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# ON GENERALIZED SLANT TOEPLITZ OPERATORS WITH CONTINUOUS SYMBOLS

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**Abstract.** In this paper, we make use of the Birkhoff's ergodic theorem to obtain some spectral properties of $k^{th}$-order slant Toeplitz operators. We also obtain the spectral radius of such an operator for continuous symbol $\varphi$.

## Introduction

Let $\varphi(z)=\sum_{i=-\infty}^{\infty}a_iz^i$ be a bounded function on the unit circle $\mathbb{T}$, where $a_i=\langle\varphi,z^i\rangle$ is the $i^{th}$ Fourier coefficient of $\varphi$ with respect to the usual standard basis $\{z^i : i\in\mathbb{Z}\}$ of the space $L^2=L^2(\mathbb{T}),\mathbb{Z}$ being the set of integers. For an integer $k\geq 2$, let $W_k : L^2\to L^2$ be an operator defined as

$$W_k(z^i)=\begin{cases} z^{i/k}, & \text{if i is divisible by k}\\ 0, & \text{otherwise.}\end{cases}$$

A $k^{th}$-order Slant Toeplitz operator $U_\varphi$ [1] is defined as $U_\varphi=W_kM_\varphi, M_\varphi$ being the multiplication operator on $L^2=L^2(\mathbb{T})$ induced by $\varphi$. In [1] it is shown that the spectrum of $U_\varphi$, with symbol $\varphi$ invertible in $L^\infty$, contains a closed disc that consists of the eigen values of $U_\varphi$ with infinite multiplicities and the radius of that disc is $(r(U_{\overline{\varphi}^{-1}}))^{-1}$. Here we find the spectral radius of $U_\varphi$ for first a trigonometric symbol $\varphi$ and then for continuous $\varphi$ in $L^\infty(\mathbb{T})$ and prove that it is same in both the cases.

## 1. Trigonometric Polynomial $\varphi$

Let $\varphi$ be a trigonometric polynomial. Suppose that

$$\varphi(z)=\sum_{l=-N}^{N}a_lz^l$$

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for some non-negative integer $N$. Assume further that $\|\varphi\|_\infty = 1$. It is proved in [1] that the spectral radius $r(U_\varphi)$, of the operator $U_\varphi$, is

$$r(U_\varphi) = \lim_{n\to\infty} \|\psi_n\|_\infty^{1/2n}$$

where $\psi_n = U^n_{|\varphi|^2}(1)$. As $W_k$ is a bounded operator on $L^2(\mathbb{T})$ with $\|W_k\| = 1$, we have $\|\psi_n\|_\infty \le 1$, for every $n$. Therefore $\|\psi_n\|_\infty \ge \|\psi_n\|_2$.

**LEMMA 1.** *Let $C_l : |l| \le M$, be complex numbers and $\psi(z) = \sum_{l=-M}^{M} C_l z^l$. Then the subspace $H$ = span of $\{z^l : |l| \le M\}$ of $L^2(\mathbb{T})$ is invariant under $U_\psi$.*

*Proof.* We see that

$$U_{z^i}(z^j) = W_k(z^{i+j}) = \begin{cases} z^{i+j/k}, & \text{if i+j is divisible by } k \\ 0, & \text{otherwise.} \end{cases}$$

Hence if $|i|, |j| \le M$, then for all $k \ge 2$

$$\frac{|i+j|}{k} \le \frac{|i|+|j|}{k} \le \frac{M+M}{k} = \frac{2M}{k} \le M.$$

Therefore $U_{z^i}(z^j) \in H$. From this we get that $U_\psi(z^j)$ is in $H$ for all $|j| \le M$. This completes the proof.

**COROLLARY 2.** *$\psi_n$ belongs to the finite - dimensional space $H$ = span $\{z^l : |l| \le 2N\}$.*

*Proof.* If $\varphi = \sum_{l=-N}^{N} a_l z^l$, then $|\varphi|^2 \in H$ and since $\psi_n = U^n_{|\varphi|^2}(1)$, therefore by Lemma-1 the result follows.

Now let $\psi_n = \sum_{l=-2N}^{2N} b_l^{(n)} z^l$ and let $(\alpha_{ij}^{(n)})$, $|i|$, $|j| \le 2N$ represent the matrix of $(U_{|\varphi|^2}|H)^n$. Since $\psi_n = (U_{|\varphi|^2}|H)^n(1)$, we have $\alpha_{00}^{(n)} = b_0^{(n)}$. But

$$\begin{aligned} b_0^{(n)} &= \langle (U_{|\varphi|^2}|H)^n(1),\ 1\rangle \\ &= \langle U^n_{|\varphi|^2}(1),\ 1\rangle \\ &= \langle 1, \prod_{l=0}^{n-1} |\varphi(z^{k^l})|^2\rangle \\ &= \langle 1, \varphi_n\rangle \\ &= \frac{1}{2\pi}\int_0^{2\pi} \varphi_n d\theta. \end{aligned}$$

where $\varphi_n = \prod_{l=0}^{n-1} |\varphi(z^{k^l})|^2$. Now, for any $|i|, |j| \leq 2N$, we have

$$\begin{aligned}\alpha_{ij}^{(n)} &= \langle (U_{|\varphi|^2}|H)^n z^j, z^i\rangle \\ &= \langle U^n_{|\varphi|^2}(z^j), z^i\rangle \\ &= \langle z^j, \varphi_n z^{k^n i}\rangle \\ &= \langle \varphi_n, z^{k^n i - j}\rangle.\end{aligned}$$

It follows that

$$\alpha_{ij}^{(n)} = \int_0^{2\pi} \varphi_n(\theta) e^{-(k^n i - j)\theta} \frac{d\theta}{2\pi}$$

and so $|\alpha_{ij}^{(n)}| \leq b_0^{(n)}$.

**THEOREM 3.** *The Hilbert Schmidt norm of $(U_{|\varphi|^2}|H)^n$ is bounded by $(4N + 1)\, b_0^{(n)}$.*

*Proof.* By definition, the Hilbert Schmidt norm of $(U_{|\varphi|^2}|H)^n$ equals $\left(\sum_{i,j=-2N}^{2N} |\alpha_{ij}^{(n)}|^2\right)^{1/2}$. Since $(U_{|\varphi|^2}|H)^n$ has at most $(4N + 1)^2$ non-zero entries, we have $||(U_{|\varphi|^2}|H)^n|| = \left(\sum_{i,j=-2N}^{2N} |\alpha_{ij}^{(n)}|^2\right)^{1/2} \leq (4N + 1)b_0^{(n)}$. This completes the proof.

**LEMMA 4.** *There exist positive integers $K_1$ and $K_2$ such that*

$$K_1||(U_{|\varphi|^2}|H)^n|| \leq ||U_\varphi^n||^2 \leq K_2||(U_{|\varphi|^2}|H)^n||.$$

*Proof.* Since $H = \{z^l : |l| \leq 2N\}$, therefore $\dim H = 4N + 1$. Thus

$$\begin{aligned}||\psi_n||_\infty \leq \sum_{l=-2N}^{2N} |b_l^{(n)}| &\leq \sqrt{4N+1}||\psi_n||_2 \qquad \text{(Cauchy Schwarz Inequality)} \\ &= \sqrt{4N+1}||U^n_{|\varphi|^2}(1)||_2 \\ &\leq \sqrt{4N+1}||(U_{|\varphi|^2}|H)^n||.\end{aligned}$$

On the other hand,

$$\begin{aligned}|\psi_n||_\infty \geq ||\psi_n||_2 \geq |b_0^{(n)}| &\geq \frac{1}{4N+1}||(U_{|\varphi|^2}|H)^n||_2 \\ &\geq \frac{1}{4N+1}||(U_{|\varphi|^2}|H)^n||.\end{aligned}$$

As $||\psi_n||_\infty = ||U_\varphi^n||^2$ we get the desired result.

As $r(U_\varphi)$, the spectral radius of $U_\varphi$, equals by [1] $\lim\limits_{n\to\infty} ||\psi_n||_\infty^{1/2n}$, we obtain the following

**COROLLARY 5.** *The spectral radius of $U_\varphi$ is equal to the square root of the spectral radius of $U_{|\varphi|^2}|H$.*

*As $H$ is finite dimensional we yet obtain the following*

**COROLLARY 6.** $(r(U_\varphi))^2 = \max\{|\lambda| : \det(U_{|\varphi|^2}|H - \lambda) = 0\}$.

**THEOREM 7.** $(r(U_\varphi))^2 = \lim\limits_{n\to\infty}\left(\int\limits_0^{2\pi}\prod\limits_{l=0}^{n-1}|\varphi(k^l\theta)|^2\frac{d\theta}{2\pi}\right)^{1/n}$

*Proof.* From the above lemma

$$b_0^{(n)} \le ||\psi_n||_\infty = ||U_{|\varphi|^2}^n|| \le \sqrt{4N+1}\,||\psi_n||_2 = \sqrt{4N+1}\left(\sum_{l=-2N}^{2N}|b_l^{(n)}|^2\right)^{1/2} \tag{1}$$

As $|b_l^{(n)}| = |\alpha_{l0}^{(n)}| \le b_0^{(n)}$, therefore $\sum\limits_{l=-2N}^{2N}|b_l^{(n)}|^2 \le |b_0^{(n)}|^2(4N+1)$. Hence

$$\sqrt{4N+1}\left(\sum_{l=-2N}^{2N}|b_l^{(n)}|^2\right)^{1/2} \le (4N+1)b_0^{(n)} \tag{2}$$

Using (1) and (2) we get

$b_0^{(n)} \le ||U_{|\varphi|^2}^n|| \le (4N+1)b_0^{(n)}$. Let $4N+1 = K$. On taking limit as $n\to\infty$ we obtain that

$$(r(U_\varphi))^2 = \lim_{n\to\infty}\left(\int_0^{2\pi}\prod_{l=0}^{n-1}|\varphi(k^l\theta)|^2\frac{d\theta}{2\pi}\right)^{1/n}$$

## 2. Continuous function $\varphi$

Let $(\mathbb{T}, \mathcal{A}, \mu)$ be a probability space and let $\tau_k : \mathbb{T}\to\mathbb{T}$ be a measure preserving continuous map. Also let $T : L^p \to L^p (1 \le p < \infty)$ be defined as

$$Tf = f\circ\tau_k$$

for any $f$ in $L^p(\mathbb{T})$, where $\tau_k : \mathbb{T}\to\mathbb{T}$ for each $k\ge 2$ is defined as $\tau_k(e^{i\theta}) = e^{ki\theta}$. We can define

$$S_nf = \frac{1}{n}\sum_{l=0}^{n-1}T^lf$$

for all $n$. Then for each $k \geq 2$, $\tau_k$ is ergodic [7], if and only if the $T$-invariant functions are constants. Consider $(U_\varphi)^*(f(z)) = M_{\overline{\varphi}} W_k^* f = \overline{\varphi}\, f \circ \tau_k$.

Therefore, $U_\varphi^*$ is a weighted composition operator on $\mathbb{T}$ as for all $f$ in $L^2$

$$(U_\varphi)^* f = \overline{\varphi}\, f \circ \tau_k$$

Now

$$\frac{1}{n}\sum_{l=0}^{n-1} T^l f = \frac{1}{n}\sum_{l=0}^{n-1} f(e^{ki\theta}) \to C$$

a.e. and in $L^p$ for all $1 \leq p < \infty$ by [7] Birkhoff's Ergodic Theorem. That is,

$$\|\frac{1}{n}\sum_{l=0}^{n-1} T^l f - C\|_p \to 0$$

as $n \to \infty$. Now for every $n$

$$\int_0^{2\pi} \frac{1}{n}\sum_{l=0}^{n-1} T^l f \frac{d\theta}{2\pi} = \frac{1}{n}\sum_{l=0}^{n-1} \int_0^{2\pi} T^l f \frac{d\theta}{2\pi}$$

$$= \frac{1}{n}\sum_{l=0}^{n-1} \int_0^{2\pi} f(e^{k^l i\theta}) \frac{d\theta}{2\pi} = \frac{1}{n}\sum_{l=0}^{n-1} \frac{1}{k^l} \int_0^{k^l 2\pi} f(e^{i\theta}) \frac{d\theta}{2\pi} = \int_0^{2\pi} f(e^{i\theta}) \frac{d\theta}{2\pi}.$$

$$\left|\int_0^{2\pi} \frac{1}{n}\sum_{l=0}^{n-1} T^l f \frac{d\theta}{2\pi} - C\right| \leq \|\frac{1}{n}\sum_{l=0}^{n-1} T^l f - C\| \leq \|\frac{1}{n}\sum_{l=0}^{n-1} T^l f - C\|_p \to 0$$

as $n \to \infty$. Hence

$$C = \int_0^{2\pi} f \frac{d\theta}{2\pi}. \tag{1}$$

Our attempt in this section is to obtain $r(U_\varphi)$ for continuous symbols. To achieve the same we begin with the following

**Lemma 8.** *If $\varphi$ is a function on $\mathbb{T}$ such that log $|\varphi|$ is integrable, then*

$$\left(\prod_{l=0}^{n-1} |\varphi(k^l\theta)|\right)^{1/n} \to \exp\left(\int_0^{2\pi} \log|\varphi| \frac{d\theta}{2\pi}\right) \text{a.e. on } \mathbb{T}.$$

*Proof.* Apply Birkhoff's ergodic theorem on $L^1$-function log $|\varphi|$ and using (1), we get that

$$\frac{1}{n}\sum_{l=0}^{n-1}\log|\varphi(k^l\theta)| \to \int_0^{2\pi}\log|\varphi|\frac{d\theta}{2\pi} \, a.e. \text{ on } \mathbb{T}.$$

The proof is complete as for almost every $\theta$ in $[0, 2\pi)$ and for all $n$

$$\left(\prod_{l=0}^{n-1}|\varphi(k^l\theta)|\right)^{1/n} = \exp\left(\frac{1}{n}\sum_{l=0}^{n-1}\log|\varphi(k^l\theta)|\right).$$

**THEOREM 9.** *For any $L^\infty$ function $\varphi$ on $\mathbb{T}$, we have*

$$\exp\left(\int_0^{2\pi}\log|\varphi|\frac{d\theta}{2\pi}\right) \leq r(U_\varphi).$$

*Proof.* Since $\varphi$ is bounded above, we have

$$-\infty \leq \int_0^{2\pi}\log|\varphi|\frac{d\theta}{2\pi} < \infty$$

If log $|\varphi|$ is not integrable then $\exp\left(\int_0^{2\pi}\log|\varphi|\frac{d\theta}{2\pi}\right) = 0$. So we assume that log $|\varphi|$ is integrable. Therefore

$$\begin{aligned} r(U_\varphi) &= \lim_{n\to\infty} \|\psi_n\|_\infty^{1/2n} \\ &\geq \lim_{n\to\infty}\sup \|\psi_n\|_1^{1/2n} \\ &= \lim_{n\to\infty}\sup\left(\int_0^{2\pi} U^n_{|\varphi|^2}(1)\frac{d\theta}{2\pi}\right)^{1/2n} \\ &= \lim_{n\to\infty}\sup\langle U^n_{|\varphi|^2}(1), 1\rangle^{1/2n} \\ &= \lim_{n\to\infty}\sup\langle 1, \prod_{l=0}^{n-1}|\varphi(k^l\theta)|^2\rangle^{1/2n}. \end{aligned}$$

As $f(x) = x^{1/2n}$ is a concave function on $[0, \infty)$ we have

$$r(U_\varphi) \geq \lim_{n\to\infty}\sup\int_0^{2\pi}\left(\prod_{l=0}^{n-1}|\varphi(k^l\theta)|\right)^{1/n}\frac{d\theta}{2\pi}$$

Also since $\prod_{l=0}^{n-1} |\varphi(k^l\theta)|^{1/n} \leq ||\varphi||_\infty$, therefore, by Lebesgue dominated convergence Theorem, we have

$$r(U_\varphi) \geq \int_0^{2\pi} \lim_{n\to\infty} \sup\left(\prod_{l=0}^{n-1} |\varphi(k^l\theta)|\right)^{1/n} = \exp\left(\int_0^{2\pi} \log|\varphi| \frac{d\theta}{2\pi}\right).$$

This completes the proof.

**Lemma 10.** *Let $\varphi$ be an $L^\infty$ function on $\mathbb{T}$ essentially bounded away from 0. If $\{\varphi_n\}$ is a sequence in $L^\infty(\mathbb{T})$ such that $||\varphi_n - \varphi||_\infty \to 0$, then*

$$\lim_{n\to\infty} r(U_{\varphi_n}) = r(U_\varphi).$$

*Proof.* Let $\rho > 1$ and $\delta < 1$. By [Lemma 3.1, 6], there exists an $\varepsilon > 0$ such that for any $L^\infty$ function $\psi$ with $||\psi - \varphi||_\infty < \varepsilon$ we have

$$\delta|\varphi| < |\psi| < \rho|\varphi| \text{ a.e. on } \mathbb{T}.$$

Since $||\varphi_n - \varphi||_\infty \to 0$. Therefore, we can find a large $N$ such that $||\varphi_n - \varphi||_\infty < \varepsilon$ if $n > N$. That is, $\delta|\varphi| < |\varphi_n| < \rho|\varphi|$ a.e. on $\mathbb{T}$ if $n > N$. Also we can see from [1] that the spectral radius function is a monotonic increasing function therefore for $n > N$ since

$$\delta|\varphi| < |\varphi_n| < \rho|\varphi| \qquad \text{a.e. on } \mathbb{T},$$

we must have

$$r(U_{\delta\varphi}) \leq r(U_{\varphi_n}) \leq r(U_{\rho\varphi})$$

The proof is complete on taking $\delta, \rho \to 1$.

**Theorem 11.** *If $\varphi$ is a continuous on $\mathbb{T}$ without zeros then*

$$r(U_\varphi) = \lim_{n\to\infty}\left(\int_0^{2\pi} \prod_{l=0}^{n-1} |\varphi(k^l\theta)|^2 \frac{d\theta}{2\pi}\right)^{1/2n}.$$

*Proof.* Since $\varphi$ is continuous, therefore, there exists a sequence $\{P_m\}$ of trigonometric polynomials, such that $||P_m - \varphi||_\infty \to 0$ as $m \to \infty$. Let $\rho > 1$ and $\delta < 1$. By [Lemma 3.1, 6], there is an $\varepsilon > 0$ such that for any $\psi$ satisfying $||\psi - \varphi||_\infty < \varepsilon$, we have $\delta|\varphi| < |\psi| < \rho|\varphi|$ a.e. on $\mathbb{T}$.

Now, since $||P_m - \varphi||_\infty \to 0$, therefore, we can choose a large $M$, such that for each $m > M$, $||P_m - \varphi||_\infty < \varepsilon$. Therefore, for $m > M$, we have

$$\delta|\varphi| < |P_m| < \rho|\varphi|.$$

Thus, for any $n$ and $m > M$, we have

$$\delta^n \prod_{l=0}^{n-1} |\varphi(k^l\theta)| \le \prod_{l=0}^{n-1} |P_m(k^l\theta)| \le \rho^n \prod_{l=0}^{n-1} |\varphi(k^l\theta)|$$

for all $\theta$ in $[0, 2\pi)$. This means that for every $n$ and for $m > M$

$$\begin{aligned}\delta^2 \left( \int_0^{2\pi} \prod_{l=0}^{n-1} |\varphi(k^l\theta)|^2 \frac{d\theta}{2\pi} \right)^{1/n} &\le \left( \int_0^{2\pi} \prod_{l=0}^{n-1} |P_m(k^l\theta)|^2 \frac{d\theta}{2\pi} \right)^{1/n} \\ &\le \rho^2 \left( \int_0^{2\pi} \prod_{l=0}^{n-1} |\varphi(k^l\theta)|^2 \frac{d\theta}{2\pi} \right)^{1/n}.\end{aligned}$$

This implies that (since the result is true for trigonometric polynomials) for $m > M$, we have

$$\begin{aligned}\delta^2 \lim_{n\to\infty} \sup \left( \int_0^{2\pi} \prod_{l=0}^{n-1} |\varphi(k^l\theta)|^2 \frac{d\theta}{2\pi} \right)^{1/n} &\le (r(U_{P_m}))^2 \\ &\le \rho^2 \lim_{n\to\infty} \inf \left( \int_0^{2\pi} \prod_{l=0}^{n-1} |\varphi(k^l\theta)|^2 \frac{d\theta}{2\pi} \right)^{1/n}.\end{aligned}$$

Again since $\|P_m - \varphi\|_\infty \to 0$, therefore, using Lemma 10, we have

$$\lim_{m\to\infty} r(U_{P_m}) = r(U_\varphi).$$

Hence,

$$\begin{aligned}\delta^2 \lim_{n\to\infty} \sup \left( \int_0^{2\pi} \prod_{l=0}^{n-1} |\varphi(k^l\theta)|^2 \frac{d\theta}{2\pi} \right)^{1/n} &\le (r(U_\varphi))^2 \\ &\le \rho^2 \lim_{n\to\infty} \inf \left( \int_0^{2\pi} \prod_{l=0}^{n-1} |\varphi(k^l\theta)|^2 \frac{d\theta}{2\pi} \right)^{1/n}.\end{aligned}$$

The proof is complete on taking $\rho, \delta \to 1$.

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