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AN OPTIMAL QUADRATURE FORMULA OF CLOSED TYPE

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Abstract. An optimal 3-point quadrature formula of closed type is derived. The optimality is considered with respect to a given way of estimation of its error.

1. Introduction

In recent years a number of authors have considered an error analysis for quadrature rules of Newton-Cotes type. In particular, the mid-point, trapezoid and Simpson rules have been investigated more recently ([1], [2], [3], [4], [9]) with the view of obtaining bounds on the quadrature rule in terms of a variety of norms involving, at most, the first derivative. In the mentioned papers explicit error bounds for the quadrature rules are given. These results are obtained from an inequalities point of view. The authors use Peano type kernels for obtaining a specific quadrature rule.

Quadrature formulas can be formed in many different ways. For example, we can integrate a Lagrange interpolating polynomial of a function f to obtain a corresponding quadrature formula (Newton-Cotes formulas). We can also seek a quadrature formula such that it is exact for polynomials of maximal degree (Gauss formulas). Gauss-like quadrature formulas are considered in [10].

Here we present a new approach to this topic. Namely, we give a type of quadrature formula. We also give a way of estimation of its error and all parameters which appear in the estimation. Then we seek a quadrature formula of the given type such that the estimation of its error is best possible. Let us consider the above described procedure with more details.

If we define

$$K(x,t)=\begin{cases} t-\alpha, t\in[a,x] \\ t-\beta, t\in(x,b] \end{cases}$$

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then, integrating by parts, we obtain

(1.1)
$$\int_a^b K(x,t)f'(t)dt = (\alpha - a)f(a) + (\beta - \alpha)f(x) + (b-\beta)f(b) - \int_a^b f(t)dt,$$

where $x \in [a,b]$. If we choose $\alpha = \beta = \frac{a+b}{2}$ then we get the trapezoid rule. If we choose $\alpha = a$, $\beta = b$ and $x = \frac{a+b}{2}$ then we get the mid-point quadrature rule. If we choose $\alpha = \frac{5a+b}{6}$, $\beta = \frac{a+5b}{6}$ and $x = \frac{a+b}{2}$ then we get Simpson's rule. In practice we cannot find an exact value of the remainder term (error) $\int_a^b K(x,t)f'(t)dt$. All we can do is to estimate the error. It can be done in different ways. For example,

$$\left|\int_a^b K(x,t)f'(t)dt\right| \leq \max_{t\in[a,b]} |f'(t)| \int_a^b |K(x,t)|\, dt \tag{1.2}$$

or

$$\left|\int_a^b K(x,t)f'(t)dt\right| \leq \|f'\|_2 \, \|K(x,\cdot)\|_2 \,, \tag{1.3}$$

where

$$\|g\|_2 = \left(\int_a^b g^2(t)dt\right)^{1/2}.$$

It is a natural question which formula of the type (1.2) is optimal, with respect to a given way of estimation of the error. The main aim of this paper is to give an answer to this question. In fact, we seek a quadrature formula of the given type such that its error bound is minimal. Note that we can only minimize the factor $\int_a^b |K(x,t)|\, dt$ in (1.2) or the factor $\|K(x,.)\|_2$ in (1.3). A general approach is: we first consider the minimization problem and then we formulate final results.

Finally, let us mention that a quadrature formula, which will be obtained in this paper, exists in the literature. Hence, here we prove that it is optimal in the described way. It is also shown that it has a better estimation of error than the Simpson's rule (see Remarks 1, 2 and 3).

2. An estimation in L_∞ and L_1 norms

We first consider the problem, described in Section 1, on the interval $[0,1]$. Let $\alpha, \beta \in R$ and $x \in [0,1]$. We define the mapping

$$K(x,t) = \begin{cases} t-\alpha, t \in [0,x] \\ t-\beta, t \in (x,1] \end{cases} . \tag{2.1}$$

Let $I \subset R$ be an open interval such that $[0,1] \subset I$ and let $f : I \to R$ be a differentiable function such that f' is bounded and integrable. We denote

$$\|f\|_\infty = \sup_{t\in[0,1]} |f(t)| . \tag{2.2}$$

Integrating by parts, we obtain

$$\int_0^1 K(x,t)f'(t)dt = \alpha f(0) + (\beta-\alpha)f(x) + (1-\beta)f(1) - \int_0^1 f(t)dt. \tag{2.3}$$

We also have

$$\left|\int_0^1 K(x,t)f'(t)dt\right| \le \|f'\|_\infty \int_0^1 |K(x,t)|\,dt \tag{2.4}$$

and

$$\int_0^1 |K(x,t)|\,dt = \int_0^x |t-\alpha|\,dt + \int_x^1 |t-\beta|\,dt. \tag{2.5}$$

We now define

$$g(\alpha,\beta,x) = \int_0^x |t-\alpha|\,dt + \int_x^1 |t-\beta|\,dt \tag{2.6}$$

and consider the problem

$$\text{minimize } g(p), p = (\alpha,\beta,x) \in R^2 \times [0,1] . \tag{2.7}$$

Hence, we should like to find a global minimizer of $g(p)$, for $p \in Q = R^2 \times [0,1]$. Recall, a global minimizer is a point $p^* = (\alpha^*, \beta^*, x^*)$ that satisfies

$$g(p^*) \le g(p), \text{for all } p \in Q. \tag{2.8}$$

The problem (2.7) we shall solve in the following few steps.

STEP I. If $x = 0$ or $x = 1$ then $K(x,t) = t - \beta$ or $K(x,t) = t - \alpha$. The problem (2.7) becomes:

$$\text{minimize } g(\alpha),\ \alpha \in R, \tag{2.9}$$

where

$$g(\alpha) = \int_0^1 |t - \alpha|\, dt. \tag{2.10}$$

If $\alpha \le 0$ then

$$g(\alpha) = \int_0^1 |t - \alpha|\, dt = \frac{1}{2} - \alpha \ge \frac{1}{2}. \tag{2.11}$$

If $\alpha \ge 1$ then

$$g(\alpha) = \int_0^1 |t - \alpha|\, dt = -\frac{1}{2} + \alpha \ge \frac{1}{2}. \tag{2.12}$$

If $\alpha \in (0,1)$ then

$$g(\alpha) = \int_0^1 |t - \alpha|\, dt = \alpha^2 - \alpha + \frac{1}{2}. \tag{2.13}$$

In this case we have

$$g'(\alpha) = 2\alpha - 1 \text{ and } g''(\alpha) = 2 > 0.$$

From the equation $g'(\alpha) = 0$ we find the stationary point $\alpha^* = \frac{1}{2}$. Since $g''(\alpha) = 2 > 0$ we conclude that $\alpha^* = \frac{1}{2}$ is a minimizer. We have

$$g(\alpha^*) = \frac{1}{4}. \tag{2.14}$$

From (2.11), (2.12) and (2.14) we see that the global minimizer, in this case, is given by $\alpha^* = \frac{1}{2}$.

On the other hand, let us consider the well-known Simpson's rule:

$$\frac{f(0) + 4f(\frac{1}{2}) + f(1)}{6} - \int_0^1 f(t)dt = \int_0^1 K(\frac{1}{2}, t) f'(t)dt, \tag{2.15}$$

where

$$K(\frac{1}{2},t) = \begin{cases} t - \frac{1}{6}, & t \in [0, \frac{1}{2}] \\ t - \frac{5}{6}, & t \in (\frac{1}{2}, 1] \end{cases} . \tag{2.16}$$

(In this case $\alpha = \frac{1}{6}$, $\beta = \frac{5}{6}$ and $x = \frac{1}{2}$.) We have

$$\int_0^1 \left| K(\frac{1}{2},t) \right| dt = \frac{5}{36}. \tag{2.17}$$

From (2.14) and (2.17) we conclude that the cases $x = 0$ or $x = 1$ cannot give a global minimizer of the function g, defined by (2.6), since $\frac{5}{36} < \frac{1}{4}$. That is, the points $p_1 = (\alpha, \beta, 0)$ and $p_2 = (\alpha, \beta, 1)$ cannot be solutions of the problem (2.7).

STEP II. We now divide our problem in the following cases:

(i) $\alpha \leq 0$, $\beta \leq x$;
(ii) $\alpha \leq 0$, $\beta \in [x, 1]$;
(iii) $\alpha \leq 0$, $\beta \geq 1$;
(iv) $\alpha \in [0, x]$, $\beta \leq x$;
(v) $\alpha \in [0, x]$, $\beta \in [x, 1]$;
(vi) $\alpha \in [0, x]$, $\beta \geq 1$;
(vii) $\alpha \geq x$, $\beta \leq x$;
(viii) $\alpha \geq x$, $\beta \in [x, 1]$;
(ix) $\alpha \geq x$, $\beta \geq 1$,

where $x \in [0, 1]$.

STEP III. Let us consider the case (i). We have

$$g(\alpha, \beta, x) = \int_0^x (t - \alpha) dt + \int_x^1 (t - \beta) dt = -\alpha x + \beta x - \beta + \frac{1}{2}. \tag{2.18}$$

We first seek stationary points of the function g, defined by (2.18). We have $\frac{\partial g}{\partial \alpha} = -x$, $\frac{\partial g}{\partial \beta} = x - 1$, $\frac{\partial g}{\partial x} = \beta - \alpha$. From the equations $\frac{\partial g}{\partial \alpha} = 0$ and $\frac{\partial g}{\partial \beta} = 0$ we find that $x = 0$ and $x = 1$. Hence, there are no stationary points in this sub-case. We must also consider the function g on the boundary ∂Q of Q, where $Q = \{(\alpha, \beta, x) : \alpha \leq 0, \beta \leq x, x \in [0, 1]\}$. We don't choose $x = 0$ and $x = 1$, for the reasons given in Step I. Let $\alpha = 0$. Then from (2.18) we get $g(0, \beta, x) = \beta x - \beta + \frac{1}{2}$. Thus, $\frac{\partial g}{\partial \beta} = x - 1$ and $\frac{\partial g}{\partial x} = \beta$. From the equation $\frac{\partial g}{\partial \beta} = 0$ we get $x = 1$. Hence, this sub-case can be omitted. Let $\beta = x$. Then (2.18) becomes: $g(\alpha, x, x) = x^2 - \alpha x - x + \frac{1}{2}$. We have: $\frac{\partial g}{\partial \alpha} = -x$ and $\frac{\partial g}{\partial x} = 2x - \alpha - 1$. From the equation $\frac{\partial g}{\partial \alpha} = 0$ we get $x = 0$. This sub-case can also be omitted.

We conclude that from the case (i) we cannot find a global minimizer. In a similar way we can consider the cases (iii), (vii) and (ix) with the same result.

STEP IV. Let us now consider the case (iv). We have

$$g(\alpha,\beta,x) = -\int_0^\alpha (t-\alpha)dt + \int_\alpha^x (t-\alpha)dt + \int_x^1 (t-\beta)dt \tag{2.19}$$
$$= \alpha^2 - \alpha x + \beta x - \beta + \frac{1}{2}.$$

We seek stationary points of the function g, defined by (2.19). We have $\frac{\partial g}{\partial \alpha} = 2\alpha - x$, $\frac{\partial g}{\partial \beta} = x - 1$, $\frac{\partial g}{\partial x} = \beta - \alpha$. From the equation $\frac{\partial g}{\partial \beta} = 0$ we find that $x = 1$. Hence, this sub-case is not interesting for us, for the reasons given in Step I. We must also consider the function g on the boundary ∂Q of Q, where $Q = \{(\alpha,\beta,x) : \alpha \in [0,x], \beta \leq x, x \in [0,1]\}$. The sub-cases $x = 0$ and $x = 1$ we omit, for the known reasons. For $\alpha = 0$, from (2.19) we get $g(0,\beta,x) = \beta x - \beta + \frac{1}{2}$. Thus, $\frac{\partial g}{\partial \beta} = x - 1$ and $\frac{\partial g}{\partial x} = \beta$. From the equation $\frac{\partial g}{\partial \beta} = 0$ we find that $x = 1$. Hence, this sub-case can be omitted. For $\alpha = x$ we have $g(x,\beta,x) = \beta x - \beta + \frac{1}{2}$. This is the same sub-case as the last one. Finally, let $\beta = x$. Then we have $g(\alpha,x,x) = \alpha^2 - \alpha x + x^2 - x + \frac{1}{2}$. Thus, $\frac{\partial g}{\partial \alpha} = 2\alpha - x$ and $\frac{\partial g}{\partial x} = -\alpha + 2x - 1$. From the equation $\frac{\partial g}{\partial \alpha} = 0$ and $\frac{\partial g}{\partial x} = 0$ we find the stationary point $p^* = (\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$. If we substitute p^* in (2.19) then we get $g(p^*) = \frac{1}{6}$. Hence, in this sub-case we have

$$\int_0^1 \left| K(\frac{2}{3}, t) \right| dt = \frac{1}{6}. \tag{2.20}$$

If we now compare (2.20) with (2.17) then we see that p^* cannot be a global minimizer, since $\frac{5}{36} < \frac{1}{6}$.

We again conclude that from the case (iv) we cannot find a global minimizer. In a similar way we can consider the cases (ii), (vi) and (viii) with the same result.

STEP V. It remains to consider the case (v). We have

$$g(\alpha,\beta,x) \tag{2.21}$$
$$= -\int_0^\alpha (t-\alpha)dt + \int_\alpha^x (t-\alpha)dt - \int_x^\beta (t-\beta)dt + \int_\beta^1 (t-\beta)dt$$
$$= \alpha^2 + \beta^2 + x^2 - \alpha x - \beta x -$$
$$\beta + \frac{1}{2}. \tag{2.22}$$

We seek stationary points of the function g, defined by (2.21). We have $\frac{\partial g}{\partial \alpha} = 2\alpha - x$, $\frac{\partial g}{\partial \beta} = 2\beta - x - 1$, $\frac{\partial g}{\partial x} = 2x - \beta - \alpha$. From the equations $\frac{\partial g}{\partial \alpha} = 0$, $\frac{\partial g}{\partial \beta} = 0$ and $\frac{\partial g}{\partial x} = 0$ we find the stationary point $p^* = (\frac{1}{4}, \frac{3}{4}, \frac{1}{2})$. The Hessian

$$H = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

is a positive definite matrix. Thus, p^* is a minimizer. We have

$$g(p^*) = \frac{1}{8}.$$

We must also consider the function g on the boundary ∂Q of Q, where $Q = \{(\alpha, \beta, x) : \alpha \in [0, x], \beta \in [x, 1], x \in [0, 1]\}$. The sub-cases $x = 0$ and $x = 1$ we omit, for the reasons given in Step I. For $\alpha = 0$, from (2.21) we get $g(0, \beta, x) = \beta^2 + x^2 - \beta x - \beta + \frac{1}{2}$. Thus, $\frac{\partial g}{\partial \beta} = 2\beta - x - 1$ and $\frac{\partial g}{\partial x} = 2x - \beta$. From the equations $\frac{\partial g}{\partial \beta} = 0$ and $\frac{\partial g}{\partial x} = 0$ we find the stationary point $p^* = (0, \frac{2}{3}, \frac{1}{3})$ and $g(p^*) = \frac{1}{6}$. Hence, this p^* cannot be a global minimizer. For $\alpha = x$ we have $g(x, \beta, x) = x^2 + \beta^2 - \beta x - \beta + \frac{1}{2}$. For $\beta = x$ we have $g(\alpha, x, x) = \alpha^2 + x^2 - \alpha x - x + \frac{1}{2}$. Finally, for $\beta = 1$ we get $g(\alpha, 1, x) = \alpha^2 + x^2 - \alpha x - x + \frac{1}{2}$. The last three sub-cases are of the same type as the last considered one. They cannot give a global minimizer.

We considered all possible cases. From the steps I-V we conclude that $p^* = (\frac{1}{4}, \frac{3}{4}, \frac{1}{2})$ is the global minimizer. Our problem is solved.

If we now substitute $\alpha = \frac{1}{4}$, $\beta = \frac{3}{4}$ and $x = \frac{1}{2}$ in (2.3) then we get

$$\int_0^1 K(\frac{1}{2}, t) f'(t) dt = \frac{1}{4}\left[f(0) + 2f(\frac{1}{2}) + f(1)\right] - \int_0^1 f(t) dt. \tag{2.23}$$

The above formula is optimal in the sense described in Section 1.

From the previous considerations we can formulate the following result.

THEOREM 1. *Let $I \subset R$ be an open interval such that $[0, 1] \subset I$ and let $f : I \to R$ be a differentiable function such that f' is bounded and integrable. Then we have*

$$\left|\frac{1}{4}\left[f(0) + 2f(\frac{1}{2}) + f(1)\right] - \int_0^1 f(t) dt\right| \leq \frac{\|f'\|_\infty}{8}. \tag{2.24}$$

REMARK 1. From (2.4), (2.15) and (2.17) we get

$$\left|\frac{1}{6}\left[f(0) + 4f(\frac{1}{2}) + f(1)\right] - \int_0^1 f(t) dt\right| \leq \frac{5\|f'\|_\infty}{36}. \tag{2.25}$$

From (2. 24) and (2. 25) we see that the quadrature formula defined by (2. 23) has a better estimate of error than the well-known Simpson formula, (since $\frac{1}{8} < \frac{5}{36}$). Note also that both formulas are of the same type, namely 3-point quadrature formulas of closed type.

We can say something more about the quadrature formula (2. 23). When we have the estimate (2. 24) then we can try to improve the first factor depending on f'. If $\alpha = \frac{1}{4}$, $\beta = \frac{3}{4}$ and $x = \frac{1}{2}$ then

$$\hat{K}(\frac{1}{2}, t) = \begin{cases} t - \frac{1}{4}, t \in [0, \frac{1}{2}] \\ t - \frac{3}{4}, t \in (\frac{1}{2}, 1] \end{cases} \tag{2. 26}$$

and

$$\int_0^1 \hat{K}(\frac{1}{2}, t)dt = 0. \tag{2. 27}$$

Thus,

$$\int_0^1 \hat{K}(\frac{1}{2}, t)f'(t)dt = \int_0^1 \hat{K}(\frac{1}{2}, t)\left[f'(t) - C\right]dt, \tag{2. 28}$$

where $C \in R$ is an arbitrary constant. If f' is bounded then there exist $\gamma, \Gamma \in R$ such that

$$\gamma \leq f'(t) \leq \Gamma, \text{ for all } t \in [0, 1]. \tag{2. 29}$$

If we choose $C = \frac{\Gamma+\gamma}{2}$ in (2. 28) then we get

$$\begin{aligned} \left|\int_0^1 \hat{K}(\frac{1}{2}, t)f'(t)dt\right| &\leq \max_{t\in[0,1]} \left|f'(t) - \frac{\Gamma+\gamma}{2}\right| \int_0^1 \left|\hat{K}(\frac{1}{2}, t)\right| dt \\ &\leq \frac{\Gamma - \gamma}{16}, \end{aligned} \tag{2. 30}$$

since $\max_{t\in[0,1]} \left|f'(t) - \frac{\Gamma+\gamma}{2}\right| \leq \frac{\Gamma-\gamma}{2}$. We got the following result.

THEOREM 2. *Under the assumptions of Theorem 1 and (2. 29) we have*

$$\left|\frac{1}{4}\left[f(0) + 2f(\frac{1}{2}) + f(1)\right] - \int_0^1 f(t)dt\right| \leq \frac{\Gamma - \gamma}{16}. \tag{2. 31}$$

REMARK 2. For Simpson's rule we have

$$\left|\frac{1}{6}\left[f(0)+4f(\frac{1}{2})+f(1)\right]-\int_0^1 f(t)dt\right| \leq \frac{5(\Gamma-\gamma)}{72}. \tag{2.32}$$

From (2.31) and (2.32) we see that the quadrature rule defined by (2.23) has a better estimate of error than the well-known Simpson's rule.

If we consider the above problem on the interval $[a,b]$ then we shall get the following results.

THEOREM 3. *Let $I \subset R$ be an open interval such that $[a,b] \subset I$ and let $f : I \to R$ be a differentiable function such that f' is bounded and integrable. Then we have*

$$\left|\frac{b-a}{4}\left[f(a)+2f(\frac{a+b}{2})+f(b)\right]-\int_a^b f(t)dt\right| \leq \frac{\|f'\|_\infty}{8}(b-a)^2. \tag{2.33}$$

THEOREM 4. *Under the assumptions of Theorem 3 we have*

$$\left|\frac{b-a}{4}\left[f(a)+2f(\frac{a+b}{2})+f(b)\right]-\int_a^b f(t)dt\right| \leq \frac{\Gamma-\gamma}{16}(b-a)^2, \tag{2.34}$$

where $\gamma \leq f'(t) \leq \Gamma$, for all $t \in [a,b]$.

3. An estimation in L_2-norm

Let $K(x,t)$ be defined by (2.1) and let the assumptions of Theorem 1 hold. We here suppose that $f' \in L_2(0,1)$ and denote

$$\|f\|_2 = \left(\int_0^1 f^2(t)dt\right)^{1/2}. \tag{3.1}$$

Then we have

$$\left|\int_0^1 K(x,t)f'(t)dt\right| \leq \|f'\|_2 \, \|K(x,\cdot)\|_2\,, \tag{3.2}$$

where

$$\|K(x,\cdot)\|_2^2 = \int_0^1 K^2(x,t)dt = \int_0^x (t-\alpha)^2 dt + \int_x^1 (t-\beta)^2 dt. \tag{3.3}$$

We define

$$g(\alpha,\beta,x) = \int_0^x (t-\alpha)^2 dt + \int_x^1 (t-\beta)^2 dt. \tag{3.4}$$

Our problem is

$$\text{minimize } g(p), p = (\alpha,\beta,x) \in R^3. \tag{3.5}$$

We find that

$$g(\alpha,\beta,x) = \frac{1}{3}\left[(x-\alpha)^3 + \alpha^3 + (1-\beta)^3 - (x-\beta)^3\right]. \tag{3.6}$$

We seek stationary points of g, defined by (3. 6). For that purpose, we must solve the system:

$$\frac{\partial g}{\partial \alpha} = \alpha^2 - (x-\alpha)^2 = 0$$
$$\frac{\partial g}{\partial \beta} = (x-\beta)^2 - (1-\beta)^2 = 0$$
$$\frac{\partial g}{\partial x} = (x-\alpha)^2 - (x-\beta)^2 = 0.$$

From the above system we get the following systems:

$$\begin{matrix} x-\alpha=\alpha & x-\alpha=\alpha & x-\alpha=\alpha & x-\alpha=\alpha \\ 1-\beta=x-\beta & 1-\beta=x-\beta & 1-\beta=\beta-x & 1-\beta=\beta-x \\ x-\alpha=x-\beta & x-\alpha=\beta-x & x-\alpha=x-\beta & x-\alpha=\beta-x \end{matrix}$$

and

$$\begin{matrix} x-\alpha=-\alpha & x-\alpha=-\alpha & x-\alpha=-\alpha & x-\alpha=-\alpha \\ 1-\beta=x-\beta & 1-\beta=x-\beta & 1-\beta=\beta-x & 1-\beta=\beta-x \\ x-\alpha=x-\beta & x-\alpha=\beta-x & x-\alpha=x-\beta & x-\alpha=\beta-x \end{matrix}$$

The above systems have the solutions: $p_1 = (-\frac{1}{2},\frac{1}{2},0)$, $p_2 = (\frac{1}{2},\frac{1}{2},0)$, $p_3 = (\frac{1}{2},\frac{1}{2},1)$, $p_4 = (\frac{1}{2},\frac{3}{2},1)$ and $p_5 = (\frac{1}{4},\frac{3}{4},\frac{1}{2})$. (Three systems have no solutions.)

We now calculate: $g(p_1) = g(p_2) = g(p_3) = g(p_4) = \frac{1}{12}$, $g(p_5) = \frac{1}{48}$. We conclude: the global minimizer is $p_5 = (\frac{1}{4},\frac{3}{4},\frac{1}{2})$. For $\alpha = \frac{1}{4}$, $\beta = \frac{3}{4}$ and $x = \frac{1}{2}$ we get (2. 23). Hence, in this case, (2. 23) is an optimal quadrature formula. We got the following result.

THEOREM 5. *Let $I \subset R$ be an open interval such that $[0,1] \subset I$ and let $f : I \to R$ be a differentiable function such that $f' \in L_2(0,1)$. Then we have*

$$\left|\frac{1}{4}\left[f(0)+2f(\frac{1}{2})+f(1)\right]-\int_0^1 f(t)dt\right| \le \frac{\|f'\|_2}{4\sqrt{3}}. \tag{3.7}$$

REMARK 3. For Simpson's formula we have

$$\left|\frac{1}{6}\left[f(0)+4f(\frac{1}{2})+f(1)\right]-\int_0^1 f(t)dt\right| \le \frac{\|f'\|_2}{6}. \tag{3.8}$$

It is obvious that (3.7) is better than (3.8).

If we consider the above problem on the interval $[a,b]$ then we get the following result.

THEOREM 6. *Let $I \subset R$ be an open interval such that $[a,b] \subset I$ and let $f : I \to R$ be a differentiable function such that $f' \in L_2(a,b)$. Then we have*

$$\left|\frac{b-a}{4}\left[f(a)+2f(\frac{a+b}{2})+f(b)\right]-\int_a^b f(t)dt\right| \le \frac{\|f'\|_2}{4\sqrt{3}}(b-a)^{3/2}. \tag{3.9}$$

REMARK 4. If we now compare (2.33) with (3.9) then we must note that the estimate (2.33) can be applied only if f' is bounded, while the estimate (3.9) can be also applied for a function whose derivative is unbounded (if $f' \in L_2(a,b)$).

EXAMPLE 1. Let us consider the integral

$$\int_0^1 \sqrt[3]{\sin t^2}dt.$$

We have

$$f(t) = \sqrt[3]{\sin t^2} and f'(t) = \frac{2t\cos t^2}{3\sqrt[3]{\sin^2 t^2}}$$

such that $f'(t) \to \infty$, $t \to 0$ i.e. $\|f'\|_\infty = \infty$ and we cannot apply the estimate (2.33). On the other hand, we have

$$\int_0^1 [f'(t)]^2\,dt \le \frac{4}{9}\max_{t\in[0,1]}\frac{t^2\cos t^2}{\sin t^2}\int_0^1 \frac{dt}{\sqrt[3]{\sin t^2}} \le \frac{16}{9},$$

i.e. $\|f'\|_2 \le \frac{4}{3}$ and we can apply the estimate (3.9).

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