YOKOHAMA MATHEMATICAL
JOURNAL VOL. 50, 2003

PROJECTIVE-PLANAR DOUBLE COVERINGS OF 3-CONNECTED GRAPHS

By

YUSUKE SUZUKI AND SEIYA NEGAMI

(Received July 23, 2003)

Abstract. We shall show that any projective-planar double covering of a 3-connected graph is planar, discussing structures of double covering of planar graphs algebraically and combinatorially.

Introduction

Our graphs are simple and finite. A graph $\tilde{G}$ is called an (*n-fold*) *covering* of a graph G with a *projection* $p : \tilde{G} \to G$ if there is an n-to-one surjection $p : V(\tilde{G}) \to V(G)$ which sends the neighbors of each vertex $v \in V(\tilde{G})$ bijectively to those of $p(v)$. A graph is said to be *projective-planar* if it can be embedded in the projective plane.

In 1986, Negami [12] has proposed the following conjecture, which is called the *1-2-∞ conjecture* or *Negami's planar cover conjecture*, recently:

CONJECTURE 1 (Negami [12], 1986). A connected graph is projective-planar if and only if it has a planar covering.

The necessity is clear since any graph embedded on the projective plane is covered doubly by a graph embedded on the sphere. The sufficiency is still open.

There are many studies [1]—[16] around this conjecture and all of them give evidences supporting it. In particular, Hliněný [6] has proposed the following conjecture and shown that it is equivalent to Conjecture 1:

CONJECTURE 2 (Hliněný [6]). A connected graph is projective-planar if and only if it has a projective-planar covering.

Recently, Negami [16] has proved the following two theorems on projective-planar coverings of graphs, related to Conjecture 2:

2000 Mathematics Subject Classification: 05C10
Key words and phrases: Planar cover conjecture, double coverings of graphs, projective-planar graphs

THEOREM 1 (Negami [16]). *A connected graph is projective-planar if and only if it has a projective-planar double covering.*

THEOREM 2 (Negami [16]). *Every projective-planar double covering of a 2-connected nonplanar graph is planar.*

He has shown the best-possibility of the latter. That is, there exist those graphs that adimit double coverings which are projective-planar but not planar if we don't assume that they are 2-connected *and* nonplanar. In this paper, we shall discuss what happes if we strengthen the assumption on the connectivity, cutting the nonplanarity. The following is our main theorem:

THEOREM 3. *Every projective-planar double covering of a 3-connected graph is planar.*

If the 3-connected graph G in the theorem is nonplanar, then the theorem follows immediately from Theorem 2. Thus, it suffices to prove the theorem when G is planar. One will be able to give a purely combinatorial proof for Theorem 3, mimiking the arguments in [16], that is, considering double coverings of K_4, instead of $K_{3,3}$, and configurations of bridges for a subdivision of K_4 in a 3-connected planar graph. However, we shall show a simple proof, avoiding complicated arguments on bridges.

We shall discuss double coverings of planar graphs with an algebraic formulation in Section 1, which will give us a general argument on those more than we need to prove our main theorem. In Section 2, we shall introduce the notion of "bridges" and prepare some technical lemmas. Section 3 is devoted to the proof of Theorem 3.

1. Double coverings of planar graphs

Let G be a connected graph. Then each double covering of $p : \tilde{G} \to G$ corresponds to a subgroup in the fundamental group $\pi_1(G)$ of index 2. Since such a subgroup is necessarily normal in $\pi_1(G)$, it can be obtained as the kernel of a homomorphism $\tilde{\sigma} : \pi_1(G) \to \boldsymbol{Z}_2$ and such a homomorphism can be determined uniquely by a homomorphism $\sigma : H_1(G; \boldsymbol{Z}_2) \to \boldsymbol{Z}_2$, where $H_1(G; \boldsymbol{Z}_2)$ is the 1-dimensional homology group with $\boldsymbol{Z}_2$-coefficients.

This is a formulation for double coverings of graphs in algebraic topology. We can also formulate them in terms of graph theory as follows, since $H_1(G; \boldsymbol{Z}_2)$ is nothing but the cycle space $C(G)$, where the addition of two elements in $H_1(G; \boldsymbol{Z}_2)$ corresponds to the symmetric difference of two sets in $C(G)$.

Let $\{C_1, \dots, C_r\}$ be a basis of $C(G)$ (or of $H_1(G; \boldsymbol{Z}_2)$). A homomorphism σ :

$H_1(G; \boldsymbol{Z}_2) \to \boldsymbol{Z}_2$ corresponds bijectively to an assignment $\sigma : \{C_1, \ldots, C_2\} \to \boldsymbol{Z}_2$. The homomorphism $\sigma : C(G) \to \boldsymbol{Z}_2$ is defined by

$$\sigma\left(\sum_{i=1}^{r} \lambda_i C_i\right) = \sum_{i=1}^{r} \lambda_i \sigma(C_i)$$

with $\lambda_i \in \boldsymbol{Z}_2 = \{0, 1\}$. There exists a double covering $p = p_\sigma : \tilde{G}_\sigma \to G$ of G such that $p^{-1}(C)$ consists of two disjoint cycles of the same length as C if $\sigma(C) = 0$ and that $p^{-1}(C)$ forms a cycle which is twice as long as C if $\sigma(C) = 1$. This double covering is said to be *derived from* σ. If σ assigns 0 to all cycles, then $\tilde{G}_\sigma$ consists of two componets each of which is isomorphic to G.

Now let G be a planar graph 2-cell embedded on the sphere S^2 with $r+1$ faces $A_0, A_1, \ldots, A_r$ and let C_i be the boundary cycle of A_i for $i = 0, 1, \ldots, r$. Then $\{C_1, \ldots, C_r\}$ becomes a basis of $C(G)$. Give any assignment $\sigma : \{C_1, \ldots, C_r\} \to \boldsymbol{Z}_2$ to define a homomorphism $\sigma : C(G) \to \boldsymbol{Z}_2$. Then we have $\sigma(C_0) = \sum_{i=1}^{r} \sigma(C_i)$. This implies that an even number of cycles $C_0, C_1, \ldots, C_r$ are assigned 1.

Consider the double covering $p : \tilde{G} \to G$ of G derived from a given σ. Pasting a 2-cell along each of components of $p^{-1}(C_i)$ for $i = 0, 1, \ldots, r$, we obtain a closed surface F^2_σ where $\tilde{G}$ is embedded so that each component of $p^{-1}(C_i)$ bounds a face. Two faces correspond to C_i with $\sigma(C_i) = 0$ while one face to C_i with $\sigma(C_i) = 1$. Furthermore, we can define naturally a branched covering projection $\tilde{p} : F^2_\sigma \to S^2$ with $\tilde{p}|_{\tilde{G}} = p$, branched over the central points of the faces bounded by C_i's with $\sigma(C_i) = 1$.

The existence of this branched covering projection forces F^2_σ to be orientable. Let n denote the number of C_i's with $\sigma(C_i) = 1$, which is an even number. Then $\chi(F^2_\sigma) = 2|V(G)| - 2|E(G)| + 2(r + 1 - n) + n = 4 - n$. This implies that $\tilde{G}$ is a planar graph embedded on the sphere F^2_σ if $n = 2$. Note that $\tilde{G}$ may be planar even if $n \geq 4$. For example, the 3-cube covers doubly K_4. This double covering is derived from the homomorphism $\sigma : C(K_4) \to \boldsymbol{Z}_2$ which assigns 1 to all four faces of K_4 embedded on the sphere.

In general, two coverings $p_1 : \tilde{G}_1 \to G$ and $p_2 : \tilde{G}_2 \to G$ of the same graph G are said to be *equivalent* modulo automorphisms if there exist an isomorphism $\tau : \tilde{G}_1 \to \tilde{G}_2$ and an automorphism $\sigma : G \to G$ with $\sigma p_1 = p_2 \tau$. Roughly speaking, two equivalent coverings modulo automorphisms look like the same in the unlabled sense. For example, Figure 1 presents two double coverings of K_4 which are not equivalent modulo automorphisms. One is isomorphic to the 3-cube and the other has 2-cuts consisting of vertices and of edges.

LEMMA 4. *There exist precisely two connected double covering of K_4, up to equivalence modulo automorphisms, as given in Figure 1.*

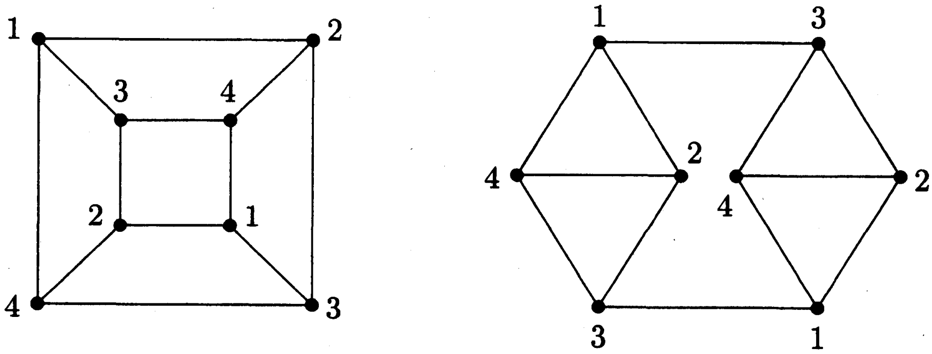

Figure 1 Inequivalent double coverings of K_4

Proof. Embed K_4 on the sphere. Then there are precisely four faces. We have only two ways to choose an even number of faces, two or four, which should be assigned 1, up to symmetry. These correspond to the double coverings of K_4 given in Figure 1. ∎

Let G be a planar graph embedded on the sphere as above and H a subgraph. Given a homomorphism $\sigma : C(G) \to \boldsymbol{Z}_2$, we can define a homomorphism $\sigma_H : C(H) \to \boldsymbol{Z}_2$ for H so that $\sigma_H(C) = \sum_{j=1}^{h} \sigma(C_{i_j})$ if $C = \sum_{j=1}^{h} C_{i_j}$ in $C(G)$. Then the double covering $\tilde{H}_\sigma$ of H derived from σ_H can be regarded naturally as a subgraph in the double covering $\tilde{G}_\sigma$ of G which covers H doubly via the projection p_σ.

LEMMA 5. *Let G be a 3-connected planar graph embedded on the sphere and $p = p_\sigma : \tilde{G} \to G$ the double covering of G derived from a homomorphism $\sigma : C(G) \to \boldsymbol{Z}_2$ which assigns 1 to at least four faces of G. Then G contains a subdivision H of K_4 such that $p^{-1}(H)$ is a subdivison of the 3-cube.*

Proof. Use induction on the number of faces of G, say $m \geq 4$. If $m = 4$, then G is isomorphic to K_4 and the lemma follows from Lemma 4. Suppose that $m > 4$.

Choose a boundary cycle C_1 of G out of those with $\sigma(C_i) = 0$, or any one if there is no such cycle. Let C_2 be another boundary cycle which shares an edge e with C_1. If $G - e$ with vertices of degree 2 neglected is 3-connected, then $G - e$ contains a subgraph H satisfying the condition in the lemma by the induction hypothesis, and H satisfies the same condition for G.

Otherwise, $G - e$ has a 2-cut $\{u, v\}$ and there is a simple closed curve γ on the sphere which crosses e transversely and passes through u and v. Then there is another boundary cycle C_3 meeting C_1 along an edge e' in a 2-cell regioin bounded by γ and we can do the same argument for C_3 as for C_2. We find the desired subgraph H or a simple closed curve γ' which meets G in a 2-cut of $G - e'$ and the middle point of e'. In the latter case, we can choose γ' so that γ'

is contained inside γ and can continue this argument. Finally, we will reach the innermost one and find H. ■

COROLLARY 6. *Let G be a 3-connected planar graph and $p : \tilde{G} \to G$ the double covering of G. If $\tilde{G}$ is nonplanar, then G contains a subdivision H of K_4 such that $p^{-1}(H)$ is a subdivison of the 3-cube.*

Proof. If $\tilde{G} = \tilde{G}_\sigma$ is nonplanar, then σ assigns 1 to at least four faces. Thus, the corollary follows from Lemma 5. ■

2. Bridges in double coverings

We shall introduce the notion of "bridges", as follows, which is often used in topological graph theory to investigate embeddings of graphs on closed surfaces. Our arguments in this section do not need the planarity of graphs although it is important in the previous section.

Let G be a graph and H a subgraph in G. A subgraph B induced by a component of $G - V(H)$ and the edges joining it to H is called a ***bridge*** for H in G. A subgraph consisting of a single edge in $E(G) - E(H)$ with both ends in H also is called a bridge for H but it is said to be ***singular***. It is clear that G decomposes into H and the bridges for H and that they are mutually edge-disjoint. A vertex of a bridge is called a vertex of *attachment* if it lies in H. If H is a subdivision of another graph H' with minimum degree at least 3, then any path joining two vertices of degree more than 2 in H corresponds to an edge of H' and is called a ***side*** of H.

Now let $p : \tilde{G} \to G$ be a double covering of G. Then there exist precisely two vertices $v_1, v_2 \in V(\tilde{G})$ with $p(v_1) = p(v_2) = v$ for each vertex $v \in V(G)$. It is clear that the mapping $\tau : V(\tilde{G}) \to V(\tilde{G})$ defined by $\tau(v_i) = v_{3-i}$ induces an automorphism of $\tilde{G}$. This automorphism $\tau : \tilde{G} \to \tilde{G}$ is called the *covering transformation* of $\tilde{G}$ (of order 2).

Put $\tilde{H} = p^{-1}(H)$ for a subgraph H in G. Then $p|_{\tilde{H}} : \tilde{H} \to H$ is a double covering of H and the covering transformation τ of $\tilde{G}$ induces that of $\tilde{H}$, that is, $\tau(\tilde{H}) = \tilde{H}$. Let $B_1, \ldots, B_s$ be the bridges for H in G. Then $p^{-1}(B_i)$ becomes either a bridge $\tilde{B}_i$ or a disjoint pair of bridges $\tilde{B}_i'$ and $\tilde{B}_i''$ for $\tilde{H}$ in $\tilde{G}$.

LEMMA 7. *Let $p : \tilde{G} \to G$ be a double covering of a 2-connected graph G and let H be a subgraph in G with a bridge B. Suppose that B has vertices of attachement only on a side S of H and that $\tilde{B} = p^{-1}(B)$ forms one bridge for $\tilde{H}$ in $\tilde{G}$. Then there exist a path Q in B joining two vertices on S such that $p^{-1}(Q)$ consists of two disjoint paths joining two distinct sides of $\tilde{H}$ which project to S.*

Proof. Since $p|_{\tilde{B}} : \tilde{B} \to B$ is a double covering of B, there exists a cycle $\tilde{C}$ in $\tilde{B}$ which covers a cycle C in B doubly. Since G is 2-conneted, we can find two disjoint paths P_1 and P_2 in B each of which joins C and S, using Menger's theorem for example. Then $p^{-1}(C \cup P_1 \cup P_2)$ consists of $\tilde{C}$ and two copies of P_1 and P_2, say $\{P_1', P_1''\}$ and $\{P_2', P_2''\}$.

Let S' and S'' be the two components of $p^{-1}(S)$. We may assume that P_1' and P_2' have their ends on S' and P_1'' and P_2'' on S''. Then we can find immediately two disjoint paths joining S' and S'', one of which runs along P_1' and P_2'' and the other along P_1'' and P_2'. They project to the same path Q in B. ■

LEMMA 8. *Let $\tilde{G}$ be a double covering of a 3-connected graph G with its covering transformation $\tau : \tilde{G} \to \tilde{G}$ and suppose that $\tilde{G}$ is 2-cell embedded on a closed surface F^2. Let $\tilde{K}$ be a graph 2-cell embedded on F^2 and let $\tilde{H}$ be a subgraph in $\tilde{G}$ satisfying the following two conditions:*

(i) *$\tilde{H}$ is a subdivision of a graph $\tilde{K}$.*
(ii) *$\tilde{H}$ is 2-cell embedded on F^2 as a subembedding of $\tilde{G}$.*

Then there exists a subgraph $\tilde{H}'$ in $\tilde{G}$ satisfying (i) *and* (ii) *such that every bridge for $\tilde{H}'$ in $\tilde{G}$ has vertices of attachment on at least two sides of $\tilde{H}'$.*

Proof. Let $\tilde{H}'$ be a subgraph with the same condition as $\tilde{H}$ which has fewest bridges among those subgraphs. Suppose that there is a bridge B for $\tilde{H}'$ in $\tilde{G}$ such that all of its vertices of attachment are contained in a side S of $\tilde{H}'$. Since $\tau(S) \cap S = \emptyset$, we have $\tau(B) \cap B = \emptyset$.

Consider the union of faces bounded by only edges lying on B or S. Then it forms a 2-cell region R which contains B since $\tilde{H}'$ is 2-cell embedded on F^2. Its boundary splits into two paths one of which is a segment of S containing all vertices of attachment in B and the other runs along the periphery of B. Let S' and Q be these paths, respectively.

It is clear that the subgraph $\tilde{H}''$ in $\tilde{G}$ obtained from $\tilde{H}'$ by replacing S' with Q and $\tau(S')$ with $\tau(Q)$ satisfies Conditions (i) and (ii). Since G is 3-connected, there is another bridge for $\tilde{H}'$ in $\tilde{G}$ which has at least one vertex of attachment on S'; otherwise, the projection of two ends of S' would form a 2-cut in G. Thus, the replacement of paths in $\tilde{H}'$ unifies $B \cup S' - E(Q)$ and those bridges to be one bridge and hence the number of bridges for $\tilde{H}''$ in $\tilde{G}$ is smaller than that for $\tilde{H}'$. This is however contrary to the assumption of $\tilde{H}'$. Therefore, there does not exsit such a bridge B for $\tilde{H}'$ in G. ■

3. Proof of the theorem

Applying general results in the previous sections, we shall give a proof of our main theorem in this section.

Proof of Theorem 3. Let G be a 3-connected graph and $p : \tilde{G} \to G$ a double covering of G which is projective-planar. It suffices to prove the theorem, assuming that G is planar, as is mentioned in introduction.

Suppose that $\tilde{G}$ is nonplanar and let $\tau : \tilde{G} \to \tilde{G}$ be the covering transformation of $\tilde{G}$. By Corollary 6, G contains a subdivision H of K_4 such that $\tilde{H} = p^{-1}(H)$ is a subdivision of the 3-cube with $\tau(\tilde{H}) = \tilde{H}$. By Lemma 8, we may assume that every bridge for $\tilde{H}$ in $\tilde{G}$ has vertices of attachment on at least two sides of $\tilde{H}$. Let v_1, v_2, v_3, v_4 be the four vertices of degree 3 in H and put $p^{-1}(v_i) = \{v_i', v_i''\}$ so that $v_1'v_2'v_3'v_4'$ and $v_1''v_2''v_3''v_4''$ form two cycles in the 3-cube as in the left hand of Figure 1, which indicates only their subscripts. Then $\tilde{H}$ contains the following six cycles, corresponding to six faces of the 3-cube:

$$C_1 = v_1'v_2'v_3'v_4', \quad C_2 = v_2'v_3'v_1''v_4'', \quad C_3 = v_3'v_4'v_2''v_1'',$$
$$C_6 = v_1''v_2''v_3''v_4'', \quad C_5 = v_2''v_3''v_1'v_4', \quad C_4 = v_3''v_4''v_2'v_1'.$$

More precisely speaking, each consecutive vertices in the above must be joined by a side of $\tilde{H}$. Note that C_i and C_{7-i} form a disjoint pair and they project to the same cycle in H.

Embed $\tilde{G}$ on the projective plane. Since any two essential cycles on the projective plane must intersect each other, at least one of C_i and C_{7-i} bounds a 2-cell region, which is a face of $\tilde{H}$. We may assume that C_1, C_2 and C_3 bound faces A_1, A_2 and A_3 of $\tilde{H}$, respectively, and that $A_1 \cup A_2 \cup A_3$ forms a 2-cell region on the projective plane. We shall show that either B or $\tau(B)$ is contained in $A_1 \cup A_2 \cup A_3$ for any bridge B for $\tilde{H}$ in $\tilde{G}$.

First, suppose that none of C_4, C_5 and C_6 is essentail on the projective plane. Then they bound faces A_4, A_5 and A_6 of $\tilde{H}$, but exactly one of them must bound a crosscap. Consider any bridge B for $\tilde{H}$ in $\tilde{G}$ which is contained in one of these faces of $\tilde{H}$, say A_6. By our assumption on the bridges for $\tilde{H}$, there is a path Q in B which joins two vertices lying on two sides of $\tilde{H}$, and these sides correspond to two distinct edges of the cycle $v_1''v_2''v_3''v_4''$ in the 3-cube. Then $\tau(Q)$ joins two sides of $\tilde{H}$ corresponding to two distinct edges of $v_1'v_2'v_3'v_4'$. We can find such a path only in the face A_1 since $A_1 \cup A_2 \cup A_3$ forms a 2-cell region. This implies that $\tau(B)$ lies in A_1.

Next, suppose that at least one of C_4, C_5 and C_6 is essential on the projective plane. We may assume that this is C_6, up to symmetry. Under the assumptions here, $\tilde{H}$ has a unique embedding, up to symmetry, as given in Figure 2, where

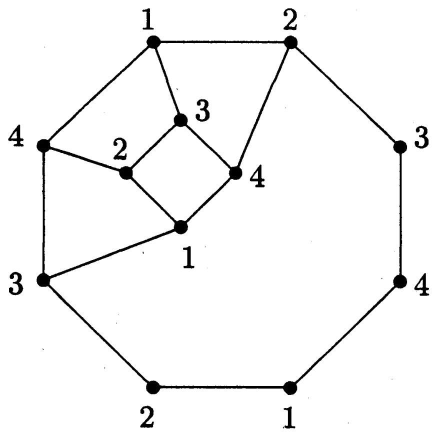

Figure 2 The 3-cube on the projective plane

C_4 bounds a face A_4 of $\tilde{H}$. The same argument as in the previous case works for any bridge contained in A_4 and it will be sent into A_3 by τ.

Let R be the face of $\tilde{H}$ bounded by the closed walk $v_1'v_4'v_2''v_3''v_4''v_1''v_2''v_3''$. Since sides $v_1'v_2'$, $v_2'v_3'$, $v_3'v_4'$, $v_2'v_4''$ and $v_3'v_1''$ lie inside the closure of $A_1 \cup A_2 \cup A_3 \cup A_4$, any bridge attached to the images of these sides by τ must be sent into one of A_1, A_2, A_3 and A_4 by τ. The only sides of $\tilde{H}$ that are not the imagas of these sides are $v_1'v_4'$ and $v_1''v_4''$.

Thus, if there were a bridge B for $\tilde{H}$ in $\tilde{G}$ such that $\tau(B)$ is not contained in $A_1 \cup A_2 \cup A_3 \cup A_4$, then B would be attached to only two sides $v_1'v_4'$ and $v_1''v_4''$. By the assumption on bridges for $\tilde{H}$, B must join both of these sides and hence $\tau(B) = B$. By Lemma 7, there exist two disjoint paths in B joining two sides $v_1'v_4'$ and $v_1''v_4''$ and they project to the same path in $p(B)$. It is however impossible to embed these two paths together in R, a contradiction. Therefore, each bridge B in R is sent into one of A_1, A_2, A_3 and A_4. However, if $\tau(B) \subset A_4$, then we would have $B = \tau(\tau(B)) \subset A_3$ by the argument in the first case. Thus, all bridges contained in A_4 or R are sent into $A_1 \cup A_2 \cup A_3$.

In either case, $\tilde{G}$ consists of the subdivision $\tilde{H}$ of the 3-cube, bridges contained in $A_1 \cup A_2 \cup A_3$ and their images by τ. Then we can construct an embedding of $\tilde{G}$ on the sphere; first embed $\tilde{H}$ on the sphere and next put a copy of the picture of A_i in each of the two faces bounded by C_i and C_{7-i} for $i = 1, 2, 3$. Therefore, $\tilde{G}$ is planar. ■

References

[1] D. Archdeacon and R.B. Richter, On the parity of planar covers, *J. Graph Theory* **14** (1990), 199–204.

[2] D. Archdeacon, Two graphs without planar covers, *J. Graph Theory* **41** (2002), 318–326.

[3] M. Fellows, Planar emulators and planar covers, manuscript, 1989.

[4] P. Hliněný $K_{4,4} - e$ has no finite planar cover, *J. Graph Theory* **27** (1998), 51–60.

[5] P. Hliněný, Planar covers of graphs: Negami's conjecture, Ph.D. Dissertation, Georgia Institute of Technology, 1999

[6] P. Hliněný, A note on possible extensions of Negami's conjecture, *J. Graph Theory* **32** (1999), 234–240.

[7] P. Hliněný, Another two graphs with no planar covers, *J. Graph Theory* **37** (2001), 227–242.

[8] P. Hliněný and R. Thomas, On possible counterexamples to Negami's planar cover conjecture, preprint.

[9] J.P. Huneke, A conjecture in topological graph theory, "Graph Structure Theory", edited by N. Robertson and P. Seymour, *Contemporary Math.* **147** (1993), 363–379.

[10] S. Kitakubo, Planar branched covering of graphs, *Yokohama Math. J.* **38** (1991), 113–120.

[11] S. Negami, Enumeration of projective-planar embeddings of graphs, *Discrete Math.* **62** (1986), 299–306.

[12] S. Negami, The spherical genus and virtually planar graphs, *Discrete Math.* **70** (1988), 159–168.

[13] S. Negami, Graphs which have no finite planar covering, *Bull. Inst. Math. Academia Sinica* **16** (1988), 377–384.

[14] S. Negami, Tower construction of planar coverings of graphs, *Information and Communication Studies of the Faculty of Information and Communication*, Bunkyo University, **25** (2000), 25–29.

[15] S. Negami, Composite planar coverings of graphs, *Discrete Math.* **268** (2003), 207–216.

[16] S. Negami, Projective-planar double coverings of graphs, to appear in *Eurpo. J. Combin.*

Department of Mathematics,
Faculty of Education and Human Sciences,
Yokohama National University,
79-2 Tokiwadai, Hodogaya-Ku,
Yokohama 240-8501,
JAPAN.
E-mail: negami@edhs.ynu.ac.jp