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# NOTES ON BERTRAND CURVES

By

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**Abstract.** Every circular helix in $E^3$ is a typical example of Bertrand curve. The circular helix is one in a family of special Frenet curves. We prove that no special Frenet curve in $E^n$ $(n \geq 4)$ is a Bertrand curve. Thus the notion of Bertrand curve stands only on $E^2$ and $E^3$. In $E^4$, we can show an idea of a generalization of Bertrand curve.

## 1. Introduction

We denote by $E^3$ a 3-dimensional Euclidean space. Let $C$ be a regular $C^{\infty}$-curve in $E^3$, that is, a $C^{\infty}$-mapping $\mathbf{c}: L \to E^3$ $(s \mapsto \mathbf{c}(s))$. Here $L \subset R$ is some interval, and $s(\in L)$ is the arc-length parameter of $C$. Following Wong and Lai [7], we call a curve $C$ a $C^{\infty}$-special Frenet curve if there exist three $C^{\infty}$-vector fields, that is, the unit tangent vector field $\mathbf{t}$, the unit principal normal vector field $\mathbf{n}$, the unit binormal vector field $\mathbf{b}$, and two $C^{\infty}$-scalar functions, that is, the curvature function $\kappa(>0)$, the torsion function $\tau(\neq 0)$. The three vector fields $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$ satisfy the Frenet equations. A $C^{\infty}$-special Frenet curve $C$ is called a Bertrand curve if there exist another $C^{\infty}$-special Frenet curve $\bar{C}$ and a $C^{\infty}$-mapping $\varphi: C \to \bar{C}$ such that the principal normal lines of $C$ and $\bar{C}$ at corresponding points coincide. Here, the principal normal line of $C$ at $\mathbf{c}(s)$ is collinear to the principal normal vector $\mathbf{n}(s)$ . It is a well-known result that a $C^{\infty}$-special Frenet curve $C$ in $E^3$ is a Bertrand curve if and only if its curvature function $\kappa$ and torsion function $\tau$ satisfy the condition $a\kappa(s) + b\tau(s) = 1$ for all $s \in L$, where $a$ and $b$ are constant real numbers.

In an $n$-dimensional Euclidean space $E^n$, let $C$ be a regular $C^{\infty}$-curve, that is, a $C^{\infty}$-mapping $\mathbf{c}: L \to E^n$ $(s \mapsto \mathbf{c}(s))$, where $s$ is the arc-length parameter of $C$. Then we can define a $C^{\infty}$-special Frenet curve $C$. That is, we define $\mathbf{t}(s) = \mathbf{c}'(s)$, $\mathbf{n}_1(s) = (1/\|\mathbf{c}''(s)\|) \cdot \mathbf{c}''(s)$, and we inductively define $\mathbf{n}_k(s)$ $(k = 2, 3, \cdots, n-1)$ by the higher order derivatives of $\mathbf{c}$ (see next section, in detail). The $n$ vector fields $\mathbf{t}, \mathbf{n}_1, \cdots, \mathbf{n}_{n-1}$ along $C$ satisfy the Frenet equations with positive curvature functions $k_1, \cdots, k_{n-2}$ of $C$ and positive or negative curvature function $k_{n-1}$ of

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$C$ . We call $\mathbf{n}_j$ the Frenet $j$-normal vector field along $C$, and the Frenet $j$-normal line of $C$ at $\mathbf{c}(s)$ is a line generated by $\mathbf{n}_j(s)$ through $\mathbf{c}(s)$ $(j = 1, 2, \cdots, n-1)$. The Frenet $(j,k)$-normal plane of $C$ at $\mathbf{c}(s)$ is a plane spanned by $\mathbf{n}_j(s)$ and $\mathbf{n}_k(s)$ through $\mathbf{c}(s)$ $(j, k = 1, 2, \cdots, n-1; j \neq k)$ . A $C^\infty$-special Frenet curve $C$ is called a Bertrand curve if there exist another $C^\infty$-special Frenet curve $\bar{C}$ and a $C^\infty$-mapping $\varphi : C \to \bar{C}$ such that the Frenet 1-normal lines of $C$ and $\bar{C}$ at corresponding points coincide. Then we obtain

**THEOREM A.** *If $n \geq 4$, then no $C^\infty$-special Frenet curve in $E^n$ is a Bertrand curve.*

This is claimed in [1] (see p. 176) with different viewpoint, thus we prove the above Theorem in section 3.

We will show an idea of generalized Bertrand curve in $E^4$. A $C^\infty$-special Frenet curve $C$ in $E^4$ is called a $(1,3)$-Bertrand curve if there exist another $C^\infty$-special Frenet curve $\bar{C}$ and a $C^\infty$-mapping $\varphi : C \to \bar{C}$ such that the Frenet $(1,3)$-normal planes of $C$ and $\bar{C}$ at corresponding points coincide. Then we obtain

**THEOREM B.** *Let $C$ be a $C^\infty$-special Frenet curve in $E^4$ with curvature functions $k_1, k_2, k_3$ . Then $C$ is a $(1,3)$-Bertrand curve if and only if there exist constant real numbers $\alpha, \beta, \gamma, \delta$ satifying*

$$\alpha k_2(s) - \beta k_3(s) \neq 0 \tag{a}$$

$$\alpha k_1(s) + \gamma\{\alpha k_2(s) - \beta k_3(s)\} = 1 \tag{b}$$

$$\gamma k_1(s) - k_2(s) = \delta k_3(s) \tag{c}$$

$$(\gamma^2 - 1) k_1(s) k_2(s) + \gamma\{(k_1(s))^2 - (k_2(s))^2 - (k_3(s))^2\} \neq 0 \tag{d}$$

*for all $s \in L$.*

This Theorem is proved in section 4.

We remark that if the Frenet $j$-normal vector fields of $C$ and $\bar{C}$ are not vector fields of same meaning then we can not consider coincidence of the Frenet 1-normal lines or the Frenet $(1,3)$-normal planes of $C$ and $\bar{C}$. Thus we consider only special Frenet curves.

In section 5, we give an example of $(1,3)$-Bertrand curve.

In the present paper, we shall work in $C^\infty$-category.

## 2. Special Frenet curves in $E^n$

Let $E^n$ be an $n$-dimentional Euclidean space with Cartesian coordinates $(x^1,x^2,\ldots,x^n)$ . By a parametrized curve $C$ of class $C^\infty$, we mean a mapping $\mathbf{c}$ of a certain interval $I$ into $E^n$ given by

$$\mathbf{c}(t) = \begin{bmatrix} x^1(t) \\ x^2(t) \\ \vdots \\ x^n(t) \end{bmatrix} \qquad \forall t \in I.$$

If $\left\| \dfrac{d\,\mathbf{c}(t)}{d\,t} \right\| = \left\langle \dfrac{d\,\mathbf{c}(t)}{d\,t}, \dfrac{d\,\mathbf{c}(t)}{d\,t} \right\rangle^{1/2} \neq 0$ for all $t \in I$, then $C$ is called a *regular curve* in $E^n$. Here $\langle \cdot,\cdot \rangle$ denotes the Euclidean inner product on $E^n$. We refer to[2] for the details of curves in $E^n$.

A regular curve $C$ is parametrized by the arc-length parameter $s$, that is, $\mathbf{c} : L \to E^n$ $(L \ni s \mapsto \mathbf{c}(s) \in E^n\ )$([1]) . Then the tangent vector field $\dfrac{d\,\mathbf{c}}{d\,s}$ along $C$ has unit length, that is, $\left\| \dfrac{d\,\mathbf{c}(s)}{d\,s} \right\| = 1$ for all $s \in L$.

Hereafter, curves considered are regular $C^\infty$-curves in $E^n$ parametrized by the arc-length parameter. Let $C$ be a curve in $E^n$, that is, $\mathbf{c}(s) \in E^n$ for all $s \in L$. Let $\mathbf{t}(s) = \dfrac{d\,\mathbf{c}(s)}{d\,s}$ for all $s \in L$. The vector field $\mathbf{t}$ is called a unit tangent vector field along $C$, and we assume that the curve $C$ satisfies the following conditions $(C_1) \sim (C_{n-1})$:

$(C_1) : k_1(s) = \left\| \dfrac{d\,\mathbf{t}(s)}{d\,s} \right\| = \left\| \dfrac{d^2\,\mathbf{c}(s)}{d\,s^2} \right\| > 0 \quad$ for all $s \in L$ .

Then we obtain a well-defined vector field $\mathbf{n}_1$ along $C$, that is, for all $s \in L$,

$$\mathbf{n}_1(s) = \frac{1}{k_1(s)} \cdot \frac{d\,\mathbf{t}(s)}{d\,s},$$

and we obtain,

$$\langle \mathbf{t}(s), \mathbf{n}_1(s) \rangle = 0, \qquad \langle \mathbf{n}_1(s), \mathbf{n}_1(s) \rangle = 1.$$

$(C_2) : k_2(s) = \left\| \dfrac{d\,\mathbf{n}_1(s)}{d\,s} + k_1(s) \cdot \mathbf{t}(s) \right\| > 0 \quad$ for all $s \in L$.

Then we obtain a well-defined vector field $\mathbf{n}_2$ along $C$, that is, for all $s \in L$,

$$\mathbf{n}_2(s) = \frac{1}{k_2(s)} \cdot \left( \frac{d\,\mathbf{n}_1(s)}{d\,s} + k_1(s) \cdot \mathbf{t}(s) \right),$$

and we obtain, for $i, j = 1, 2$,

$$\langle \mathbf{t}(s), \mathbf{n}_i(s) \rangle = 0, \qquad \langle \mathbf{n}_i(s), \mathbf{n}_j(s) \rangle = \delta_{ij},$$

where $\delta_{ij}$ denotes the Kronecker's symbol.
By an inductive procedure, for $\ell = 3, 4, \cdots, n-2$,

$$(C_\ell) : k_\ell(s) = \left\| \frac{d\ \mathbf{n}_{\ell-1}(s)}{d\ s} + k_{\ell-1}(s) \cdot \mathbf{n}_{\ell-2}(s) \right\| > 0 \text{ for all } s \in L.$$

Then we obtain, for $\ell = 3, 4, \cdots, n-2$, a well-defined vector field $\mathbf{n}_\ell$ along $C$, that is, for all $s \in L$

$$\mathbf{n}_\ell(s) = \frac{1}{k_\ell(s)} \cdot \left( \frac{d\ \mathbf{n}_{\ell-1}(s)}{d\ s} + k_{\ell-1}(s) \cdot \mathbf{n}_{\ell-2}(s) \right),$$

and for $i, j = 1, 2, \cdots, n-2$

$$\langle \mathbf{t}(s), \mathbf{n}_i(s) \rangle = 0, \qquad \langle \mathbf{n}_i(s), \mathbf{n}_j(s) \rangle = \delta_{ij}.$$

And

$$(C_{n-1}) : k_{n-1}(s) = \left\langle \frac{d\ \mathbf{n}_{n-2}(s)}{d\ s}, \mathbf{n}_{n-1}(s) \right\rangle \neq 0 \text{ for all } s \in L,$$

where the unit vector field $\mathbf{n}_{n-1}$ along $C$ is determined by the fact that the frame $\{\mathbf{t}, \mathbf{n}_1, \cdots, \mathbf{n}_{n-1}\}$ is of orthonormal and of positive orientation. We remark that the functions $k_1, \cdots, k_{n-2}$ are of positive and the function $k_{n-1}$ is of non-zero. Such a curve $C$ is called a *special Frenet curve* in $E^n$ ([7]). The term "special" means that the vector field $\mathbf{n}_{i+1}$ is inductively defined by the vector fields $\mathbf{n}_i$ and $\mathbf{n}_{i-1}$ and the positive functions $k_i$ and $k_{i-1}$. Each function $k_i$ is called the *i-curvature function* of $C$ $(i = 1, 2, \cdots, n-1)$. The orthonormal frame $\{\mathbf{t}, \mathbf{n}_1, \cdots, \mathbf{n}_{n-1}\}$ along $C$ is called the *special Frenet frame* along $C$ ([7]).

Thus we obtain the Frenet equations ([1], [2], [3], [4]):

$$\begin{aligned}
\frac{d\ \mathbf{t}(s)}{d\ s} &= k_1(s) \cdot \mathbf{n}_1(s) \\
\frac{d\ \mathbf{n}_1(s)}{d\ s} &= -k_1(s) \cdot \mathbf{t}(s) + k_2(s) \cdot \mathbf{n}_2(s) \\
&\cdots \\
\frac{d\ \mathbf{n}_\ell(s)}{d\ s} &= -k_\ell(s) \cdot \mathbf{n}_{\ell-1} + k_{\ell+1}(s) \cdot \mathbf{n}_{\ell+1}(s) \\
&\cdots \\
\frac{d\ \mathbf{n}_{n-2}(s)}{d\ s} &= -k_{n-2}(s) \cdot \mathbf{n}_{n-3}(s) + k_{n-1}(s) \cdot \mathbf{n}_{n-1}(s) \\
\frac{d\ \mathbf{n}_{n-1}(s)}{d\ s} &= -k_{n-1}(s) \cdot \mathbf{n}_{n-2}(s)
\end{aligned}$$

for all $s \in L$. And, for $j = 1, 2, \cdots, n-1$, the unit vector field $\mathbf{n}_j$ along $C$ is called the *Frenet j-normal vector field* along $C$. A straight line is called the *Frenet j-normal line* of $C$ at $\mathbf{c}(s)$ $(j = 1, 2, \cdots, n-1$ and $s \in L)$, if it passes throught the point $\mathbf{c}(s)$ and is collinear to the $j$-normal vector $\mathbf{n}_j(s)$ of $C$ at $\mathbf{c}(s)$.

*Remark.* In the case of Euclidean 3-space, the Frenet 1-normal vector fields $\mathbf{n}_1$ is already called the ***principal normal vector field*** along $C$, and the Frenet 1-normal line is already called the ***principal normal line*** of $C$ at $\mathbf{c}(s)$ ([3], [4]).

For each point $\mathbf{c}(s)$ of $C$, a plane throught the point $\mathbf{c}(s)$ is called the *Frenet* $(j,k)$*-normal plane* of $C$ at $\mathbf{c}(s)$ if it is spaned by the two vectors $\mathbf{n}_j(s)$ and $\mathbf{n}_k(s)$ $(j,k = 1,2,\cdots,n-1;\ j<k)$.

*Remark.* In the case of Euclidean 3-space, 1-curvature function $k_1$ is called the *curvature* of $C$, 2-curvature function $k_2$ is called the *torsion* of $C$, and $(1,2)$-normal plane is already called the *normal plane* of $C$ at $\mathbf{c}(s)$ ([3], [4]).

## 3. Bertrand curves in $E^n$

A $C^\infty$-special Frenet curve $C$ in $E^n$ $(\mathbf{c}: L \to E^n)$ is called a *Bertrand curve* if there exist a $C^\infty$-special Frenet curve $\bar{C}$ $(\bar{\mathbf{c}}: \bar{L} \to E^n)$ , distinct from $C$, and a regular $C^\infty$-map $\varphi: L \to \bar{L}$ $\left(\bar{s} = \varphi(s), \dfrac{d\,\varphi(s)}{d\,s} \neq 0 \text{ for all } s \in L\right)$ such that curves $C$ and $\bar{C}$ have the same 1-normal line at each pair of corresponding points $\mathbf{c}(s)$ and $\bar{\mathbf{c}}(\bar{s}) = \bar{\mathbf{c}}(\varphi(s))$ under $\varphi$. Here $s$ and $\bar{s}$ are arc-lenght parameters of $C$ and $\bar{C}$ respectively. In this case, $\bar{C}$ is called a Bertrand mate of $C$ ([3], [4]). The following results are well-known ([3], [4]):

**THEOREM** (the case of $n=2$). *Every $C^\infty$-plane curve is a Bertrand curve.*

**THEOREM** (the case of $n=3$). *A $C^\infty$-special Frenet curve in $E^3$ with 1-curvature function $k_1$ and 2-curvature function $k_2$ is a Bertrand curve if and only if there exists a linear relation*

$$ak_1(s) + bk_2(s) = 1$$

*for all $s \in L$, where $a$ and $b$ are nonzero constant real numbers.*

The typical example of Bertrand curves in $E^3$ is a circular helix. A circular helix has infinitely many Bertrand mates ([3], [4]).
We consider the case of $n \geq 4$. Then we obtain Theorem A.

*Proof of Theorem A.* Let $C$ be a Bertrand curve in $E^n$ $(n \geq 4)$ and $\bar{C}$ a Bertrand mate of $C$. $\bar{C}$ is distinct from $C$. Let the pair of $\mathbf{c}(s)$ and $\bar{\mathbf{c}}(\bar{s}) = \bar{\mathbf{c}}(\varphi(s))$ be of corresponding points of $C$ and $\bar{C}$. Then the curve $\bar{C}$ is given by

$$\bar{\mathbf{c}}(\bar{s}) = \bar{\mathbf{c}}(\varphi(s)) = \mathbf{c}(s) + \alpha(s) \cdot \mathbf{n}_1(s) \tag{3.1}$$

where $\alpha$ is a $C^\infty$-function on $L$. Differentiating (3.1) with respect to $s$, we obtain

$$\varphi'(s) \cdot \left.\frac{d\,\bar{\mathbf{c}}(\bar{s})}{d\,\bar{s}}\right|_{\bar{s}=\varphi(s)} = \mathbf{c}'(s) + \alpha'(s) \cdot \mathbf{n}_1(s) + \alpha(s) \cdot \mathbf{n}_1'(s).$$

Here and hereafter, the prime denotes the derivative with respect to $s$. By the Frenet equations, it holds that

$$\varphi'(s) \cdot \bar{\mathbf{t}}(\varphi(s)) = (1 - \alpha(s)k_1(s)) \cdot \mathbf{t}(s) + \alpha'(s) \cdot \mathbf{n}_1(s) + \alpha(s)k_2(s) \cdot \mathbf{n}_2(s).$$

Since $\langle \bar{\mathbf{t}}(\varphi(s)), \bar{\mathbf{n}}_1(\varphi(s)) \rangle = 0$ and $\bar{\mathbf{n}}_1(\varphi(s)) = \pm \mathbf{n}_1(s)$, we obtain, for all $s \in L$,

$$\alpha'(s) = 0,$$

that is, $\alpha$ is a constant function on $L$ with value $\alpha$ (we can use the same letter without confusion). Thus (3.1) are rewritten as

$$\bar{\mathbf{c}}(\bar{s}) = \bar{\mathbf{c}}(\varphi(s)) = \mathbf{c}(s) + \alpha \cdot \mathbf{n}_1(s), \tag{3.1$'$}$$

and we obtain

$$\varphi'(s) \cdot \bar{\mathbf{t}}(\varphi(s)) = (1 - \alpha k_1(s)) \cdot \mathbf{t}(s) + \alpha k_2(s) \cdot \mathbf{n}_2(s) \tag{3.2}$$

for all $s \in L$. By (3.2), we can set

$$\bar{\mathbf{t}}(\varphi(s)) = (\cos\theta(s)) \cdot \mathbf{t}(s) + (\sin\theta(s)) \cdot \mathbf{n}_2(s), \tag{3.3}$$

where $\theta$ is a $C^\infty$-function on $L$ and

$$\cos\theta(s) = (1 - \alpha k_1(s))/\varphi'(s) \tag{3.4.1}$$

$$\sin\theta(s) = \alpha k_2(s)/\varphi'(s). \tag{3.4.2}$$

Differentiating (3.3) and using the Frenet equations, we obtain

$$\begin{aligned} &\bar{k}_1(\varphi(s))\varphi'(s) \cdot \bar{\mathbf{n}}_1(\varphi(s)) \\ &= \frac{d\ \cos\theta(s)}{d\ s} \cdot \mathbf{t}(s) \\ &+ (k_1(s)\cos\theta(s) - k_2(s)\sin\theta(s)) \cdot \mathbf{n}_1(s) \\ &+ \frac{d\ \sin\theta(s)}{d\ s} \cdot \mathbf{n}_2(s) \\ &+ k_3(s)\sin\theta(s) \cdot \mathbf{n}_3(s). \end{aligned}$$

Since $\bar{\mathbf{n}}_1(\varphi(s)) = \pm\mathbf{n}_1(s)$ for all $s \in L$, we obtain

$$k_3(s)\sin\theta(s) \equiv 0. \tag{3.5}$$

By $k_3(s) \neq 0(\forall s \in L)$ and (3.5), we obtain that $\sin\theta(s) \equiv 0$. Thus, by $k_2(s) > 0(\forall s \in L)$ and (3.4.2), we obtain that $\alpha = 0$. Therefore, (3.1)′ implies that $\bar{C}$ coincides with $C$. This is a contradiction. This completes the proof of Theorem A.

## 4. (1, 3)-Bertrand curves in $E^4$

By the results in the previous section, the notion of Bertrand curve stands only on $E^2$ and $E^3$. Thus we will try to get the notion of generalization of Bertrand curve in $E^n (n \geq 4)$.

Let $C$ and $\bar{C}$ be $C^\infty$-special Frenet curves in $E^4$ and $\varphi : L \to \bar{L}$ a regular $C^\infty$-map such that each point $\mathbf{c}(s)$ of $C$ corresponds to the point $\bar{\mathbf{c}}(\bar{s}) = \bar{\mathbf{c}}(\varphi(s))$ of $\bar{C}$ for all $s \in L$. Here $s$ and $\bar{s}$ are arc-length parameters of $C$ and $\bar{C}$ respectively. If the Frenet (1, 3)-normal plane at each point $\mathbf{c}(s)$ of $C$ coincides with the Frenet (1, 3)-normal plane at corresponding point $\bar{\mathbf{c}}(\bar{s}) = \bar{\mathbf{c}}(\varphi(s))$ of $\bar{C}$ for all $s \in L$, then $C$ is called a (1, 3)-*Bertrand curve* in $E^4$ and $\bar{C}$ is called a (1, 3)-*Bertrand mate* of $C$. We obtain a characterization of (1, 3)-Bertrand curve, that is, we obtain Theorem B.

*Proof of Theorem B.* (i) We assume that $C$ is a (1, 3)-Bertrand curve parametrized by arc-length $s$. The (1, 3)-Bertrand mate $\bar{C}$ is given by

$$\bar{\mathbf{c}}(\bar{s}) = \bar{\mathbf{c}}(\varphi(s)) = \mathbf{c}(s) + \alpha(s)\cdot\mathbf{n}_1(s) + \beta(s)\cdot\mathbf{n}_3(s) \tag{4.1}$$

for all $s \in L$. Here $\alpha$ and $\beta$ are $C^\infty$-functions on $L$, and $\bar{s}$ is the arc-legnth parameter of $\bar{C}$. Differentiating (4.1) with respect to $s$, and using the Frenet equations, we obtain

$$\begin{aligned}\varphi'(s)\cdot\bar{\mathbf{t}}(\varphi(s)) = (1-\alpha(s)k_1(s))\cdot\mathbf{t}(s) + \alpha'(s)\cdot\mathbf{n}_1(s)\\ +(\alpha(s)k_2(s) - \beta(s)k_3(s))\cdot\mathbf{n}_2(s) + \beta'(s)\cdot\mathbf{n}_3(s)\end{aligned}$$

for all $s \in L$.

Since the plane spanned by $\mathbf{n}_1(s)$ and $\mathbf{n}_3(s)$ coincides with the plane spanned by $\bar{\mathbf{n}}_1(\varphi(s))$ and $\bar{\mathbf{n}}_3(\varphi(s))$, we can put

$$\bar{\mathbf{n}}_1(\varphi(s)) = (\cos\theta(s))\cdot\mathbf{n}_1(s) + (\sin\theta(s))\cdot\mathbf{n}_3(s) \tag{4.2.1}$$

$$\bar{\mathbf{n}}_3(\varphi(s)) = (-\sin\theta(s))\cdot\mathbf{n}_1(s) + (\cos\theta(s))\cdot\mathbf{n}_3(s) \tag{4.2.2}$$

and we notice that $\sin\theta(s) \neq 0$ for all $s \in L$. By the following facts

$$0 = \langle \varphi'(s) \cdot \bar{\mathbf{t}}(\varphi(s)), \bar{\mathbf{n}}_1(\varphi(s)) \rangle = \alpha'(s) \cdot (\cos\theta(s)) + \beta'(s) \cdot (\sin\theta(s))$$

$$0 = \langle \varphi'(s) \cdot \bar{\mathbf{t}}(\varphi(s)), \bar{\mathbf{n}}_3(\varphi(s)) \rangle = -\alpha'(s) \cdot (\sin\theta(s)) + \beta'(s) \cdot (\cos\theta(s)),$$

we obtain

$$\alpha'(s) \equiv 0, \quad \beta'(s) \equiv 0,$$

that is, $\alpha$ and $\beta$ are constant functions on $L$ with values $\alpha$ and $\beta$, respectively. Therefore, for all $s \in L$, (4.1) is rewritten as

$$\bar{\mathbf{c}}(\bar{s}) = \bar{\mathbf{c}}(\varphi(s)) = \mathbf{c}(s) + \alpha \cdot \mathbf{n}_1(s) + \beta \cdot \mathbf{n}_3(s), \tag{4.1$'$}$$

and we obtain

$$\varphi'(s) \cdot \bar{\mathbf{t}}(\varphi(s)) = (1 - \alpha k_1(s)) \cdot \mathbf{t}(s) + (\alpha k_2(s) - \beta k_3(s)) \cdot \mathbf{n}_2(s). \tag{4.3}$$

Here we notice that

$$(\varphi'(s))^2 = (1 - \alpha k_1(s))^2 + (\alpha k_2(s) - \beta k_3(s))^2 \neq 0 \tag{4.4}$$

for all $s \in L$. Thus we can set

$$\bar{\mathbf{t}}(\varphi(s)) = (\cos\tau(s)) \cdot \mathbf{t}(s) + (\sin\tau(s)) \cdot \mathbf{n}_2(s) \tag{4.5}$$

and

$$\begin{aligned} \cos\tau(s) &= (1 - \alpha k_1(s))/(\varphi'(s)) \\ \sin\tau(s) &= (\alpha k_2(s) - \beta k_3(s))/(\varphi'(s)) \end{aligned}$$

where $\tau$ is a $C^\infty$-function on $L$. Differentiating (4.5) with respect to $s$ and using the Frenet equations, we obtain

$$\begin{aligned} \varphi'(s)\bar{k}_1(\varphi(s)) \cdot \bar{\mathbf{n}}_1(\varphi(s)) = {} & \frac{d\ \cos(\tau(s))}{d\ s} \cdot \mathbf{t}(s) \\ & + \{k_1(s)\cos(\tau(s)) - k_2(s)\sin(\tau(s))\} \cdot \mathbf{n}_1(s) \\ & + \frac{d\ \sin(\tau(s))}{d\ s} \cdot \mathbf{n}_2(s) \\ & + k_3(s)\sin(\tau(s)) \cdot \mathbf{n}_3(s). \end{aligned}$$

Since $\bar{\mathbf{n}}_1(\varphi(s))$ is expressed by linear combination of $\mathbf{n}_1(s)$ and $\mathbf{n}_3(s)$, it holds that

$$\frac{d\ \cos\tau(s)}{d\ s} \equiv 0, \quad \frac{d\ \sin\tau(s))}{d\ s} \equiv 0,$$

that is, $\tau$ is a constant function on $L$ with value $\tau_0$. Thus we obtain

$$\bar{\mathbf{t}}(\varphi(s)) = (\cos\tau_0)\cdot\mathbf{t}(s) + (\sin\tau_0)\cdot\mathbf{n}_2(s) \tag{4.5)'$$

$$\varphi'(s)\cos\tau_0 = 1 - \alpha k_1(s) \tag{4.6.1}$$

$$\varphi'(s)\sin\tau_0 = \alpha k_2(s) - \beta k_3(s) \tag{4.6.2}$$

for all $s \in L$. Therefore we obtain

$$(1 - \alpha k_1(s))\sin\tau_0 = (\alpha k_2(s) - \beta k_3(s))\cos\tau_0 \tag{4.7}$$

for all $s \in L$.

If $\sin\tau_0 = 0$, then it holds $\cos\tau_0 = \pm 1$. Thus $(4.5)'$ implies that $\bar{\mathbf{t}}(\varphi(s)) = \pm\mathbf{t}(s)$. Differentiating this equality, we obtain

$$\varphi'(s)\bar{k}_1(\varphi(s))\cdot\bar{\mathbf{n}}_1(\varphi(s)) = \pm k_1(s)\cdot\mathbf{n}_1(s),$$

that is,

$$\bar{\mathbf{n}}_1(\varphi(s)) = \pm\mathbf{n}_1(s),$$

for all $s \in L$. By Theorem A, this fact is a contradiction . Thus we must consider only the case of $\sin\tau_0 \neq 0$. Then (4.6.2) implies

$$\alpha k_2(s) - \beta k_3(s) \neq 0 \quad (s \in L),$$

that is, we obtain the relation $(a)$.

The fact $\sin\tau_0 \neq 0$ and (4.7) imply

$$\alpha k_1(s) + \{(\cos\tau_0)(\sin\tau_0)^{-1}\}(\alpha k_2(s) - \beta k_3(s)) = 1.$$

From this, we obtain

$$\alpha k_1(s) + \gamma(\alpha k_2(s) - \beta k_3(s)) = 1$$

for all $s \in L$, where $\gamma = (\cos\tau_0)(\sin\tau_0)^{-1}$ is a constant number. Thus we obtain the relation $(b)$.

Differentiating $(4.5)'$ with respect to $s$ and using the Frenet equations, we obtain

$$\begin{aligned}\varphi'(s)\bar{k}_1(\varphi(s))\cdot\bar{\mathbf{n}}_1(\varphi(s)) = (k_1(s)\cos\tau_0 - k_2(s)\sin\tau_0)\cdot\mathbf{n}_1(s)\\ +k_3(s)\sin\tau_0\cdot\mathbf{n}_3(s)\end{aligned}$$

for all $s \in L$. From the above equality, (4.6.1), (4.6.2) and $(b)$, we obtain

$$\begin{aligned}&\{\varphi'(s)\bar{k}_1(\varphi(s))\}^2\\&=\{k_1(s)\cos\tau_0-k_2(s)\sin\tau_0\}^2+\{k_3(s)\sin\tau_0\}^2\\&=(\alpha k_2(s)-\beta k_3(s))^2\left[(\gamma k_1(s)-k_2(s))^2+(k_3(s))^2\right](\varphi'(s))^{-2}.\end{aligned}$$

for all $s \in L$. From (4.4) and $(b)$, it holds

$$(\varphi'(s))^2=(\gamma^2+1)(\alpha k_2(s)-\beta k_3(s))^2.$$

Thus we obtain

$$\{\varphi'(s)\bar{k}_1(\varphi(s))\}^2=\frac{1}{\gamma^2+1}\{(\gamma k_1(s)-k_2(s))^2+(k_3(s))^2\}. \tag{4.8}$$

By (4.6.1) , (4.6.2) and $(b)$ , we can set

$$\bar{\mathbf{n}}_1(\varphi(s))=(\cos\eta(s))\cdot\mathbf{n}_1(s)+(\sin\eta(s))\cdot\mathbf{n}_3(s), \tag{4.9}$$

where

$$\cos\eta(s)=\frac{(\alpha k_2(s)-\beta k_3(s))(\gamma k_1(s)-k_2(s))}{\bar{k}_1(\varphi(s))(\varphi'(s))^2} \tag{4.10.1}$$

$$\sin\eta(s)=\frac{(\alpha k_2(s)-\beta k_3(s))k_3(s)}{\bar{k}_1(\varphi(s))(\varphi'(s))^2} \tag{4.10.2}$$

for all $s \in L$. Here, $\eta$ is a $C^\infty$-function on $L$.

Differentiating (4.9) with respect to $s$ and using the Frenet equations, we obtain

$$\begin{aligned}&-\varphi'(s)\bar{k}_1(\varphi(s))\cdot\bar{\mathbf{t}}(\varphi(s))+\varphi'(s)\bar{k}_2(\varphi(s))\cdot\bar{\mathbf{n}}_2(\varphi(s))\\&\qquad=\frac{d\ \cos\eta(s)}{d\ s}\cdot\mathbf{n}_1(s)+\frac{d\ \sin\eta(s)}{d\ s}\cdot\mathbf{n}_3(s)\\&\qquad\quad-k_1(s)(\cos\eta(s))\cdot\mathbf{t}(s)\\&\qquad\quad+(k_2(s)(\cos\eta(s))-k_3(s)(\sin\eta(s))\cdot\mathbf{n}_2(s)\end{aligned}$$

for all $s \in L$. From the above fact, it holds

$$\frac{d\ \cos\eta(s)}{d\ s}\equiv 0,\quad \frac{d\ \sin\eta(s)}{d\ s}\equiv 0,$$

that is, $\eta$ is a constant function on $L$ with value $\eta_0$. Let $\delta=(\cos\eta_0)(\sin\eta_0)^{-1}$ be a constant number. Then (4.10.1) and (4.10.2) imply

$$\gamma k_1(s)-k_2(s)=\delta k_3(s)\quad(\forall s\in L),$$

that is, we obtain the relation $(c)$.

Moreover, we obtain

$$-\varphi'(s)\bar{k}_1(\varphi(s))\cdot\bar{\mathbf{t}}(\varphi(s))+\varphi'(s)\bar{k}_2(\varphi(s))\cdot\bar{\mathbf{n}}_2(\varphi(s))$$
$$=-k_1(s)(\cos\eta(s))\cdot\mathbf{t}(s)+\{k_2(s)(\cos\eta(s))-k_3(s)(\sin\eta(s))\}\cdot\mathbf{n}_2(s)$$

By the above equality and (4.3), we obtain

$$\begin{aligned}\varphi'(s)\bar{k}_2(\varphi(s))\cdot\bar{\mathbf{n}}_2(\varphi(s)) &= \varphi'(s)\bar{k}_1(\varphi(s))\cdot\bar{\mathbf{t}}(\varphi(s))\\ &\quad -k_1(s)(\cos\eta_0)\cdot\mathbf{t}(s)\\ &\quad +\{k_2(s)(\cos\eta_0)-k_3(s)(\sin\eta_0)\}\cdot\mathbf{n}_2(s)\\ &= (\varphi'(s))^{-2}\{\bar{k}_1(\varphi(s))\}^{-1}\\ &\quad \cdot\{A(s)\cdot\mathbf{t}(s)+B(s)\cdot\mathbf{n}_2(s)\},\end{aligned}$$

where

$$\begin{aligned}A(s) &= \{\varphi'(s)\bar{k}_1(\varphi(s))\}^2(1-\alpha k_1(s))\\ &\quad -k_1(s)(\alpha k_2(s)-\beta k_3(s))(\gamma k_1(s)-k_2(s))\end{aligned}$$

$$\begin{aligned}B(s) &= \{\varphi'(s)\bar{k}_1(\varphi(s))\}^2(\alpha k_2(s))-\beta k_3(s))\\ &\quad +(\alpha k_2(s)-\beta k_3(s))(\gamma k_1(s)-k_2(s))k_2(s)\\ &\quad -(\alpha k_2(s)-\beta k_3(s))(k_3(s))^2\end{aligned}$$

for all $s\in L$. From $(b)$ and (4.8), $A(s)$ and $B(s)$ are rewritten as:

$$\begin{aligned}A(s) &= -(\gamma^2+1)^{-1}(\alpha k_2(s)-\beta k_3(s))\\ &\quad\times\left[(\gamma^2-1)k_1(s)k_2(s)+\gamma\{(k_1(s))^2-(k_2(s))^2-(k_3(s))^2\}\right]\end{aligned}$$

$$\begin{aligned}B(s) &= \gamma(\gamma^2+1)^{-1}(\alpha k_2(s)-\beta k_3(s))\\ &\quad\times\left[(\gamma^2-1)k_1(s)k_2(s)+\gamma\{(k_1(s))^2-(k_2(s))^2-(k_3(s))^2\}\right].\end{aligned}$$

Since $\varphi'(s)\bar{k}_2(\varphi(s))\cdot\bar{\mathbf{n}}_2(\varphi(s))\neq\mathbf{o}$ for all $s\in L$, it holds

$$(\gamma^2-1)k_1(s)k_2(s)+\gamma\{(k_1(s))^2-(k_2(s))^2-(k_3(s))^2\}\neq 0$$

for all $s\in L$. Thus we obtain the relation $(d)$.

(ii) We assume that $C$ ( $\mathbf{c}:L\to E^4$ ) is a $C^\infty$-special Frenet curve in $E^4$ with curvature functions $k_1, k_2$ and $k_3$ satisfying the relation $(a),(b),(c)$ and $(d)$ for constant numbers $\alpha,\beta,\gamma$ and $\delta$. Then we define a $C^\infty$ -curve $\bar{C}$ by

$$\bar{\mathbf{c}}(s)=\mathbf{c}(s)+\alpha\cdot\mathbf{n}_1(s)+\beta\cdot\mathbf{n}_3(s) \tag{4.11}$$

for all $s \in L$, where $s$ is the arc-length parameter of $C$. Differentiating (4.11) with respect to $s$ and using the Frenet equations, we obtain

$$\frac{d\,\bar{\mathbf{c}}(s)}{d\,s} = (1 - \alpha k_1(s)) \cdot \mathbf{t}(s) + (\alpha k_2(s) - \beta k_3(s)) \cdot \mathbf{n}_2(s)$$

for all $s \in L$. Thus, by the relation $(b)$, we obtain

$$\frac{d\,\bar{\mathbf{c}}(s)}{d\,s} = (\alpha k_2(s) - \beta k_3(s)) \cdot (\gamma \cdot \mathbf{t}(s) + \mathbf{n}_2(s)) \tag{4.12}$$

for all $s \in L$. Since the relation $(a)$ holds,the curve $\bar{C}$ is a regular curve. Then there exists a regular map $\varphi : L \to \bar{L}$ defined by

$$\bar{s} = \varphi(s) = \int_0^s \left\| \frac{d\,\bar{\mathbf{c}}(t)}{d\,t} \right\| dt \quad (\forall s \in L),$$

where $\bar{s}$ denotes the arc-length parameter of $\bar{C}$, and we obtain

$$\varphi'(s) = \varepsilon\sqrt{\gamma^2 + 1}(\alpha k_2(s) - \beta k_3(s)) > 0, \tag{4.13}$$

where $\varepsilon = 1$ if $\alpha k_2(s) - \beta k_3(s) > 0$, and $\varepsilon = -1$ if $\alpha k_2(s) - \beta k_3(s) < 0$, for all $s \in L$. Thus the curve $\bar{C}$ is rewritten as

$$\begin{aligned} \bar{\mathbf{c}}(\bar{s}) &= \bar{\mathbf{c}}(\varphi(s)) \\ &= \mathbf{c}(s) + \alpha \cdot \mathbf{n}_1(s) + \beta \cdot \mathbf{n}_3(s) \end{aligned}$$

for all $s \in L$. Differentiating the above equality with respect to $s$, we obtain

$$\varphi'(s) \cdot \left.\frac{d\,\bar{\mathbf{c}}(\bar{s})}{d\,\bar{s}}\right|_{\bar{s}=\varphi(s)} = (\alpha k_2(s) - \beta k_3(s)) \cdot \{\gamma \cdot \mathbf{t}(s) + \mathbf{n}_2(s)\}. \tag{4.14}$$

We can define a unit vector field $\bar{\mathbf{t}}$ along $\bar{C}$ by $\bar{\mathbf{t}}(\bar{s}) = d\,\bar{\mathbf{c}}(\bar{s})/d\,\bar{s}$ for all $\bar{s} \in \bar{L}$. By (4.13) and (4.14), we obtain

$$\bar{\mathbf{t}}(\varphi(s)) = \varepsilon(\gamma^2 + 1)^{-1/2} \cdot \{\gamma \cdot \mathbf{t}(s) + \mathbf{n}_2(s)\} \tag{4.15}$$

for all $s \in L$. Differentiating (4.15) with respect to $s$ and using the Frenet equations, we obtain

$$\begin{aligned} \varphi'(s) \cdot \left.\frac{d\,\bar{\mathbf{t}}(\bar{s})}{d\,\bar{s}}\right|_{\bar{s}=\varphi(s)} &= \varepsilon(\gamma^2 + 1)^{-1/2} \cdot \{(\gamma k_1(s) - k_2(s)) \cdot \mathbf{n}_1(s) \\ &\quad + k_3(s) \cdot \mathbf{n}_3(s)\} \end{aligned}$$

and

$$\left\| \frac{d\,\bar{\mathbf{t}}(\bar{s})}{d\,\bar{s}}\bigg|_{\bar{s}=\varphi(s)} \right\| = \frac{\sqrt{(\gamma k_1(s) - k_2(s))^2 + (k_3(s))^2}}{\varphi'(s)\sqrt{\gamma^2+1}}.$$

By the fact that $k_3(s) > 0$ for all $s \in L$, we obtain

$$\bar{k}_1(\varphi(s)) = \left\| \frac{d\,\bar{\mathbf{t}}(\bar{s})}{d\,\bar{s}}\bigg|_{\bar{s}=\varphi(s)} \right\| > 0 \tag{4.16}$$

for all $s \in L$. Then we can define a unit vector field $\bar{\mathbf{n}}_1$ along $\bar{C}$ by

$$\begin{aligned}\bar{\mathbf{n}}_1(\bar{s}) &= \bar{\mathbf{n}}_1(\varphi(s)) \\ &= \frac{1}{\bar{k}_1(\varphi(s))} \cdot \frac{d\,\bar{\mathbf{t}}(\bar{s})}{d\,\bar{s}}\bigg|_{\bar{s}=\varphi(s)} \\ &= \frac{1}{\varepsilon\sqrt{(\gamma k_1(s) - k_2(s))^2 + (k_3(s))^2}} \\ &\qquad \cdot\{(\gamma k_1(s) - k_2(s)) \cdot \mathbf{n}_1(s) + k_3(s) \cdot \mathbf{n}_3(s)\}\end{aligned}$$

for all $s \in L$ . Thus we can put

$$\bar{\mathbf{n}}_1(\varphi(s)) = (\cos\xi(s)) \cdot \mathbf{n}_1(s) + (\sin\xi(s)) \cdot \mathbf{n}_3(s), \tag{4.17}$$

where

$$\cos\xi(s) = \frac{\gamma k_1(s) - k_2(s)}{\varepsilon\sqrt{(\gamma k_1(s) - k_2(s))^2 + (k_3(s))^2}} \tag{4.18.1}$$

$$\sin\xi(s) = \frac{k_3(s)}{\varepsilon\sqrt{(\gamma k_1(s) - k_2(s))^2 + (k_3(s))^2}} > 0 \tag{4.18.2}$$

for all $s \in L$. Here, $\xi$ is a $C^\infty$-function on $L$. Differentiating (4.17) with respect to $s$ and using the Frenet equations, we obtain

$$\begin{aligned}\varphi'(s) \cdot \frac{d\,\bar{\mathbf{n}}_1(\bar{s})}{d\,\bar{s}}\bigg|_{\bar{s}=\varphi(s)} &= -k_1(s)(\cos\xi(s)) \cdot \mathbf{t}(s) \\ &\quad + \frac{d\ \cos\xi(s)}{d\,s} \cdot \mathbf{n}_1(s) \\ &\quad + \{k_2(s)(\cos\xi(s)) - k_3(s)(\sin\xi(s))\} \cdot \mathbf{n}_2(s) \\ &\quad + \frac{d\ \sin\xi(s)}{d\,s} \cdot \mathbf{n}_3(s).\end{aligned}$$

Differentiating $(c)$ with respect to $s$ , we obtain

$$(\gamma k_1'(s) - k_2'(s))k_3(s) - (\gamma k_1(s) - k_2(s))k_3'(s) \equiv 0. \tag{4.19}$$

Differentiating (4.18.1) and (4.18.2) with respect to $s$ and using (4.19), we obtain

$$\frac{d\ \cos\xi(s)}{d\ s} \equiv 0, \qquad \frac{d\ \sin\xi(s)}{d\ s} \equiv 0,$$

that is, $\xi$ is a constant function on $L$ with value $\xi_0$. Thus we obtain

$$\frac{\gamma k_1(s) - k_2(s)}{\varepsilon\sqrt{(\gamma k_1(s) - k_2(s))^2 + (k_3(s))^2}} = \cos\xi_0, \tag{4.18.1$'$}$$

$$\frac{k_3(s)}{\varepsilon\sqrt{(\gamma k_1(s) - k_2(s))^2 + (k_3(s))^2}} = \sin\xi_0 > 0. \tag{4.18.2$'$}$$

From (4.17) , it holds

$$\bar{\mathbf{n}}_1(\varphi(s)) = (\cos\xi_0)\cdot\mathbf{n}_1(s) + (\sin\xi_0)\cdot\mathbf{n}_3(s). \tag{4.20}$$

Thus we obtain, by (4.15) and (4.16),

$$\begin{aligned}&\bar{k}_1(\varphi(s))\cdot\bar{\mathbf{t}}(\varphi(s))\\ &= \frac{(\gamma k_1(s) - k_2(s))^2 + (k_3(s))^2}{\varepsilon\varphi'(s)(\gamma^2+1)\sqrt{(\gamma k_1(s) - k_2(s))^2 + (k_3(s))^2}}\cdot(\gamma\cdot\mathbf{t}(s) + \mathbf{n}_2(s)),\end{aligned}$$

and by (4.18.1)$'$,(4.18.2)$'$ and (4.20),

$$\begin{aligned}\left.\frac{d\ \bar{\mathbf{n}}_1(\bar{s})}{d\ \bar{s}}\right|_{\bar{s}=\varphi(s)} &= \frac{-k_1(s)(\gamma k_1(s) - k_2(s))}{\varepsilon\varphi'(s)\sqrt{(\gamma k_1(s) - k_2(s))^2 + (k_3(s))^2}}\cdot\mathbf{t}(s)\\ &\quad+\frac{k_2(s)(\gamma k_1(s) - k_2(s)) - (k_3(s))^2}{\varepsilon\varphi'(s)\sqrt{(\gamma k_1(s) - k_2(s))^2 + (k_3(s))^2}}\cdot\mathbf{n}_2(s),\end{aligned}$$

for all $s \in L$. By the above equalities, we obtain

$$\begin{aligned}&\left.\frac{d\ \bar{\mathbf{n}}_1(\bar{s})}{d\ \bar{s}}\right|_{\bar{s}=\varphi(s)} + \bar{k}_1(\varphi(s))\cdot\bar{\mathbf{t}}(\varphi(s))\\ &\qquad = \frac{P(s)}{R(s)}\cdot\mathbf{t}(s) + \frac{Q(s)}{R(s)}\cdot\mathbf{n}_2(s),\end{aligned}$$

where

$$P(s) = -\left[\gamma\{(k_1(s))^2 - (k_2(s))^2 - (k_3(s))^2\} + (\gamma^2-1)k_1(s)k_2(s)\right]$$

$$Q(s) = \gamma\left[\gamma\{(k_1(s))^2 - (k_2(s))^2 - (k_3(s))^2\} + (\gamma^2-1)k_1(s)k_2(s)\right]$$

$$R(s) = \varepsilon(\gamma^2+1)\varphi'(s)\sqrt{(\gamma k_1(s) - k_2(s))^2 + (k_3(s))^2} \neq 0$$

for all $s \in L$. We notice that, by $(c)$, $P(s) \neq 0$ for all $s \in L$. Thus we obtain

$$\begin{aligned}
&\bar{k}_2(\varphi(s)) \\
&= \left\| \left.\frac{d\,\bar{\mathbf{n}}_1(\bar{s})}{d\,\bar{s}}\right|_{\bar{s}=\varphi(s)} + \bar{k}_1(\varphi(s)) \cdot \bar{\mathbf{t}}(\varphi(s)) \right\| \\
&= \frac{|\gamma\{(k_1(s))^2 - (k_2(s))^2 - (k_3(s))^2\} + (\gamma^2 - 1)k_1(s)k_2(s)|}{\varphi'(s)\sqrt{\gamma^2+1}\sqrt{(\gamma k_1(s) - k_2(s))^2 + (k_3(s))^2}} \\
&> 0
\end{aligned}$$

for all $s \in L$. Thus we can define a unit vector field $\bar{\mathbf{n}}_2(\bar{s})$ along $\bar{C}$ by

$$\begin{aligned}
\bar{\mathbf{n}}_2(\bar{s}) &= \bar{\mathbf{n}}_2(\varphi(s)) \\
&= \frac{1}{\bar{k}_2(\varphi(s))} \cdot \left( \left.\frac{d\,\bar{\mathbf{n}}_1(\bar{s})}{d\,\bar{s}}\right|_{\bar{s}=\varphi(s)} + \bar{k}_1(\varphi(s)) \cdot \bar{\mathbf{t}}(\varphi(s)) \right),
\end{aligned}$$

that is,

$$\bar{\mathbf{n}}_2(\varphi(s)) = \frac{1}{\varepsilon\sqrt{\gamma^2+1}} \cdot (-\mathbf{t}(s) + \gamma \cdot \mathbf{n}_2(s)) \tag{4.21}$$

for all $s \in L$. Next we can define a unit vector field $\bar{\mathbf{n}}_3$ along $\bar{C}$ by

$$\begin{aligned}
\bar{\mathbf{n}}_3(\bar{s}) &= \bar{\mathbf{n}}_3(\varphi(s)) \\
&= \frac{1}{\varepsilon\sqrt{(\gamma k_1(s) - k_2(s))^2 + (k_3(s))^2}} \\
&\quad \cdot\{-k_3(s) \cdot \mathbf{n}_1(s) + (\gamma k_1(s) - k_2(s)) \cdot \mathbf{n}_3(s)\},
\end{aligned}$$

that is,

$$\bar{\mathbf{n}}_3(\varphi(s)) = -(\sin\xi_0) \cdot \mathbf{n}_1(s) + (\cos\xi_0) \cdot \mathbf{n}_3(s) \tag{4.22}$$

for all $s \in L$. Now we obtain, by (4.15), (4.20), (4.21) and (4.22),

$$\begin{aligned}
&\det[\bar{\mathbf{t}}(\varphi(s)), \bar{\mathbf{n}}_1(\varphi(s)), \bar{\mathbf{n}}_2(\varphi(s)), \bar{\mathbf{n}}_3(\varphi(s))] \\
&= \det[\mathbf{t}(s), \mathbf{n}_1(s), \mathbf{n}_2(s), \mathbf{n}_3(s)] = 1
\end{aligned}$$

for all $s \in L$. And we obtain

$$\langle \bar{\mathbf{t}}(\varphi(s), \bar{\mathbf{n}}_i(\varphi(s)) \rangle = 0, \quad \langle \bar{\mathbf{n}}_i(\varphi(s)), \bar{\mathbf{n}}_j(\varphi(s)) \rangle = \delta_{ij}$$

for all $s \in L$ and $i, j = 1, 2, 3$. Thus the frame $\{\bar{\mathbf{t}}, \bar{\mathbf{n}}_1, \bar{\mathbf{n}}_2, \bar{\mathbf{n}}_3\}$ along $\bar{C}$ is of orthonormal and of positive. And we obtain

$$\begin{aligned}\bar{k}_3(\varphi(s)) &= \left\langle \frac{d\,\bar{\mathbf{n}}_2(\bar{s})}{d\,\bar{s}}\bigg|_{\bar{s}=\varphi(s)}, \bar{\mathbf{n}}_3(\varphi(s)) \right\rangle \\ &= \frac{\sqrt{\gamma^2+1}k_1(s)k_3(s)}{\varphi'(s)\sqrt{(\gamma k_1(s)-k_2(s))^2+(k_3(s))^2}} \\ &> 0\end{aligned}$$

for all $s \in L$. Thus curve $\bar{C}$ is a $C^\infty$-special Frenet curve in $E^4$ . And it is trivial that the Frenet $(1,3)$-normal plane at each point $\mathbf{c}(s)$ of $C$ coincides with the Frenet $(1,3)$-normal plane at corresponding point $\bar{\mathbf{c}}(\bar{s}) = \bar{\mathbf{c}}(\varphi(s))$ of $\bar{C}$. Therefore $C$ is a $(1,3)$-Bertrand curve in $E^4$.

Thus (i) and (ii) complete the proof of Theorem B.

## 5. An example of $(1,3)$-Bertrand curve

Let $a$ and $b$ be positive numbers, and let $r$ be an integer greater than 1. We consider a $C^\infty$-curve $C$ in $E^4$ defined by $\mathbf{c} : L \to E^4$ ;

$$\mathbf{c}(s) = \begin{bmatrix} a\cos\left(\dfrac{r}{\sqrt{r^2a^2+b^2}}s\right) \\ a\sin\left(\dfrac{r}{\sqrt{r^2a^2+b^2}}s\right) \\ b\cos\left(\dfrac{1}{\sqrt{r^2a^2+b^2}}s\right) \\ b\sin\left(\dfrac{1}{\sqrt{r^2a^2+b^2}}s\right) \end{bmatrix}$$

for all $s \in L$. The curve $C$ is a regular curve and $s$ is the arc-length parameter of $C$. Then $C$ is a special Frenet curve in $E^4$ and its curvature functions are as follows:

$$\begin{aligned} k_1(s) &= \frac{\sqrt{r^4a^2+b^2}}{r^2a^2+b^2}, \\ k_2(s) &= \frac{r(r^2-1)ab}{(r^2a^2+b^2)\sqrt{r^4a^2+b^2}}, \\ k_3(s) &= \frac{r}{\sqrt{r^4a^2+b^2}}. \end{aligned}$$

We take constants $\alpha$, $\beta$, $\gamma$ and $\delta$ defined by

$$\alpha = \frac{-(r^2aA + bB) + (r^2a^2 + b^2)}{\sqrt{r^4a^2 + b^2}},$$
$$\beta = \frac{-(r^2aB - bA) + (r^2 - 1)ab}{\sqrt{r^4a^2 + b^2}},$$
$$\gamma = \frac{r^2aA + bB}{r(aB - bA)},$$
$$\delta = \frac{r^4aA + bB}{r^2(aB - bA)}.$$

Here $A$ and $B$ are positive numbers such that $aB \neq bA$. Then it is trival that $(a)$, $(b)$, $(c)$ and $(d)$ hold. Therefore, the curve $C$ is a Bertrand curve in $E^4$, and its Bertrand mate curve $\bar{C}$ in $E^4$ ( $\bar{\mathbf{c}} : \bar{L} \to E^4$ ) is given by

$$\bar{\mathbf{c}}(\bar{s}) = \begin{bmatrix} A\cos\left(\frac{r}{\sqrt{r^2A^2 + B^2}}\bar{s}\right) \\ A\sin\left(\frac{r}{\sqrt{r^2A^2 + B^2}}\bar{s}\right) \\ B\cos\left(\frac{1}{\sqrt{r^2A^2 + B^2}}\bar{s}\right) \\ B\sin\left(\frac{1}{\sqrt{r^2A^2 + B^2}}\bar{s}\right) \end{bmatrix}$$

for all $\bar{s} \in \bar{L}$, where $\bar{s}$ is the arc-length parameter of $\bar{C}$, and a regular $C^\infty$-map $\varphi : L \to \bar{L}$ is given by

$$\bar{s} = \varphi(s) = \frac{\sqrt{r^2A^2 + B^2}}{\sqrt{r^2a^2 + b^2}}s \qquad (\forall s \in L).$$

*Remark.* If $a^2 + b^2 = 1$, then the curve $C$ in $E^4$ is a leaf of Hopf $r$-foliation on $S^3$ ([6], [8]).

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