 and (21), we get

$$||L_2||_{qr} \le \frac{M_7(q,r)}{n^s}||f_h'||_{qr} \le \frac{M_7(q,r)}{n^s h}\omega_1(f; C_{qr}; h) \quad \text{for } h > 0,\ n \in N.$$

Consequently

$$||A_n(f; q, r, s; \cdot) - f(\cdot)||_{qr} \le \left(1 + e^{qr} + \frac{M_7(q,r)}{n^s h}\right)\omega_1(f; C_{qr}; h).$$

Now, for fixed $n \in N$, setting $h = \frac{1}{n^s}$, we obtain

$$||A_n(f; q, r, s; \cdot) - f(\cdot)||_{qr} \le M_{10}(q,r)\omega_1(f; C_{qr}; 1/n^s).$$

This completes the proof of Theorem 2.

From Theorem 1 and Theorem 2 we get

COROLLARY. Fix $q, s > 0$ and $r \in R_2$. Then

$$\lim_{n\to\infty} ||A_n(f; q, r, s; \cdot) - f(\cdot)||_{qr} = 0$$

for $f \in C_{qr}$. In particular

$$||A_n(f; q, r, s; \cdot) - f(\cdot)||_{qr} = O(1/n^s)$$

for $f \in C^1_{qr}$.

Remark. It is easily verified that analogous approximation properties hold for the following operators on C_{qr}.

$$B_n(f; q, r, s; x) := e^{-(n^s x+1)^r}\sum_{k=0}^{\infty}\frac{(n^s x+1)^{rk}}{k!}.$$

$$[n^s(n^s x+1)^{r-1} + qr]\int_{b_k}^{b_{k+1}} f(t)dt \qquad (f \in C_{qr})$$

for fixed $q, s > 0$, $x \in R_0$, $n \in N$ and $r \in R_2$, where $b_k = \frac{k}{n^s(n^s x+1)^{r-1}+qr}$.

Acknowledgment

The author would like to thank the referee for his valuable comments.

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Institute of Mathematics
Poznań University of Technology
Piotrowo 3A
60-965 Poznań, POLAND
E-mail: zwalczak@math.put.poznan.pl