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STABILITY RESULTS FOR ϕ-STRONGLY PSEUDOCONTRACTIVE MAPPINGS

By

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Abstract. We study the stability of a recently introduced iteration procedure for the class of ϕ-strongly pseudocontractive mappings in real Banach spaces. Our main results improve and extend a lot of recent results.

1. Introduction

Suppose X is a real Banach space and T is a selfmap of X. Suppose $x_0 \in X$ and $x_{n+1} = f(T, x_n)$ defines an iteration procedure which yields a sequence of points (x_n) in X. Suppose $F(T) = \{x \in X \mid Tx = x\} \neq \emptyset$, and that (x_n) converges strongly to $p \in F(T)$. Suppose (y_n) is a sequence of points in X and (ε_n) is a sequence in $[0, +\infty)$ given by $\varepsilon_n = ||y_{n+1} - f(T, y_n)||$. If $\lim_{n\to\infty} \varepsilon_n = 0$ implies that $\lim_{n\to\infty} y_n = p$, then the iteration procedure defined by $x_{n+1} = f(T, x_n)$ is said to be *$T-$stable or stable with respect to T* (see [11]).

We say that the iteration procedure (x_n) is *almost T-stable or almost stable with respect to T* if $\sum_{n=0}^{\infty} \varepsilon_n < \infty$ implies that $\lim_{n\to\infty} y_n = p$ (see [21, p.319]). It is clear that an iteration procedure (x_n) which is T-stable is almost T-stable. In [21, p.328] an example of almost T-stable mapping which is not T-stable was presented.

Stability results for several iteration procedures for certain classes of nonlinear mappings have been established in the recent papers by several authors see, for example, [10], [11], [17], [19-21], [23], [29], [30] and the references therein. Harder and Hicks [10] showed how such sequences (y_n) could arise in practice and demonstrated the importance of investigating the stability of various iteration procedures for various classes of nonlinear mappings.

Let X be a real Banach space. The normalized duality mapping $J : X \to 2^{X^*}$ is defined by

$$J(x) = \{f \in X^* \mid \langle x, f\rangle = ||x||^2,\ ||x|| = ||f||\}.$$

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where X^* denotes the dual space of X and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that J is bounded, $J(\alpha x) = \alpha J(x)$ for all $\alpha \in [0, +\infty)$, $x \in X$ and that X is uniformly smooth (or equivalently, X^* is a uniformly convex Banach space) if and only if J is single-valued and uniformly continuous on bounded subsets of X.

An operator T with domain $D(T)$ and range $R(T)$ in X is called ***strongly pseudocontractive*** if for all $x, y \in D(T)$ there exist $j(x-y) \in J(x-y)$ and $t > 1$ such that

$$\langle Tx - Ty, j(x-y) \rangle \leq \frac{1}{t} ||x-y||^2. \tag{1}$$

If, in the above definition, $t = 1$, then T is said to be ***pseudocontractive operator.***

T is called ***ϕ-strongly pseudocontractive*** if for all $x, y \in D(T)$ there exist $j(x-y) \in J(x-y)$ and a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$\langle Tx - Ty, j(x-y) \rangle \leq ||x-y||^2 - \phi(||x-y||)||x-y||. \tag{2}$$

It is easy to see that the class of strongly pseudocontractive operators is a proper subset of the class of ϕ-strongly pseudocontractive operators ([18]).

An operator T is called ***strongly accretive*** if for all $x, y \in D(T)$ there exist $j(x-y) \in J(x-y)$ and a constant $k > 0$ such that

$$\langle Tx - Ty, j(x-y) \rangle \geq k||x-y||^2. \tag{3}$$

If, in the above definition, $k = 0$, then T is said to be ***accretive operator.***

T is called ***ϕ-strongly accretive*** if for all $x, y \in D(T)$ there exist $j(x-y) \in J(x-y)$ and a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$\langle Tx - Ty, j(x-y) \rangle \geq \phi(||x-y||)||x-y||. \tag{4}$$

If I denotes the identity operator, then it follows from inequalities (1)-(4) that T is strongly pseudocontractive (respectively, ϕ-strongly pseudocontractive) if and only if $I - T$ is strongly accretive (respectively, ϕ-strongly accretive). Hence the mapping theory for strongly accretive operators (respectively, ϕ-strongly accretive) is closely related to the fixed point theory for strongly pseudocontractive operators (respectively, ϕ-strongly pseudocontractive). These classes of operators have been studied extensively by several authors (see, for example [1-6, 8, 14, 15, 17-22, 30, 32, 33, 35]).

In [17], [19-21] Osilike studied stability of certain ***Mann*** [16] and ***Ishikawa*** [12] iteration procedures for fixed points of ***Lipschitz strong pseudocontractions*** as well as ϕ-strong pseudocontractions, and solutions of nonlinear accretive operators and respectively ϕ- accretive operators. In [23] the authors among other topics studied also stability of *Kirk* iteration procedure for fixed points of certain contractive-type mappings which were studied by Harder and Hiks [10], Rhoades [26-28] and Osilike [17]. In [29] the author generalized results from [23] using a new iteration procedure.

That new iteration procedure was introduced in [31] (see also [29]), for investigating of approximations of fixed points for nonexpansive mappings. This procedure is defined by

$$x_{n+1}=t_n^{(1)}T(t_n^{(2)}T(\dots T(t_n^{(k)}Tx_n+(1-t_n^{(k)})x_n+u_n^{(k)})+\dots)+(1-t_n^{(2)})x_n+u_n^{(2)})+(1-t_n^{(1)})x_n+u_n^{(1)}$$

$$x_0 \in X, \tag{5}$$

$n = 1,2,3...$, where $(t_n^{(j)})$ are real sequences in $[0,1]$ and $(u_n^{(j)})$, $j = \overline{1,k}$ are given sequences in X for a fixed natural number k.

The procedure generalizes Mann and Ishikawa iteration processes. In [31] we proved that under some conditions on $(u_n^{(j)})$, $j = \overline{1,k}$ the iteration procedure (5) converges strongly to a fixed point of nonexpansive mapping.

In this note we study the stability of iteration procedure (5) for the class of ϕ-strongly pseudocontractive mappings in arbitrary real Banach spaces. We were motivated by [5] and [21].

2. Auxiliary results

In this section we gather several auxiliary results which we need in the sequel and which are useful in general.

The following lemma is well known.

LEMMA A. *Suppose that (a_n) is a sequence of real numbers bounded from below, such that*

$$(\forall\varepsilon > 0)(\exists n_0 \in \mathbf{N})(\forall n \geq n_0)(\forall k \in \mathbf{N})\ a_{n+k} < a_n + \varepsilon.$$

Then the finite limit $\lim_{n\to\infty} a_n$ exists.

COROLLARY 1. *([32]) Suppose that $\{a_n\}$ and $\{b_n\}$ are two sequences of nonnegative numbers such that $a_{n+1} \leq a_n + b_n$ for all $n \geq 1$. If $\sum_{n=1}^{\infty} b_n < \infty$, then the finite limit $\lim_{n\to\infty} a_n$ exists.*

The following simple lemma is essentially proved in [2] and [9] together, see also [15].

LEMMA B. *Let (a_n), (b_n) and (c_n) be three non-negative real sequences satisfying the difference inequality*

$$a_{n+1} \le (1 - t_n)a_n + b_n + c_n$$

with $t_n \in [0,1]$, $\sum_{n=0}^{\infty} t_n = \infty$, $b_n = o(t_n)$ and $\sum_{n=0}^{\infty} c_n < \infty$. Then $\lim_{n\to\infty} a_n = 0$.

LEMMA 1. *Let (a_n), (b_n) and (c_n) be three non-negative real sequences satisfying the difference inequality*

$$a_{n+1} \le (1 - t_n)a_n + b_n + c_n$$

with $t_n \in [0,1]$, $\sum_{n=0}^{\infty} t_n = \infty$, $b_n = \mathcal{O}(t_n)$ and $\sum_{n=0}^{\infty} c_n < \infty$. Then the sequence (a_n) is bounded.

Proof. Let $M > 0$ be such that $|b_n| \le Mt_n$ for all $n \in \mathbf{N} \cup \{0\}$. It is easy to prove by induction that

$$a_n \le \max\{M, a_0\} + \sum_{i=0}^{n-1} c_i, \qquad n \in \mathbf{N},$$

from which the result follows.

LEMMA 2. *Let $(a_n), (b_n), (t_n), (\delta_n)$ and (ε_n) be nonnegative sequences of real numbers such that*

(a) $t_n \in [0,1], n \in \mathbf{N} \cup \{0\}$, *and* $t_n \to 0$ *as* $n \to \infty$;
(b) $\delta_n \to 0$ *as* $n \to \infty$;
(c) $\sum_{n=0}^{\infty} \varepsilon_n < \infty$;
(d) (a_n) *is bounded and* $\liminf a_n = 0$;
(e) $\lim_{n\to\infty}(a_n - b_n) = 0$;
(f) $\lim_{n\to\infty}(a_{n+1} - a_n) = 0$;
(g)

$$a_{n+1} \le a_n \left(1 - t_n \frac{f_1(b_n)}{f_2(b_n)}\right) + t_n\delta_n + \varepsilon_n,$$

where f_1 and f_2 are nonnegative increasing function on $[0,\infty)$ and $f_2(0) > 0$.
Then $\lim_{n\to\infty} a_n = 0$.

Proof. From the boundedness of the sequence (a_n) and *(e)* it follows that there is a constant $M > 0$ such that $b_n \le M$ for $n \in \mathbf{N} \cup \{0\}$. From the conditions of the theorem we have that for each $\varepsilon > 0$ there is an $n_0 \in \mathbf{N}$ such that $a_{n_0} < \varepsilon$ and

$$a_n - a_{n+1} > -\varepsilon/2; \quad b_n - a_n > -\varepsilon/4; \quad \sum_{k=n}^{\infty} \varepsilon_k < \varepsilon; \quad \delta_n < \frac{\varepsilon f_1(\varepsilon/4)}{2f_2(M)}$$

for all $n \geq n_0$.

Using this and *(g)* we obtain

$$a_{n_0+1} \leq \varepsilon + t_{n_0}\left(\frac{\varepsilon f_1(\varepsilon/4) - 2a_{n_0} f_1(b_{n_0})}{2f_2(M)}\right) + \varepsilon_{n_0}. \tag{6}$$

We show that $a_{n_0+1} \leq \varepsilon + \varepsilon_{n_0}$. Assume the contrary that $a_{n_0+1} > \varepsilon + \varepsilon_{n_0}$. Then we have $a_{n_0} > a_{n_0+1} - \varepsilon/2 > \varepsilon/2$ and consequently $b_{n_0} > a_{n_0} - \varepsilon/4 > \varepsilon/4$. From this and (6) we obtain $a_{n_0+1} \leq \varepsilon + \varepsilon_{n_0}$, which is a contradiction.

Similarly by induction we can obtain

$$a_{n_0+k} \leq \varepsilon + \sum_{i=0}^{k-1} \varepsilon_{n_0+i} < 2\varepsilon$$

for all $k \in \mathbf{N}$. From this the result follows.

In the following lemmas we give some estimates in Banach spaces. For given sequences $(x_n) \subset X$, $(u_n^{(i)}) \subset X$, $i = 1, \ldots, k$, real sequences $(t_n^{(i)})$, $i = 1, \ldots, k$ and operator $T : X \to X$, let us define

$$x_n^{(i)} = t_n^{(i+1)} T(x_n^{(i+1)}) + (1 - t_n^{(i+1)}) x_n + u_n^{(i+1)}, \quad i = 1, \ldots, k-1,$$
$$x_n^{(k)} = x_n,$$

for $n \in \mathbf{N}$.

Similarly for given sequences $(y_n) \subset X$, $(u_n^{(i)}) \subset X$, $i = 1, \ldots, k$, real sequences $(t_n^{(i)})$, $i = 1, \ldots, k$ and operator $T : X \to X$, we can define the sequences $y_n^{(i)}$, $i = 1, \ldots, k$.

We can easily prove the following lemma.

Lemma 3. *Let X be a normed space and let T be a selfmap of X. Assume a sequence (p_n) in X satisfies the following recurrent formula*

$$p_n = (1 - t_n^{(1)}) y_n + t_n^{(1)} T(y_n^{(1)}) + u_n^{(1)}, \quad n = 0, 1, \ldots, \tag{7}$$

where (y_n) is a sequence in X, $(u_n^{(i)})$, $i = \overline{1,k}$, are k sequences in X and $(t_n^{(i)})$, $i = \overline{1,k}$, are k real sequences. Then

$$\begin{aligned} y_n = {} & (1 + t_n^{(1)}) p_n + t_n^{(1)} (I - T - \gamma I) p_n - (1 - \gamma) t_n^{(1)} y_n \\ & + (2 - \gamma)(t_n^{(1)})^2 (y_n - T y_n^{(1)}) + t_n^{(1)} (T p_n - T y_n^{(1)}) - (1 + (2 - \gamma) t_n^{(1)}) u_n^{(1)} \end{aligned} \tag{8}$$

for all $n \geq 1$ and $\gamma \in \mathbf{R}$.

COROLLARY 2. *Let X be a normed space and let T be a selfmap of X. Assume a sequence (x_n) in X satisfies recurrent formula* (5), *where $(u_n^{(i)})$, $i=\overline{1,k}$, are k sequences in X and $(t_n^{(i)})$, $i=\overline{1,k}$, are k real sequences. Then*

$$\begin{aligned} x_n = (1+t_n^{(1)})x_{n+1} &+ t_n^{(1)}(I-T-\gamma I)x_{n+1} - (1-\gamma)t_n^{(1)}x_n \\ &+ (2-\gamma)(t_n^{(1)})^2(x_n - Tx_n^{(1)}) + t_n^{(1)}(Tx_{n+1} - Tx_n^{(1)}) - (1+(2-\gamma)t_n^{(1)})u_n^{(1)} \end{aligned} \tag{9}$$

for all $n \geq 1$ and $\gamma \in \mathbf{R}$.

Proof. Set $p_n = x_{n+1}$ and $y_n = x_n$ in Lemma 3.

The following lemma is essentially proved in [21, p.323] by Kato's lemma [13].

LEMMA 4. *Let X be a Banach space, T be a ϕ-strongly pseudocontractive selfmap of X. Then for all $x, y \in X$ and $r > 0$ holds*

$$||x-y|| \leq ||x-y+r((I-T-\gamma(x,y)I)x - (I-T-\gamma(x,y)I)y)||,$$

where $\gamma(x,y) = \phi(||x-y||)/(1+\phi(||x-y||)+||x-y||)$.

LEMMA 5. *Let X be a Banach space, (p_n) be the sequence in Lemma 3 and let T be a ϕ-strongly pseudocontractive selfmap of X with $F(T) \neq \emptyset$. If $p \in F(T)$, then*

$$\begin{aligned} ||p_n - p|| \leq & \frac{1+t_n^{(1)}(1-\gamma_n)}{1+t_n^{(1)}}||y_n - p|| + \frac{(2-\gamma_n)(t_n^{(1)})^2}{1+t_n^{(1)}}||y_n - Ty_n^{(1)}|| \\ & + \frac{t_n^{(1)}}{1+t_n^{(1)}}||Tp_n - Ty_n^{(1)}|| + \frac{1+t_n^{(1)}(2-\gamma_n)}{1+t_n^{(1)}}||u_n^{(1)}||, \end{aligned} \tag{10}$$

where $\gamma_n = \gamma(p_n, p)$.

Proof. We have

$$p = (1+t_n^{(1)})p + t_n^{(1)}(I-T-\gamma_n I)p - (1-\gamma_n)t_n^{(1)}p. \tag{11}$$

By (8) and (11) we obtain

$$\begin{aligned} y_n - p = (1+t_n^{(1)})(p_n - p) &+ t_n^{(1)}(\,(I-T-\gamma_n I)p_n - (I-T-\gamma_n I)p)\,) \\ &- (1-\gamma_n)t_n^{(1)}(y_n - p) + (2-\gamma_n)(t_n^{(1)})^2(y_n - Ty_n^{(1)}) + t_n^{(1)}(Tp_n - Ty_n^{(1)}) \\ &- (1+t_n^{(1)}(2-\gamma_n))u_n^{(1)}. \end{aligned}$$

From this and by Lemma 4 we obtain

$$\begin{aligned}||y_n - p|| \geq (1 + t_n^{(1)})||p_n - p|| - (1 - \gamma_n)t_n^{(1)}||y_n - p|| \\ -(2 - \gamma_n)(t_n^{(1)})^2||y_n - Ty_n^{(1)}|| - t_n^{(1)}||Tp_n - Ty_n^{(1)}|| \\ -(1 + (2 - \gamma_n)t_n^{(1)})||u_n^{(1)}||,\end{aligned} \tag{12}$$

from which the result follows.

LEMMA 6. *Let X be a normed space and $T : X \to X$ be a Lipschitzian mapping with $F(T) \neq \emptyset$. Assume a sequence (x_n) in X satisfies recurrent formula (5), where $(u_n^{(i)})$, $i = \overline{1,k}$, are k sequences in X and $(t_n^{(i)})$, $i = \overline{1,k}$, are k real sequences in $[0,1]$. Then*

$$\begin{aligned}||x_{n+1} - p|| \leq [(1 - t_n^{(1)}) + t_n^{(1)}L((1 - t_n^{(2)}) + t_n^{(2)}L((1 - t_n^{(3)}) + \cdots \\ +t_n^{(k-1)}L((1 - t_n^{(k)}) + t_n^{(k)}L)...)] \, ||x_n - p|| \\ +||u_n^{(1)}|| + t_n^{(1)}L(||u_n^{(2)}|| + t_n^{(2)}L(||u_n^{(3)}|| + \cdots + t_n^{(k-1)}L||u_n^{(k)}||)...)\end{aligned} \tag{13}$$

for all $n \geq 1$ and all $p \in F(T)$.

Proof. Let p be a fixed point of T. Since T is a Lipschitzian mapping, by (5) we get

$$\begin{aligned}||x_{n+1} - p|| &= ||(1 - t_n^{(1)})x_n + t_n^{(1)}Tx_n^{(1)} + u_n^{(1)} - p|| \\ &= ||(1 - t_n^{(1)})(x_n - p) + t_n^{(1)}(Tx_n^{(1)} - Tp) + u_n^{(1)}|| \\ &\leq (1 - t_n^{(1)})||x_n - p|| + t_n^{(1)}||Tx_n^{(1)} - Tp|| + ||u_n^{(1)}|| \\ &\leq (1 - t_n^{(1)})||x_n - p|| + t_n^{(1)}L||x_n^{(1)} - p|| + ||u_n^{(1)}||,\end{aligned} \tag{14}$$

and similarly, for $i = \overline{1, k-1}$, we obtain

$$\begin{aligned}||x_n^{(i)} - p|| &= ||(1 - t_n^{(i+1)})x_n + t_n^{(i+1)}Tx_n^{(i+1)} + u_n^{(i+1)} - p|| \\ &\leq (1 - t_n^{(i+1)})||x_n - p|| + t_n^{(i+1)}L||x_n^{(i+1)} - p|| + ||u_n^{(i+1)}||.\end{aligned} \tag{15}$$

Especially for $i = k - 1$ we have

$$\begin{aligned}||x_n^{(k-1)} - p|| &\leq (1 - t_n^{(k)})||x_n - p|| + t_n^{(k)}L||x_n^{(k)} - p|| + ||u_n^{(k)}|| \\ &= (1 - t_n^{(k)})||x_n - p|| + t_n^{(k)}L||x_n - p|| + ||u_n^{(k)}||,\end{aligned} \tag{16}$$

since $x_n^{(k)} = x_n$ for $n \in \mathbf{N}$.

From (14), (15) and (16) the result follows.

COROLLARY 3. *Let X be a normed space and $T : X \to X$ be a Lipschitzian mapping with constant $L \geq 1$ and with $F(T) \neq \emptyset$. Assume a sequence (x_n) in X satisfies recurrent formula* (5), *where $(t_n^{(i)})$, $i = \overline{1,k}$ are real sequences in* $[0,1]$ *and $(u_n^{(i)})$, $i = \overline{1,k}$ are sequences in X satisfying*

$$t_n^{(1)} \to 0 \quad as \quad n \to \infty$$
$$||u_n^{(i)}|| \to 0 \quad as \quad n \to \infty, \qquad for \quad i = \overline{2,k}.$$

Then

$$||x_{n+1} - p|| \leq L^k \, ||x_n - p|| + ||u_n^{(1)}|| + o(t_n^{(1)})$$

for all $n \geq 1$ and all $p \in F(T)$.

Similarly to Lemma 6 we can prove the following lemma.

LEMMA 6 a). *Let X be a normed space and $T : X \to X$ be a Lipschitzian mapping with $F(T) \neq \emptyset$. Assume a sequence (x_n) in X satisfies recurrent formula* (5), *where $(u_n^{(i)})$, $i = \overline{1,k}$, are k sequences in X and $(t_n^{(i)})$, $i = \overline{1,k}$, are k real sequences in* $[0,1]$. *Then*

$$\begin{aligned} ||x_n^{(i)} - p|| \leq {} & [(1 - t_n^{(i+1)}) + t_n^{(i+1)} L((1 - t_n^{(i+2)}) + t_n^{(i+2)} L((1 - t_n^{(i+3)}) + \cdots \\ & + t_n^{(k-1)} L((1 - t_n^{(k)}) + t_n^{(k)} L)...)] \, ||x_n - p|| \\ & + ||u_n^{(i+1)}|| + t_n^{(i+1)} L(||u_n^{(i+2)}|| + \cdots + t_n^{(k-1)} L||u_n^{(k)}||) \ldots) \end{aligned} \tag{17}$$

for $i = \overline{1, k-1}$ and for all $n \geq 1$ and all $p \in F(T)$.

LEMMA 7. *Let X be a normed space and $T : X \to X$ be a Lipschitzian mapping with $F(T) \neq \emptyset$. Assume a sequence (x_n) in X satisfies recurrent formula* (5), *where $(u_n^{(i)})$, $i = \overline{1,k}$, are k sequences in X and $(t_n^{(i)})$, $i = \overline{1,k}$, are k real sequences in* $[0,1]$. *Then*

$$\begin{aligned} ||x_n - Tx_n^{(1)}|| \leq {} & (1 + L((1 - t_n^{(2)}) + t_n^{(2)} L((1 - t_n^{(3)}) + \cdots \\ & t_n^{(k-1)} L((1 - t_n^{(k)}) + t_n^{(k)} L) \ldots)) \, ||x_n - p|| \\ & + L[\, ||u_n^{(2)}|| + t_n^{(2)} L(||u_n^{(3)}|| + \cdots + t_n^{(k-1)} L||u_n^{(k)}||) \ldots)\,] \end{aligned} \tag{18}$$

for $i = \overline{1, k-1}$ and for all $n \geq 1$ and all $p \in F(T)$.

Proof. Since T is a Lipschitzian mapping, we have

$$||x_n - Tx_n^{(1)}|| \leq ||x_n - p|| + L||x_n^{(1)} - p||.$$

By Lemma 6 a) the result follows.

LEMMA 8. *Let X be a normed space and $T : X \to X$ be a mapping with $F(T) \neq \emptyset$. Assume a sequence (p_n) is defined as in Lemma 3, where $(u_n^{(i)})$, $i = \overline{1,k}$, are k sequences in X and $(t_n^{(i)})$, $i = \overline{1,k}$, are k real sequences in $[0,1]$. Then*

$$\begin{aligned}||p_n - y_n^{(1)}|| \leq |t_n^{(2)} - t_n^{(1)}|\,||y_n - p|| + t_n^{(1)}||Ty_n^{(1)} - p|| + \\ +t_n^{(2)}||Ty_n^{(2)} - p|| + ||u_n^{(1)}|| + ||u_n^{(2)}||.\end{aligned} \tag{19}$$

Proof. We have

$$\begin{aligned}||p_n - y_n^{(1)}|| &= ||(t_n^{(2)} - t_n^{(1)})(y_n - p) + t_n^{(1)}(Ty_n^{(1)} - p) - \\ &\quad -t_n^{(2)}(Ty_n^{(2)} - p) + u_n^{(1)} - u_n^{(2)}|| \\ &\leq |t_n^{(2)} - t_n^{(1)}|\,||y_n - p|| + t_n^{(1)}||Ty_n^{(1)} - p|| + \\ &\quad +t_n^{(2)}||Ty_n^{(2)} - p|| + ||u_n^{(1)}|| + ||u_n^{(2)}||,\end{aligned}$$

as desired.

COROLLARY 4. *Let X be a normed space and $T : X \to X$ be a Lipschitzian mapping with $F(T) \neq \emptyset$. Assume a sequence (x_n) in X satisfies recurrent formula* (5), *where $(u_n^{(i)})$, $i = \overline{1,k}$, are k sequences in X and $(t_n^{(i)})$, $i = \overline{1,k}$, are k real sequences in $[0,1]$. Then*

$$\begin{aligned}||x_{n+1} - x_n^{(1)}|| \leq |t_n^{(2)} - t_n^{(1)}|\,||x_n - p|| + t_n^{(1)}L||x_n^{(1)} - p|| + \\ +t_n^{(2)}L\,||x_n^{(2)} - p|| + ||u_n^{(1)}|| + ||u_n^{(2)}||.\end{aligned}$$

3. Main results

We are now in a position to formulate and to prove our results. The following theorem is the main result in this paper. The proof of this result relies among other things on Lemma 2. Our starting point for this result was Theorem 2.1 in [5]. Before we formulate and prove the result we want to point out that it is not only a generalization of Theorem 2.1 in [5] for the new iteration method (5). In fact, the most important fact is that it generalizes the theorem in the case of Ishikawa iteration process with errors ($k = 2$).

THEOREM 1. *Let X be an arbitrary Banach space and $T : X \to X$ be an uniformly continuous ϕ-strongly pseudo-contractive mapping with bounded range and $F(T) \neq \emptyset$. Let $(u_n^{(i)})$, $i = \overline{1,k}$, be k sequences in X and $(t_n^{(i)})$, $i = \overline{1,k}$, be k real sequences in $[0,1]$ satisfying the following conditions:*

(a) $\sum_{n=0}^{\infty} ||u_n^{(1)}|| < \infty$;

(b) $\lim_{n\to\infty} ||u_n^{(i)}|| = 0,\ i = \overline{2,k}$;
(c) $\lim_{n\to\infty} t_n^{(i)} = 0,\ i = \overline{1,k}$;
(d) $\sum_{n=1}^{\infty} t_n^{(1)} = \infty$.

Let x_0 be an arbitrary point in X. Suppose (x_n) is a sequence in X which satisfies recurrent formula (5), *(y_n) is a sequence in X,*

$$\varepsilon_n = ||y_{n+1} - (1 - t_n^{(1)})y_n - t_n^{(1)}Ty_n^{(1)} - u_n^{(1)}||, \quad n = 0, 1, \dots .$$

Then
(I) *The sequence (x_n) is almost T stable.*
(II) $\lim_{n\to\infty} y_n = p \in F(T)$ *implies* $\lim_{n\to\infty} \varepsilon_n = 0$.

Proof. (I) Assume that $\sum_{n=0}^{\infty} \varepsilon_n < \infty$. First we show that the operator T has a unique fixed point. Otherwise there are $p, q \in F(T)$, $p \neq q$. From (2) we have

$$\langle Tp - Tq, j(p-q)\rangle \leq ||p-q||^2 - \phi(||p-q||)||p-q||$$

and consequently $\phi(||p-q||)||p-q|| \leq 0$. Since ϕ is nonnegative strictly increasing we obtain $||p-q|| = 0$ i.e. $p = q$, which is a contradiction with the assumption $p \neq q$.

Now we show that the sequence (y_n) is bounded. We have

$$\begin{aligned} ||y_{n+1} - p|| &= ||y_{n+1} - (1 - t_n^{(1)})y_n - t_n^{(1)}Ty_n^{(1)} - u_n^{(1)}|| \\ &\quad + ||(1 - t_n^{(1)})(y_n - p) + t_n^{(1)}(Ty_n^{(1)} - p) + u_n^{(1)}|| \\ &\leq (1 - t_n^{(1)})||y_n - p|| + t_n^{(1)}||Ty_n^{(1)} - p|| + ||u_n^{(1)}|| + \varepsilon_n \\ &\leq (1 - t_n^{(1)})||y_n - p|| + t_n^{(1)}M + ||u_n^{(1)}|| + \varepsilon_n \end{aligned}$$

where $M = \sup_{n\in\mathbf{N}\cup\{0\}} ||Ty_n^{(1)} - p|| < \infty$. By Lemma 1 we get the boundedness of (y_n). Let $M_1 = \sup\{||y_n^{(i)} - p||, ||Ty_n^{(i)} - p||\} < \infty$, where we take supremum over $i \in \{1, \dots, k\}$ and $n \in \mathbf{N} \cup \{0\}$.

Note that

$$\begin{aligned} ||y_{n+1} - p|| &\leq ||y_{n+1} - (1 - t_n^{(1)})y_n - t_n^{(1)}Ty_n^{(1)} - u_n^{(1)}|| \\ &\quad + ||(1 - t_n^{(1)})y_n + t_n^{(1)}Ty_n^{(1)} + u_n^{(1)} - p|| \\ &= ||p_n - p|| + \varepsilon_n. \end{aligned}$$

Form (10) and the conditions of the theorem we obtain

$$\begin{aligned} ||y_{n+1} - p|| &\leq \frac{1 + t_n^{(1)}(1 - \gamma_n)}{1 + t_n^{(1)}}||y_n - p|| + (2 - \gamma_n)(t_n^{(1)})^2||y_n - Ty_n^{(1)}|| \\ &\quad + t_n^{(1)}||Tp_n - Ty_n^{(1)}|| + (1 + t_n^{(1)}(2 - \gamma_n))||u_n^{(1)}|| + \varepsilon_n \end{aligned} \tag{20}$$

where $\gamma_n = \gamma(p_n, p)$.

Using the fact that $\gamma_n \in [0,1), n \in \mathbf{N}$, (20) and the inequality

$$\frac{1+t_n^{(1)}(1-\gamma_n)}{1+t_n^{(1)}} \leq 1 - t_n^{(1)}\gamma_n + (t_n^{(1)})^2 \tag{21}$$

we obtain that for sufficiently large n the following inequality holds

$$\begin{aligned} ||y_{n+1}-p|| \leq (1-\gamma_n t_n^{(1)} + 3(t_n^{(1)})^2)||y_n-p|| + t_n^{(1)}||Tp_n - Ty_n^{(1)}|| \\ +4(t_n^{(1)})^2 M_1 + 3||u_n^{(1)}|| + \varepsilon_n. \end{aligned} \tag{22}$$

By Lemma 8 and the conditions of the theorem we see that $||p_n - y_n^{(1)}|| \to 0$ as $n \to \infty$. Hence from the uniform continuity of T it follows

$$||Tp_n - Ty_n^{(1)}|| \to 0 \quad \text{as} \quad n \to \infty.$$

Now we show that $\liminf \gamma_n = 0$. Otherwise $\liminf \gamma_n = \gamma > 0$ and by (22) we have that for all $\varepsilon \in (0, \gamma/2)$ and for sufficiently large n

$$\begin{aligned} ||y_{n+1}-p|| \leq (1-(\gamma-\varepsilon)t_n^{(1)})||y_n-p|| + t_n^{(1)}\delta_n \\ +4(t_n^{(1)})^2 M_1 + 3||u_n^{(1)}|| + \varepsilon_n, \end{aligned} \tag{23}$$

holds, where $\delta_n = ||Tp_n - Ty_n^{(1)}||$. By Lemma B we obtain $||y_n - p|| \to 0$ as $n \to \infty$. By Lemma 5 and the conditions of the theorem we obtain $||p_n - p|| \to 0$ as $n \to \infty$. Hence $\gamma = 0$, arriving at a contradiction.

Since $\liminf \gamma_n = 0$ we have $\liminf ||p_n - p|| = 0$. Furthermore, it follows from the inequality

$$\begin{aligned} ||p_n-p|| - t_n^{(1)}(||Ty_n^{(1)}-p|| + ||y_n-p||) - ||u_n^{(1)}|| \leq ||y_n-p|| \\ \leq ||p_n-p|| + t_n^{(1)}(||Ty_n^{(1)}-p|| + ||y_n-p||) + ||u_n^{(1)}|| \end{aligned}$$

that $\liminf ||y_n - p|| = 0$, moreover

$$\lim_{n\to\infty}(||y_n-p|| - ||p_n-p||) = 0.$$

On the other hand we have

$$\varepsilon_n - ||t_n^{(1)}(y_n - Ty_n^{(1)}) - u_n^{(1)}|| \leq ||y_{n+1}-y_n|| \leq \varepsilon_n + ||t_n^{(1)}(y_n - Ty_n^{(1)}) - u_n^{(1)}||.$$

Hence $\lim_{n\to\infty} ||y_{n+1} - y_n|| = 0$ and consequently

$$\lim_{n\to\infty}(||y_{n+1}-p|| - ||y_n-p||) = 0.$$

Let $a_n = ||y_n - p||$ and $b_n = ||p_n - p||$. Then (22) can be written in the following form

$$a_{n+1} \le a_n \left(1 - t_n^{(1)} \frac{\phi(b_n)}{1 + \phi(b_n) + b_n}\right) + t_n^{(1)} \delta_n^{(1)} + \varepsilon_n^{(1)},$$

where $\delta_n^{(1)} \to 0$ as $n \to \infty$ and $\sum_{n=0}^{\infty} \varepsilon_n^{(1)} < \infty$.

Since all conditions of Lemma 2 are satisfied we obtain $\lim_{n\to\infty} a_n = 0$, as desired.

(II) Let $\lim_{n\to\infty} y_n = p \in F(T)$, then by some simple calculations and by the conditions of the theorem we obtain

$$\begin{aligned} \varepsilon_n &= ||y_{n+1} - (1 - t_n^{(1)}) y_n - t_n^{(1)} T y_n^{(1)} - u_n^{(1)}|| \\ &\le ||y_{n+1} - p|| + (1 - t_n^{(1)}) ||y_n - p|| + t_n^{(1)} ||T y_n^{(1)} - p|| + ||u_n^{(1)}|| \to 0 \end{aligned}$$

as $n \to \infty$, as desired.

The following theorem is a natural generalization of Theorem 1 in [21]. The proof of the theorem follows the lines of the proof of that Theorem with necessary modifications. Some details in the proof are shorter since we use lemmas from section 2.

THEOREM 2. *Let X be an arbitrary Banach space and $T : X \to X$ be a Lipschitz ϕ-strongly pseudo-contractive mapping with $F(T) \ne \emptyset$. Let $(u_n^{(i)})$, $i = \overline{1,k}$, be k sequences in X and $(t_n^{(i)})$, $i = \overline{1,k}$, be k real sequences in $[0,1]$ satisfying the following conditions:*

(a) $\sum_{n=0}^{\infty} ||u_n^{(i)}|| < \infty$, $i = \overline{1,2}$;
(b) $\lim_{n\to\infty} ||u_n^{(i)}|| = 0$, $i = \overline{2,k}$;
(c) $\lim_{n\to\infty} t_n^{(i)} = 0$, $i = \overline{1,k}$;
(d) $\sum_{n=1}^{\infty} t_n^{(1)} t_n^{(2)} < \infty$;
(e) $\sum_{n=1}^{\infty} (t_n^{(1)})^2 < \infty$.

Let x_0 be an arbitrary point in X. Suppose (x_n), $(y_n^{(i)})$ $i = 1, \dots, k$ and (ε_n) are as in Theorem 1.

Then

(I) *If $p \in F(T)$, then*

$$\begin{aligned} ||y_{n+1} - p|| \le (1 - t_n^{(1)} \gamma(p_n, p)) ||y_n - p|| + C_1 (t_n^{(1)})^2 ||y_n - p|| \\ + C_2 t_n^{(1)} t_n^{(2)} ||y_n - p|| + 3||u_n^{(1)}|| + 3L||u_n^{(2)}|| + \varepsilon_n, \end{aligned} \tag{24}$$

for some $C_1, C_2 > 0$.

(II) *The sequence (x_n) is almost T stable.*

(III) $\lim_{n\to\infty} y_n = p \in F(T)$ *implies* $\lim_{n\to\infty} \varepsilon_n = 0$

Proof. (I) As in Theorem 1 we obtain $F(T)$ is a singleton if $F(T) \neq \emptyset$. Also

$$||y_{n+1} - p|| \leq ||p_n - p|| + \varepsilon_n.$$

By Lemma 5 and the conditions of the theorem we obtain

$$\begin{aligned}||y_{n+1} - p|| \leq \frac{1 + t_n^{(1)}(1 - \gamma_n)}{1 + t_n^{(1)}}||y_n - p|| + (2 - \gamma_n)(t_n^{(1)})^2||y_n - Ty_n^{(1)}|| \\ + t_n^{(1)}L||p_n - y_n^{(1)}|| + (1 + t_n^{(1)}(2 - \gamma_n))||u_n^{(1)}|| + \varepsilon_n\end{aligned}$$

where $\gamma_n = \gamma(p_n, p)$.

Applying Lemma 7 and Lemma 8 to (24) we obtain the result.

(II) From (23) we have

$$||y_{n+1} - p|| \leq (1 + \delta_n)||y_n - p|| + 3||u_n^{(1)}|| + 3L||u_n^{(2)}|| + \varepsilon_n,$$

where $\sum_{n=0}^{\infty} \delta_n < \infty$. From this and Lemma 1 we obtain that the sequence (y_n) is bounded above by some K. By Corollary 1 we obtain that there is a finite limit $\lim_{n\to\infty} ||y_n - p|| = a$. We prove that $a = 0$. Assume that $a > 0$. By Lemma 6 a) and since (y_n) is bounded, the sequence $||Ty_n^{(1)} - p||$ is also bounded. From the inequality

$$(1-t_n^{(1)})||y_n-p||-t_n^{(1)}||Ty_n^{(1)}-p|| \leq ||p_n-p|| \leq (1-t_n^{(1)})||y_n-p||+t_n^{(1)}||Ty_n^{(1)}-p||$$

and since $t_n^{(1)} \to 0$ as $n \to \infty$ we obtain that

$$\lim_{n\to\infty} ||p_n - p|| = a.$$

Hence $||p_n - p||$ is bounded i.e. there are constants $m, M > 0$ such that

$$m \leq ||p_n - p|| \leq M \quad \text{for} \quad n \geq n_0$$

then

$$\gamma(p_n, p) \geq \frac{\phi(m)}{1 + \phi(M) + M} = M_1 \quad \text{for} \quad n \geq n_0$$

so that (24) implies that

$$\begin{aligned}||y_{n+1} - p|| \leq (1 - t_n^{(1)}M_1)||y_n - p|| + C_1(t_n^{(1)})^2K \\ + C_2t_n^{(1)}t_n^{(2)}K + 3||u_n^{(1)}|| + 3L||u_n^{(2)}|| + \varepsilon_n.\end{aligned} \tag{25}$$

From (25) and Lemma B the result follows.

(III) Let $\lim_{n\to\infty} y_n = p \in F(T)$, then as in previous theorem we obtain

$$\begin{aligned}\varepsilon_n &= ||y_{n+1} - (1 - t_n^{(1)})y_n - t_n^{(1)}Ty_n^{(1)} - u_n^{(1)}|| \\ &\leq ||y_{n+1} - p|| + (1 - t_n^{(1)})||y_n - p|| + t_n^{(1)}L||y_n^{(1)} - p|| + ||u_n^{(1)}|| \to 0\end{aligned}$$

as $n \to \infty$, as desired.

REMARK 1. For $\varepsilon_n = 0,\ n \geq 0$, we obtain that under conditions of Theorem 1 or Theorem 2 the sequence (x_n) converges strongly to $p \in F(T)$ and $F(T)$ is a single set.

REMARK 2. The reader can state and prove the corresponding results for ϕ-accretive operators.

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