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MOTION OF CHARGED PARTICLES IN KÄHLER C-SPACES

By

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Abstract. Let $(M, \langle\ ,\ \rangle, J)$ be a Hermitian symmetric space or a Kähler C-space with second betti number 1 and with a certain condition. We concretely solve the differential equation of the motion of a charged particle under electromagnetic field κJ, which is given by

$$\nabla_{\dot{x}}\dot{x} = \kappa J\dot{x}.$$

Applying this, we show that if the motion of a charged particle intersects itself, then it is simply closed.

Introduction

Let $(M, \langle\ ,\ \rangle)$ be a Riemannian manifold and F a 2-form on M. We denote by $\iota(X) : \bigwedge^m(M) \to \bigwedge^{m-1}(M)$ the interior product operator induced from X and by $\mathfrak{L} : T(M) \to T^*(M)$, the Legendre transformation defined by

$$\mathfrak{L} : T(M) \to T^*(M); u \mapsto \mathfrak{L}(u), \quad \mathfrak{L}(u)(v) = \langle u, v\rangle \quad (v \in T(M)).$$

A curve $x(t)$ in M is called the *motion of a charged particle under electromagnetic field F* if it satisfies the following differential equation:

$$\nabla_{\dot{x}}\dot{x} = -\mathfrak{L}^{-1}(\iota(\dot{x})F),$$

where ∇ is the Levi-Civita connection of M. This equation originated in theory of general relativity (see [11, p. 112, (19.15)]). When $F = 0$, then $x(t)$ is a geodesic. If $x(t)$ is the motion of a charged particle under electromagnetic field F, then the norm $||\dot{x}||$ of its velocity vector is a constant. If $x(t)$ is the motion of a charged particle under F, then $y(t) = x(at)$ (a : constant) is the motion of a charged particle under aF. If F has an electromagnetic potential A, that is $F = dA$, then we define a functional E_A by

$$E_A(x) = \frac{1}{2}\int_0^1 (||\dot{x}||^2 + A(\dot{x}))dt.$$

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The Euler-Lagrange equation of E_A is nothing but the motion of a charged particle under F. When $(M, J, \langle\ ,\ \rangle)$ is a Hermitian manifold, it is natural to take a scalar multiple of Kähler form Ω defined by $\Omega(X,Y) = \langle X, JY\rangle$ as electromagnetic field F. Since $-\mathfrak{L}^{-1}(\iota(X)\Omega) = JX$, a curve $x(t)$ is the motion of a charged particle under electromagnetic field $\kappa\Omega$ if and only if

$$\nabla_{\dot{x}}\dot{x} = \kappa J\dot{x}. \tag{0.1}$$

It is an interesting question, in general, whether a given equation of motion has a periodic solution or not. In this paper, we describe the solution of the equation (0.1) in Hermitian symmetric spaces and Kähler C-spaces with certain conditions (Theorem 1.1, Corollaries 2.1 and 3.1). Applying this we show that if the motion $x(t)$ of a charged particle intersects itself, then it is simply closed (it is known by S. Kobayashi that if a geodesic in a Riemannian homogeneous space intersects itself, then it is simply closed [9, p. 321]). These results are a generalization of a theorem of R. Dohira ([5]).

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1. Motion of charged particles

In this section we shall construct a Riemannian homogeneous space M with an invariant $(1,1)$-tensor I and consider the motion of charged particles under electromagnetic field κI.

Let G be a connected Lie group and K a compact subgroup of G. We consider the coset manifold $M = G/K$. We denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K, respectively. Since K is compact, there exists an $\mathrm{Ad}(K)$-invariant subspace $\mathfrak{m}$ of $\mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} \quad \text{(direct sum)}. \tag{1.1}$$

We denote by π the natural projection from G onto M, and by $o = \pi(e)$, the origin of M. Then we can identify $\mathfrak{m}$ with $T_o(M)$ through π_*. We assume that there exist $\mathrm{Ad}(K)$-invariant subspaces $\mathfrak{m}_1$ and $\mathfrak{m}_2$ of $\mathfrak{m}$ such that

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \qquad \text{(direct sum)} \tag{1.2}$$

and such that

$$[\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{k} \oplus \mathfrak{m}_2, \quad [\mathfrak{m}_2, \mathfrak{m}_2] \subset \mathfrak{k}, \quad [\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{m}_1. \tag{1.3}$$

For X in $\mathfrak{g}$, we denote by X_i the $\mathfrak{m}_i$-component of X. Moreover we assume that there exist a nonzero constant $c \in \boldsymbol{R}$ and $\mathrm{Ad}(K)$-invariant inner product $\langle\ ,\ \rangle$ of $\mathfrak{m}$ such that

$$\mathfrak{m}_1 \perp \mathfrak{m}_2 \text{ and that } \langle [X,Y]_2, Z\rangle + c\langle X, [Z,Y]\rangle = 0 \ (X, Y \in \mathfrak{m}_1, Z \in \mathfrak{m}_2). \tag{1.4}$$

If we extend the inner product $\langle\ ,\ \rangle$ to a G-invariant Riemannian metric $\langle\ ,\ \rangle$ on M, then M is a Riemannian homogeneous space and G acts on M isometrically. We denote by $\mathfrak{c}$ the center of $\mathfrak{k}$. For W in $\mathfrak{c}$, we define an endomorphism I of $\mathfrak{m}$ by

$$I : \mathfrak{m} \to \mathfrak{m}; X_1 + X_2 \mapsto [W, X_1] + \frac{1}{c}[W, X_2] \quad (X_1 \in \mathfrak{m}_1, X_2 \in \mathfrak{m}_2). \quad (1.5)$$

Since $\mathrm{Ad}(k)I = I\mathrm{Ad}(k)$ for any k in K, we can extend I to a G-invariant $(1,1)$-tensor I on M. We then have

$$\langle IX, Y\rangle + \langle X, IY\rangle = 0 \qquad (X, Y \in \mathfrak{X}(M)).$$

Let κ be a constant. A curve $x(t)$ is called the *motion of a charged particle under electromagnetic field* κI, if it satisfies the following differential equation:

$$\nabla_{\dot{x}}\dot{x} = \kappa I \dot{x}. \quad (1.6)$$

When $\kappa = 0$, then $x(t)$ is a geodesic.

THEOREM 1.1. *Let $M = (G/K, \langle\ ,\ \rangle)$ be a Riemannian homogeneous space with a G-invariant skew-symmetric $(1,1)$-tensor I satisfying the conditions* (1.1), (1.2), (1.3), (1.4) *and* (1.5). *Let $x(t)$ be the motion of a charged particle defined by* (1.6) *under electromagnetic field κI with initial conditions $x(0) = o$ and $\dot{x}(0) = X_1 + X_2$ $(X_1 \in \mathfrak{m}_1, X_2 \in \mathfrak{m}_2)$. Then $x(t)$ is given by*

$$x(t) = \pi\left(\exp t(X_1 + cX_2 + \kappa W)\exp t(1-c)\left(X_2 + \frac{\kappa}{c}W\right)\right).$$

If $x(t)$ intersects itself, then it is simply closed.

Remark. In the case where $\kappa = 0$, this is a theorem of R. Dohira ([5]).

EXAMPLE 1.2 (geodesics in compact 4-symmetric spaces). Let G be a compact connected Lie group and θ an automorphism of G of order 4. We also denote by θ the differential of θ. We define a closed subgroup K of G by $K = \{g \in G \mid \theta(g) = g\}$. We define a subspace $\mathfrak{m}$ in the Lie algebra $\mathfrak{g}$ of G by

$$\mathfrak{m} = \{X \in \mathfrak{g} \mid (\theta^3 + \theta^2 + \theta + 1)(X) = 0\}.$$

We define subspaces $\mathfrak{m}_1$ and $\mathfrak{m}_2$ in $\mathfrak{m}$ by

$$\mathfrak{m}_1 = \{X \in \mathfrak{m} \mid \theta^2(X) = -X\} = \{X \in \mathfrak{g} \mid \theta^2(X) = -X\},$$
$$\mathfrak{m}_2 = \{X \in \mathfrak{m} \mid \theta^2(X) = X\} = \{X \in \mathfrak{g} \mid \theta(X) = -X\}.$$

Let $F : G/K \to G$ be a Cartan embedding, which is defined by

$$F : G/K \to G; gK \mapsto g\theta(g^{-1}).$$

Take an $\mathrm{Ad}(G)$ and θ invariant inner product $(\ ,\ )$ on $\mathfrak{g}$. Then F and $(\ ,\ )$ induce a G-invariant Riemannian metric $\langle\ ,\ \rangle$ on G/K. Since $F_*X = X - \theta X \quad (X \in \mathfrak{m})$, we have

$$\langle X, Y\rangle = (X - \theta X, Y - \theta Y) \qquad (X, Y \in \mathfrak{m}).$$

If we set $c = 2$, then the conditions (1.1), (1.2), (1.3) and (1.4) are satisfied. Hence a curve $x(t)$ in $(G/K, \langle\ ,\ \rangle)$ is a geodesic such that $x(0) = o$ and that $\dot{x}(0) = X_1 + X_2 (X_i \in \mathfrak{m}_i)$ if and only if

$$x(t) = \pi\left(\exp t(X_1 + 2X_2)\exp(-tX_2)\right).$$

In order to prove the theorem above, we show the following lemma.

LEMMA 1.3. *Let $x(t)$ be a curve in M such that $x(0) = o$. Let $\alpha(t)$ be a curve in G such that $\alpha(0) = e$ and that $\pi(\alpha(t)) = x(t)$. Then*

$$\alpha(t)_*^{-1}\nabla_{\dot{x}}\dot{x} = \frac{d}{dt}\alpha(t)_*^{-1}\dot{x}(t) + (c-1)[(\alpha(t)_*^{-1}\dot{x}(t))_1, (\alpha(t)_*^{-1}\dot{x}(t))_2]$$
$$+[(\alpha(t)_*^{-1}\dot{\alpha}(t))_{\mathfrak{k}}, \alpha(t)_*^{-1}\dot{x}(t)],$$

where we denote by $X_{\mathfrak{k}}$ the $\mathfrak{k}$-component of $X \in \mathfrak{g}$.

Proof of Lemma 1.3. For $X \in \mathfrak{g}$, we define a Killing vector field X^* on M by

$$X_p^* = \frac{d}{dt}\exp tXp_{|t=0} \in T_p(M).$$

Then for $X, Y \in \mathfrak{g}$ and $g \in G$ we have

$$[X^*, Y^*] = -[X, Y]^*, \qquad g_*X^* = (\mathrm{Ad}(g)X)^*. \tag{1.7}$$

By a formula of Koszul ([9, p. 61, Theorem 11]) and the first equation of (1.7) we have $(\nabla_{X^*}X^*)_o = (c-1)[X_1, X_2]$ for $X \in \mathfrak{m}$, where we used (1.3) and (1.4). The decomposition (1.1) defines an invariant connection ∇^0 on the reductive homogeneous space M (see [8]). Then $(\nabla^0_{X^*}X^*)_o = 0$ for $X \in \mathfrak{m}$. We define a tensor field T on M of type (1,2) by $T = \nabla - \nabla^0$. Then T is a Riemannian homogeneous structure on M (see [10] for the definition) and $T_XX = (c-1)[X_1, X_2]$ for $X \in \mathfrak{m}$. Since

$$\nabla^0_{\dot{x}}\dot{x} = \alpha(t)_*\left(\frac{d}{dt}\alpha(t)_*^{-1}\dot{x} + [(\alpha(t)_*^{-1}\dot{\alpha}(t))_{\mathfrak{k}}, \alpha(t)_*^{-1}\dot{x}]\right),$$

the lemma is proved.

Proof of Theorem 1.1. To begin with, we prove the first half of the theorem. We define a curve $\alpha(t)$ in G by

$$\alpha(t) = \exp t(X_1 + cX_2 + \kappa W) \exp t(1-c)\left(X_2 + \frac{\kappa}{c}W\right).$$

We further define a curve $x(t)$ in M by $x(t) = \pi(\alpha(t))$. It is sufficient to show $\nabla_{\dot{x}}\dot{x} = \kappa I(\dot{x})$. Since

$$\begin{aligned}\alpha(t)_*^{-1}\dot{\alpha}(t) &= \mathrm{Ad}(\alpha(t)^{-1})(X_1 + cX_2 + \kappa W) + (1-c)\left(X_2 + \frac{\kappa}{c}W\right)\\ &= \mathrm{Ad}\left(\exp t(c-1)\left(X_2 + \frac{\kappa}{c}W\right)\right)X_1 + \left(X_2 + \frac{\kappa}{c}W\right),\end{aligned}$$

we have

$$\alpha(t)_*^{-1}\dot{x}(t) = \mathrm{Ad}\left(\exp t(c-1)\left(X_2 + \frac{\kappa}{c}W\right)\right)X_1 + X_2,$$

which implies that

$$\frac{d}{dt}\alpha(t)_*^{-1}\dot{x}(t) = (c-1)\left[X_2 + \frac{\kappa}{c}W, \mathrm{Ad}\left(\exp t(c-1)\left(X_2 + \frac{\kappa}{c}W\right)\right)X_1\right].$$

By using the equations above and Lemma 1.3 we have

$$\begin{aligned}\alpha(t)_*^{-1}\nabla_{\dot{x}}\dot{x} &= \kappa\left[W, \mathrm{Ad}\left(\exp t(c-1)\left(X_2 + \frac{\kappa}{c}W\right)\right)X_1\right] + \frac{\kappa}{c}[W, X_2]\\ &= \kappa I(\alpha(t)_*^{-1}\dot{x}(t)).\end{aligned}$$

Next we prove the latter half of the theorem. The velocity vector $\dot{x}(t)$ of $x(t)$ is given by

$$\dot{x}(t) = \alpha(t)_*\left(\mathrm{Ad}\left(\exp t(c-1)\left(X_2 + \frac{\kappa}{c}W\right)\right)X_1 + X_2\right).$$

We assume that there exists a real number t_0 such that $x(t_0) = o$; that is, $\alpha(t_0) \in K$. Then

$$\dot{x}(t_0) = \mathrm{Ad}(\exp t_0(X_1 + cX_2 + \kappa W))X_1 + \mathrm{Ad}(\alpha(t_0))X_2.$$

By the way we have

$$\begin{aligned}&X_1 + cX_2 + \kappa W\\ &= \mathrm{Ad}(\exp t_0(X_1 + cX_2 + \kappa W))(X_1 + cX_2 + \kappa W)\\ &= \mathrm{Ad}\left(\alpha(t_0)\exp t_0(c-1)\left(X_2 + \frac{\kappa}{c}W\right)\right)(X_1 + cX_2 + \kappa W)\\ &= \mathrm{Ad}\left(\alpha(t_0)\exp t_0(c-1)\left(X_2 + \frac{\kappa}{c}W\right)\right)X_1 + \mathrm{Ad}(\alpha(t_0))(cX_2 + \kappa W).\end{aligned}$$

Here we obtain

$$\mathrm{Ad}\left(\alpha(t_0)\exp t_0(c-1)\left(X_2+\frac{\kappa}{c}W\right)\right)X_1 \in \mathfrak{m}_1,$$
$$\mathrm{Ad}(\alpha(t_0))cX_2 \in \mathfrak{m}_2, \quad \mathrm{Ad}(\alpha(t_0))\kappa W \in \mathfrak{k},$$

which implies that

$$\mathrm{Ad}(\exp t_0(X_1+cX_2+\kappa W))X_1 = X_1, \quad \mathrm{Ad}(\alpha(t_0))X_2 = X_2.$$

Hence we have $\dot{x}(t_0) = \dot{x}(0)$.

2. Charged particles in Hermitian symmetric spaces

In this section we shall apply Theorem 1.1 to the motion of charged particles in Hermitian symmetric spaces. We know that every motion of a charged particle in a Hermitian symmetric space under Kähler electromagnetic field is simple (see [1], [6] or [7]). Let $(G, K, \theta, \langle\ ,\ \rangle, J)$ be an almost effective Hermitian symmetric pair. Then the coset manifold $M = G/K$ is a Hermitian symmetric space. Conversely, every Hermitian symmetric space is obtained in this way. Let

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$$

be the canonical decomposition of the Lie algebra $\mathfrak{g}$ of G. We denote by $\mathfrak{c}$ the center of $\mathfrak{k}$. There exists an element J_o in $\mathfrak{c}$ such that $J = \mathrm{ad}(J_o)$ is a complex structure on $\mathfrak{m}$. Setting $\mathfrak{m}_2 = \{0\}$ and $W = J_o$ in Theorem 1.1, we redemonstrate the following.

COROLLARY 2.1 (Adachi-Maeda-Udagawa[1]). *Let $M = G/K$ be a Hermitian symmetric space . Let $x(t)$ be the motion of a charged particle defined by (0.1) under the electromagnetic field κJ with initial conditions $x(0) = o$ and $\dot{x}(0) = X \in \mathfrak{m}$. Then $x(t)$ is given by*

$$x(t) = \pi(\exp t(\kappa J_o + X)). \tag{2.1}$$

COROLLARY 2.2. *Let $x(t)$ be the motion of a charged particle in a Hermitian symmetric space. Its velocity vector $\dot{x}(t)$ can then be extended to a Killing vector field which is an infinitesimal automorphism of J.*

Proof. For the motion of a charged particle (2.1), we set $Y = \kappa J_o + X \in \mathfrak{g}$. Then we have $Y^*_{x(t)} = \dot{x}(t)$.

3. Charged particles in Kähler C-spaces

In this section we shall apply Theorem 1.1 to the motion of charged particles in Kähler C-spaces with certain conditions. We know that every motion of a charged particle in a Kähler C-space under Kähler electromagnetic field is simple (see [6] or [7]). By a C-space we mean a compact simply connected complex homogeneous space, and by a Kähler C-space, a C-space M which admits a Kähler metric such that a group of holomorphic isometries acts transitively on M. We shall construct Kähler C-spaces according to [2, Chap. 8]. Let G be a compact connected semisimple Lie group and W in its Lie algebra $\mathfrak{g}$. We define a closed subgroup K of G by

$$K = \{g \in G \mid \mathrm{Ad}(g)W = W\}.$$

Then K is connected, and coset manifold $M = G/K$ is compact and simply connected, which is called a generalized flag manifold. We can identify the tangent space $T_o(M)$ at the origin o with $\mathfrak{m} = \mathrm{Im}\ \mathrm{ad}(W)$.

In order to define a G-invariant complex structure J on M, take a maximal torus T of G such that W is in its Lie algebra $\mathfrak{t}$. Take a biinvariant Riemannian metric $(\ ,\ )$ on G. We denote by Δ the set of nonzero roots of $\mathfrak{g}^{\mathbf{C}}$ with respect to $\mathfrak{t}^{\mathbf{C}}$. Take a lexicographic ordering on $\mathfrak{t}$ such that $(W, \alpha) \geq 0$ for any positive root α. We denote by Δ^+ the set of positive roots. We have the following direct sum decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{t} \oplus \sum_{\alpha \in \Delta^+} (\boldsymbol{R}F_\alpha \oplus \boldsymbol{R}G_\alpha),$$

where for each $H \in \mathfrak{t}$, $[H, F_\alpha] = (\alpha, H)G_\alpha, [H, G_\alpha] = -(\alpha, H)F_\alpha$. Set

$$\Delta_W = \{\alpha \in \Delta \mid (\alpha, W) = 0\}, \quad \Delta_W^+ = \Delta_W \cap \Delta^+;$$

then we have

$$\mathfrak{k} = \mathfrak{t} \oplus \sum_{\alpha \in \Delta_W^+} (\boldsymbol{R}F_\alpha \oplus \boldsymbol{R}G_\alpha), \quad \mathfrak{m} = \sum_{\alpha \in \Delta^+ - \Delta_W^+} (\boldsymbol{R}F_\alpha \oplus \boldsymbol{R}G_\alpha).$$

We define a complex structure J on $\mathfrak{m}$ by

$$JF_\alpha = G_\alpha, \quad JG_\alpha = -F_\alpha \qquad (\alpha \in \Delta^+ - \Delta_W^+).$$

Since $\mathrm{Ad}(k)J = J\mathrm{Ad}(k)$ for any k in K, we can extend J to a G-invariant almost complex structure on M. This almost complex structure J is integrable.

We assume that G is simple. We denote by $\Pi = \{\alpha_1, \cdots, \alpha_r\}$ the set of simple roots, and by $\alpha_0 = \sum m_j \alpha_j$, the highest root. If we set

$$\Pi_W = \{\alpha_j \in \Pi \mid (\alpha_j, W) > 0\} = \{\alpha_{i_1}, \cdots, \alpha_{i_s}\},$$

then it is known that the second betti number $b_2(M)$ of M is given by $b_2(M) = s = \#(\Pi_W)$ ([3]). We assume that $b_2(M) = 1$; that is, $\Pi_W = \{\alpha_i\}$. For a natural number n, set

$$\Delta^+(\alpha_i; n) = \{\alpha = \sum n_j \alpha_j \in \Delta^+ \mid n_i = n\}, \quad \mathfrak{m}_n = \sum_{\alpha \in \Delta^+(\alpha_i; n)} (\boldsymbol{R}F_\alpha \oplus \boldsymbol{R}G_\alpha);$$

then we have

$$\Delta^+ - \Delta_W^+ = \Delta^+(\alpha_i) = \bigcup_{n \geq 1} \Delta^+(\alpha_i; n), \quad \mathfrak{m} = \sum_{n \geq 1} \mathfrak{m}_n.$$

We set $\mathfrak{m}_0 = \mathfrak{k}$ for simplicity; then for $n, m \geq 0$ we have $[\mathfrak{m}_n, \mathfrak{m}_m] \subset \mathfrak{m}_{n+m} + \mathfrak{m}_{|n-m|}$. If we normalize W so that $(W, \alpha_i) = 1$, then we have $nJ = \mathrm{ad}W$ on $\mathfrak{m}_n$. We define a G-invariant Kähler metric $\langle\ ,\ \rangle$ on M by

$$\langle X_n, X_m \rangle = n\delta_{nm}(X_n, X_m) \qquad (X_n \in \mathfrak{m}_n, X_m \in \mathfrak{m}_m).$$

We assume that $m_i = 2$. If we set $c = 2$, then conditions (1.1), (1.2), (1.3), (1.4) and (1.5) are satisfied.

COROLLARY 3.1. *Let $M = (G/K, J)$ be a Kähler C-space with $b_2(M) = 1$. We assume that G is a compact connected simple Lie group. Further, we assume that there exists a simple root α_i such that $\Pi_W = \{\alpha_i\}$ and that $m_i = 2$, where $\alpha_0 = \sum_j m_j \alpha_j$ is the highest root. Let $x(t)$ be a motion of charged particle defined by (0.1) under the electromagnetic field κJ with initial conditions $x(0) = o$ and $\dot{x}(0) = X_1 + X_2$ $(X_1 \in \mathfrak{m}_1, X_2 \in \mathfrak{m}_2)$. Then $x(t)$ is given by*

$$x(t) = \pi\left(\exp t(X_1 + 2X_2 + \kappa W) \exp\left(-t\left(X_2 + \frac{\kappa}{2}W\right)\right)\right),$$

where W is in the center of the Lie algebra $\mathfrak{k}$ of K.

For instance, (G, K, α_i)'s in the following table satisfy the assumption of Corollary 3.1. Here we adopt the same notations and numberings of simple roots given in the Bourbaki's table[4].

G	K	α_i
$Sp(r)$	$U(i) \times Sp(r-i)$	$\epsilon_i - \epsilon_{i+1}$ $(1 \leq i \leq r-1)$
$SO(2r)$	$U(i) \times SO(2(r-i))$	$\epsilon_i - \epsilon_{i+1}$ $(2 \leq i \leq r-2)$
$SO(2r+1)$	$U(i) \times SO(2(r-i)+1)$	$\epsilon_i - \epsilon_{i+1}$ $(2 \leq i \leq r-1), \epsilon_r (i = r)$

Here we set

$$Sp(r) = \{g \in U(2r) \mid {}^t g J_r g = J_r\},\ J_r = \begin{pmatrix} & I_r \\ -I_r & \end{pmatrix}.$$

The imbedding of $U(i) \times Sp(r-i)$ into $Sp(r)$ is given by

$$\left\{ \left(\begin{array}{cccc} \mathrm{Re}(x) & & \mathrm{Im}(x) & \\ & y_{11} & & y_{12} \\ -\mathrm{Im}(x) & & \mathrm{Re}(x) & \\ & y_{21} & & y_{22} \end{array} \right) \middle| \begin{array}{l} x \in U(i), \\ \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \in Sp(r-i) \end{array} \right\} \subset Sp(r).$$

The imbedding of $U(i) \times SO(j)$ into $SO(2i+j)$ is given by

$$U(i) \times SO(j) = \left\{ \left(\begin{array}{ccc} \mathrm{Re}(x) & \mathrm{Im}(x) & \\ -\mathrm{Im}(x) & \mathrm{Re}(x) & \\ & & y \end{array} \right) \middle| \begin{array}{l} x \in U(i), \\ y \in SO(j) \end{array} \right\} \subset SO(2i+j),$$

where $j = 2(r-i)$ or $j = 2(r-i)+1$.

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