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# ORBIT SPACES OF JORDAN ALGEBRAS WITH RESPECT TO EXCEPTIONAL LIE GROUPS AND THEIR ORBIT TYPES

By

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**Abstract.** Compact exceptional Lie groups of types $F_4$ and $E_6$ act on exceptional Jordan algebras $\mathfrak{J}$ and $\mathfrak{J}^C$, respectively. Similarly, maximal compact subgroups of noncompact exceptional Lie groups of types $E_{6(-26)}$ and $E_{6(6)}$ act on exceptional Jordan algebras $\mathfrak{J}^1$ and $\mathfrak{J}'$, respectively. We give the orbit spaces of the Jordan algebras under these actions, respectively. Moreover, we give the orbit types of a maximal compact subgroup of $E_{6(6)}$ in $\mathfrak{J}'$.

## 1. Introduction

It is known that any element of the exceptional Jordan algebra $\mathfrak{J}$ (resp. $\mathfrak{J}^1, \mathfrak{J}^C, \mathfrak{J}'$) can be transformed to a diagonal form by some element of the group $F_4$ (resp. $F_4, E_6, (E_{6(6)})_K$ ($\cong Sp(4)/\boldsymbol{Z}_2$)) ([1], [2], [3], [5]).

In this paper, by virtue of these results of the diagonalization, we shall give the orbit spaces of the Jordan algebras with respect to the groups above. The results are as follows:

$$\begin{aligned} \mathfrak{J}/F_4 &= \{(\xi_1, \xi_2, \xi_3) \in \boldsymbol{R}^3 \,|\, \xi_1 \le \xi_2 \le \xi_3\}, \\ \mathfrak{J}^1/F_4 &= \{(\xi_1, \xi_2, \xi_3) \in \boldsymbol{R}^3 \,|\, -\xi_1 \le \xi_2 \le \xi_3\}, \\ \mathfrak{J}^C/E_6 &= \Big\{(\xi_0, \xi_1, \xi_2, \xi_3) \in \boldsymbol{R}^4 \,\Big|\, \sqrt{{\xi_0}^2 + {\xi_1}^2} \le \xi_2 \le \xi_3\Big\}, \\ \mathfrak{J}'/Sp(4) &= \{(\xi_1, \xi_2, \xi_3) \in \boldsymbol{R}^3 \,|\, |\xi_3| \le \xi_2 \le \xi_1\}. \end{aligned}$$

Moreover, we have known that the orbit types of the groups $F_4$ and $E_6$ in the Jordan algebras $\mathfrak{J}$ and $\mathfrak{J}^C$, respectively ([6]). In the latter half of this paper, we determine the orbit types of symplectic group $Sp(4)$ in the split Jordan algebra $\mathfrak{J}'$. There are five orbit types as follows:

$$Sp(4)/(Sp(1) \times Sp(1) \times Sp(1) \times Sp(1)), \quad Sp(4)/(Sp(1) \times Sp(1) \times Sp(2)),$$
$$Sp(4)/(Sp(2) \times Sp(2)), \quad Sp(4)/(Sp(1) \times Sp(3)), \quad Sp(4)/Sp(4).$$

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## 2. Preliminaries

### 2.1 Cayley algebras $\mathfrak{C}, \mathfrak{C}'$ and $\mathfrak{C}^C$

We denote by $\boldsymbol{R}, \boldsymbol{C} = \{a_0 + a_1 e_1 \,|\, a_k \in \boldsymbol{R}\}$ $({e_1}^2 = -1)$ and $\boldsymbol{H} = \{a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 \,|\, a_k \in \boldsymbol{R}\}$ $({e_k}^2 = -1, e_1 e_2 = e_3, e_2 e_3 = e_1, e_3 e_1 = e_2, e_k e_l = -e_l e_k \ (k \neq l))$, the field of real numbers, complex numbers and quaternions, respectively. Next, let $\mathfrak{C} = \boldsymbol{H} \oplus \boldsymbol{H} e$ (resp. $\mathfrak{C}' = \boldsymbol{H} \oplus \boldsymbol{H} e'$) be the Cayley algebra (resp. the split Cayley algebra) over $\boldsymbol{R}$ with the multiplication

$$(a + be)(c + de) = (ac - \overline{d}b) + (b\overline{c} + da)e, \quad a + be, c + de \in \mathfrak{C},$$
$$(\text{resp. } (a + be')(c + de') = (ac + \overline{d}b) + (b\overline{c} + da)e', \quad a + be', c + de' \in \mathfrak{C}'),$$

where $\overline{c}$, $\overline{d}$ are the conjugate elements of $c, d \in \boldsymbol{H}$, respectively. In $\mathfrak{C}$ (resp. $\mathfrak{C}'$), the conjugation and the inner product are defined as follows. $\overline{a + be} = \overline{a} - be$ (resp. $\overline{a + be'} = \overline{a} - be'$) and $(x, y) = \frac{1}{2}(x\overline{y} + y\overline{x}), x, y \in \mathfrak{C}$ (resp. $\mathfrak{C}'$).

Now, let the complex Cayley algebra $\mathfrak{C}^C := \{x_1 + ix_2 \,|\, x_1, x_2 \in \mathfrak{C}\}$ be the complexification of the Cayley algebra $\mathfrak{C}$, where the coefficient field of complex numbers is denoted by $C := \{a + bi \,|\, a, b \in \boldsymbol{R}\}$ $(i^2 = -1)$. Note that we use the usual font $C$ instead of the bold face $\boldsymbol{C}$ above, in order to distinguish two kinds of complex numbers. $\mathfrak{C}^C$ has also the multiplication $xy$, the conjugation $\overline{x}$ and the inner product $(x, y)$ in the same manner as in $\mathfrak{C}$. Furthermore, $\mathfrak{C}^C$ has the complex conjugation $\tau$ defined by $\tau(x_1 + ix_2) = x_1 - ix_2$, $x_1, x_2 \in \mathfrak{C}$.

### 2.2 Jordan algebras $\mathfrak{J}, \mathfrak{J}', \mathfrak{J}^C$ and $\mathfrak{J}^1$

Let

$$\begin{aligned}
\mathfrak{J}(3, K) &= \{X \in M(3, K) \,|\, X^* = X\} \\
&= \left\{ X = X(\xi_k, x_k) = \begin{pmatrix} \xi_1 & x_3 & \overline{x}_2 \\ \overline{x}_3 & \xi_2 & x_1 \\ x_2 & \overline{x}_1 & \xi_3 \end{pmatrix} \,\middle|\, \xi_k \in \boldsymbol{R}, x_k \in K \right\}, \quad K = \mathfrak{C}, \mathfrak{C}',
\end{aligned}$$

$$\begin{aligned}
\mathfrak{J}(3, \mathfrak{C}^C) &= \{X \in M(3, \mathfrak{C}^C) \,|\, X^* = X\} \\
&= \left\{ X = X(\xi_k, x_k) = \begin{pmatrix} \xi_1 & x_3 & \overline{x}_2 \\ \overline{x}_3 & \xi_2 & x_1 \\ x_2 & \overline{x}_1 & \xi_3 \end{pmatrix} \,\middle|\, \xi_k \in C, x_k \in \mathfrak{C}^C \right\},
\end{aligned}$$

$$\begin{aligned}
\mathfrak{J}(1, 2, \mathfrak{C}) &= \{X \in M(3, \mathfrak{C}) \,|\, I_1 X^* I_1 = X\} \\
&= \left\{ X = X(\xi_k, x_k) = \begin{pmatrix} \xi_1 & x_3 & -\overline{x}_2 \\ -\overline{x}_3 & \xi_2 & x_1 \\ x_2 & \overline{x}_1 & \xi_3 \end{pmatrix} \,\middle|\, \xi_k \in \boldsymbol{R}, x_k \in \mathfrak{C} \right\},
\end{aligned}$$

where $I_1 = \mathrm{diag}(-1, 1, 1)$, be the Jordan algebras with the Jordan multiplication

$$X \circ Y = \frac{1}{2}(XY + YX).$$

Hereafter, we shall briefly denote $\mathfrak{J}(3, \mathfrak{C}), \mathfrak{J}(3, \mathfrak{C}'), \mathfrak{J}(3, \mathfrak{C}^C)$ and $\mathfrak{J}(1, 2, \mathfrak{C})$ by $\mathfrak{J}$, $\mathfrak{J}'$, $\mathfrak{J}^C$ and $\mathfrak{J}^1$, respectively. In $\mathfrak{J}, \mathfrak{J}', \mathfrak{J}^C$ and $\mathfrak{J}^1$, we define the inner product $(X, Y)$, the Freudenthal multiplication $X \times Y$ and the determinant $\mathrm{det}X$ respectively by

$$(X, Y) = \mathrm{tr}(X \circ Y),$$
$$X \times Y = \frac{1}{2}(2X \circ Y - \mathrm{tr}(X)Y - \mathrm{tr}(Y)X + (\mathrm{tr}(X)\mathrm{tr}(Y) - (X, Y))E),$$
$$\mathrm{det}X = \frac{1}{3}(X, X \times X),$$

where $E$ is the unit matrix.

### 2.3 Exceptional Lie groups $F_4, E_6, E_{6(-26)}$ and $E_{6(6)}$

We shall recall the following exceptional Lie groups.

$$\begin{aligned}
F_4 &= \{\alpha \in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}) \,|\, \alpha(X \circ Y) = \alpha X \circ \alpha Y\}, \\
E_6 &= \{\alpha \in \mathrm{Iso}_C(\mathfrak{J}^C) \,|\, \mathrm{det}\alpha X = \mathrm{det}X, \langle \alpha X, \alpha Y\rangle = \langle X, Y\rangle\}, \\
&= \{\alpha \in \mathrm{Iso}_C(\mathfrak{J}^C) \,|\, \alpha X \times \alpha Y = {}^t\alpha^{-1}(X \times Y), \langle \alpha X, \alpha Y\rangle = \langle X, Y\rangle\}, \\
E_{6(-26)} &= \{\alpha \in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}^1) \,|\, \mathrm{det}\alpha X = \mathrm{det}X\}, \\
E_{6(6)} &= \{\alpha \in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}') \,|\, \mathrm{det}\alpha X = \mathrm{det}X\},
\end{aligned}$$

where $\langle X, Y\rangle$ is the positive definite inner product of $\mathfrak{J}^C$ defined by $\langle X, Y\rangle = (\tau X, Y)$ and ${}^t\alpha$ is the transpose of $\alpha : ({}^t\alpha X, Y) = (X, \alpha Y)$. $F_4$ and $E_6$ are simply connected compact simple Lie groups of type $F_{4(-52)}$ and $E_{6(-78)}$, respectively. $E_{6(-26)}$ and $E_{6(6)}$ are noncompact simple Lie groups of type $E_{6(-26)}$ and $E_{6(6)}$, respectively.

## 3. Orbit spaces

### 3.1 Orbit spaces $\mathfrak{J}/F_4$ and $\mathfrak{J}^1/F_4$

We have known the following proposition and lemma.

**PROPOSITION 1** ([2], [4]). *A maximal compact subgroup $(E_{6(-26)})_K$ of the group $E_{6(-26)}$ is isomorphic to the group $F_4$ by the isomorphism $\varphi : F_4 \to$*

$(E_{6(-26)})_K \subset E_{6(-26)}$,

$$\varphi(\alpha)(X) = (f^{-1}\alpha f)X, \quad X \in \mathfrak{J}^1,$$

*where* $f : \mathfrak{J}^1 \to \mathfrak{J}, fX = I_1X$.

**Lemma 2** ([1], [2]). (1) *Any element* $X \in \mathfrak{J}$ *can be transformed to the following diagonal form by some element* $\alpha \in F_4$:

$$\alpha X = \begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix}, \quad \xi_1 \le \xi_2 \le \xi_3.$$

*Moreover, this form is uniquely determined by* $X$ *independent of the choice of* $\alpha \in F_4$.

(2) *Any element* $X \in \mathfrak{J}^1$ *can be transformed to the following diagonal form by some element* $\alpha = \varphi(\alpha') \in (E_{6(-26)})_K, \alpha' \in F_4$:

$$\alpha X = \begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix}, \quad -\xi_1 \le \xi_2 \le \xi_3.$$

*Moreover, this form is uniquely determined by* $X$ *independent of the choice of* $\alpha \in (E_{6(-26)})_K$.

From this lemma, we immediately get the following theorem.

**Theorem 3.** (1) *The orbit space* $\mathfrak{J}/F_4$ *of* $\mathfrak{J}$ *with respect to the group* $F_4$ *is given as follows*:

$$\mathfrak{J}/F_4 = \{(\xi_1, \xi_2, \xi_3) \in \boldsymbol{R}^3 \mid \xi_1 \le \xi_2 \le \xi_3\}.$$

(2) *The orbit space* $\mathfrak{J}^1/F_4$ *of* $\mathfrak{J}^1$ *with respect to the group* $F_4$ *of which the action is defined through* $\varphi : F_4 \to (E_{6(-26)})_K \subset E_{6(-26)}$, *is given as follows*:

$$\mathfrak{J}^1/F_4 = \{(\xi_1, \xi_2, \xi_3) \in \boldsymbol{R}^3 \mid -\xi_1 \le \xi_2 \le \xi_3\}.$$

### 3.2 Orbit space $\mathfrak{J}^C/E_6$

**Lemma 4.** *Any element* $X \in \mathfrak{J}^C$ *can be transformed to the following diagonal form by some element* $\alpha \in E_6$:

$$\alpha X = \begin{pmatrix} \zeta & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad \zeta \in C,\ \lambda_k \in \boldsymbol{R},\ |\zeta| \le \lambda_2 \le \lambda_3.$$

*Moreover, for a given* $X \in \mathfrak{J}^C$, $\alpha X$ *is uniquely expressed in the following form.*

*In the case* $\det X = 0$,

$$\alpha X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad 0 \le \lambda_2 \le \lambda_3.$$

*In the case* $\det X \neq 0$,

$$\alpha X = \begin{pmatrix} \zeta & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad 0 < |\zeta| \le \lambda_2 \le \lambda_3.$$

*Proof.* We can easily obtain the first half of the lemma from Proposition 5 in [5]. We shall now show the latter half. Let $X \in \mathfrak{J}^C$ be transformed to the following diagonal form

$$\alpha X = \mathrm{diag}(\zeta, \lambda_2, \lambda_3), \quad 0 \le |\zeta| \le \lambda_2 \le \lambda_3,$$

by $\alpha \in E_6$. Denote $|\zeta|$ by $\lambda_1$. Then we have

$$\langle X, X\rangle = \langle \alpha X, \alpha X\rangle = |\zeta|^2 + {\lambda_2}^2 + {\lambda_3}^2 = {\lambda_1}^2 + {\lambda_2}^2 + {\lambda_3}^2,$$

$$\begin{aligned}\langle X \times X, X \times X\rangle &= \langle {}^t\alpha^{-1}(X \times X), {}^t\alpha^{-1}(X \times X)\rangle = \langle \alpha X \times \alpha X, \alpha X \times \alpha X\rangle \\ &= (\lambda_2\lambda_3)^2 + |\lambda_3\zeta|^2 + |\zeta\,\lambda_2|^2 = {\lambda_2}^2{\lambda_3}^2 + {\lambda_3}^2{\lambda_1}^2 + {\lambda_1}^2{\lambda_2}^2,\end{aligned}$$

$$\det X = \det \alpha X = \zeta\,\lambda_2\lambda_3, \quad \text{in particular,} \quad |\det X| = \lambda_1\lambda_2\lambda_3.$$

Hence, ${\lambda_1}^2, {\lambda_2}^2, {\lambda_3}^2$, namely, $\lambda_1, \lambda_2, \lambda_3$ $(\lambda_k \ge 0)$ are uniquely determined as the solutions of the following equations:

$$\begin{cases} {\lambda_1}^2 + {\lambda_2}^2 + {\lambda_3}^2 = \langle X, X\rangle, \\ {\lambda_2}^2{\lambda_3}^2 + {\lambda_3}^2{\lambda_1}^2 + {\lambda_1}^2{\lambda_2}^2 = \langle X \times X, X \times X\rangle, \\ {\lambda_1}^2{\lambda_2}^2{\lambda_3}^2 = |\det X|^2. \end{cases}$$

Therefore, in the case $\det X = 0$, it readily follows that $\alpha X$ is uniquely expressed in the form of the lemma. Next, in the case $\det X \neq 0$, for unique $\lambda_2, \lambda_3$, we can uniquely obtain $\zeta$ from $\det X = \zeta\,\lambda_2\lambda_3$.

From this lemma, we immediately get the following theorem.

**THEOREM 5.** *The orbit space* $\mathfrak{J}^C/E_6$ *of* $\mathfrak{J}^C$ *with respect to the group* $E_6$ *is given by*

$$\mathfrak{J}^C/E_6 = \left\{(\xi_0, \xi_1, \xi_2, \xi_3) \in \boldsymbol{R}^4 \;\middle|\; \sqrt{{\xi_0}^2 + {\xi_1}^2} \le \xi_2 \le \xi_3\right\}$$

*with the cross section* $\sigma : \mathfrak{J}^C/E_6 \to \mathfrak{J}^C$,

$$\sigma(\xi_0, \xi_1, \xi_2, \xi_3) = \mathrm{diag}(\xi_0 + i\xi_1, \xi_2, \xi_3).$$

We can also express the orbit space $\mathfrak{J}^C/E_6$ in the following:

$$\mathfrak{J}^C/E_6 = \left\{(\lambda_0, \lambda_1, \lambda_2, \lambda_3) \in \boldsymbol{R}^4 \,\middle|\, \begin{cases} 0 \le \lambda_0 < 1, \\ 0 < \lambda_1 \le \lambda_2 \le \lambda_3 \end{cases} \text{or } 0 = \lambda_0 = \lambda_1 \le \lambda_2 \le \lambda_3 \right\}$$

with the cross section $\sigma : \mathfrak{J}^C/E_6 \to \mathfrak{J}^C$,

$$\sigma(\lambda_0, \lambda_1, \lambda_2, \lambda_3) = \mathrm{diag}(\lambda_1 e^{2\pi i \lambda_0}, \lambda_2, \lambda_3),$$

and furthermore express it as

$$\begin{aligned} \mathfrak{J}^C/E_6 = \{(\lambda_1, \lambda_2, \lambda_3) \in \boldsymbol{R}^3 \,|\, 0 < \lambda_1 \le \lambda_2 \le \lambda_3\} \times S^1 \\ \cup \{(0, \lambda_2, \lambda_3) \in \boldsymbol{R}^3 \,|\, 0 \le \lambda_2 \le \lambda_3\}, \end{aligned}$$

where $S^1$ is a one dimensional unit sphere and the symbol $\cup$ denotes disjoint sum.

### 3.3 Orbit space $\mathfrak{J}'/Sp(4)$

We have known the following proposition.

**PROPOSITION 6** ([3]). *A maximal compact subgroup $(E_{6(6)})_K$ of the group $E_{6(6)}$ is isomorphic to the group $Sp(4)/\boldsymbol{Z}_2$ by the isomorphism induced by the homomorphism $\psi : Sp(4) \to (E_{6(6)})_K \subset E_{6(6)}$,*

$$\psi(A)X = g^{-1}(A(gX)A^{-1}), \quad X \in \mathfrak{J}',$$

*where $g : \mathfrak{J}' \to \mathfrak{J}(4, \boldsymbol{H})_0 := \{X \in M(4, \boldsymbol{H}) \,|\, X^* = X, \mathrm{tr}(X) = 0\}$ is defined by*

$$g\left(\begin{pmatrix} \xi_1 & a_3 + b_3 e' & \overline{a}_2 - b_2 e' \\ \overline{a}_3 - b_3 e' & \xi_2 & a_1 + b_1 e' \\ a_2 + b_2 e' & \overline{a}_1 - b_1 e' & \xi_3 \end{pmatrix}\right) = \begin{pmatrix} \lambda_1 & b_1 & b_2 & b_3 \\ \overline{b}_1 & \lambda_2 & a_3 & \overline{a}_2 \\ \overline{b}_2 & \overline{a}_3 & \lambda_3 & a_1 \\ \overline{b}_3 & a_2 & \overline{a}_1 & \lambda_4 \end{pmatrix},$$

$$\lambda_1 = \frac{1}{2}(\xi_1 + \xi_2 + \xi_3),\ \lambda_2 = \frac{1}{2}(\xi_1 - \xi_2 - \xi_3),\ \lambda_3 = \frac{1}{2}(\xi_2 - \xi_3 - \xi_1),\ \lambda_4 = \frac{1}{2}(\xi_3 - \xi_1 - \xi_2).$$

Using this proposition, we can prove the following lemma.

**LEMMA 7.** *Any element $X \in \mathfrak{J}'$ can be transformed to the following form by some element $\alpha \in (E_{6(6)})_K$:*

$$\alpha X = \begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix}, \quad |\xi_3| \le \xi_2 \le \xi_1.$$

*Moreover, this form is uniquely determined by $X \in \mathfrak{J}'$ independent of the choice of $\alpha \in (E_{6(6)})_K$.*

*Proof.* For a given $X \in \mathfrak{J}'$, as is well known, there exists $A \in Sp(4)$ such that $gX \in \mathfrak{J}(4, \boldsymbol{H})_0$ is transformed to the following form:

$$A(gX)A^{-1} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4), \quad \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4,\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0,$$

and this form is uniquely determined by $gX \in \mathfrak{J}(4, \boldsymbol{H})_0$, i.e., by $X \in \mathfrak{J}'$. Therefore, by the definition of $\psi$, we have

$$\begin{aligned}\psi(A)X &= g^{-1}(A(gX)A^{-1}) \\ &= \mathrm{diag}(\lambda_1 + \lambda_2, \lambda_1 + \lambda_3, \lambda_1 + \lambda_4) \underset{\text{put}}{=} \mathrm{diag}(\xi_1, \xi_2, \xi_3).\end{aligned}$$

Then obviously $\lambda_1 = \frac{1}{2}(\xi_1+\xi_2+\xi_3), \lambda_2 = \frac{1}{2}(\xi_1-\xi_2-\xi_3), \lambda_3 = \frac{1}{2}(\xi_2-\xi_3-\xi_1), \lambda_4 = \frac{1}{2}(\xi_3 - \xi_1 - \xi_2)$. Thus, substituting the above $\lambda_k, k = 1, 2, 3, 4$ into inequalities $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \cdots (*)$, we obtain

$$\begin{cases} \xi_1 \geq \xi_2 \geq \xi_3, \\ \xi_2 \geq -\xi_3, \end{cases} \quad \text{that is,} \quad \xi_1 \geq \xi_2 \geq |\xi_3|,$$

which is equivalent to $(*)$.

From this lemma, we immediately get the following theorem.

**THEOREM 8.** *The orbit space $\mathfrak{J}'/Sp(4)$ of $\mathfrak{J}'$ with respect to the group $Sp(4)$ of which the action is defined through $\psi : Sp(4) \to (E_{6(6)})_K \subset E_{6(6)}$, is given as follows:*

$$\mathfrak{J}'/Sp(4) = \{(\xi_1, \xi_2, \xi_3) \in \boldsymbol{R}^3 \mid |\xi_3| \leq \xi_2 \leq \xi_1\}.$$

## 4. Orbit types of $Sp(4)$ in $\mathfrak{J}'$

In order to prove the following theorem, we observe that for $\alpha = \psi(A), A \in Sp(4)$ and $X \in \mathfrak{J}'$,

$$\alpha X = X \quad \text{if and only if} \quad A(gX) = (gX)A.$$

Hereafter, for the sake of brevity we denote the matrix $\mathrm{diag}(\xi_1, \xi_2, \xi_3)$ by $(\xi_1, \xi_2, \xi_3)$.

**THEOREM 9.** *The orbit types of the group $Sp(4)$ in $\mathfrak{J}'$ under the action defined through $\psi : Sp(4) \to (E_{6(6)})_K \subset E_{6(6)}$, are given as follows:*

$$Sp(4)/(Sp(1) \times Sp(1) \times Sp(1) \times Sp(1)), \quad Sp(4)/(Sp(1) \times Sp(1) \times Sp(2)),$$
$$Sp(4)/(Sp(2) \times Sp(2)), \quad Sp(4)/(Sp(1) \times Sp(3)), \quad Sp(4)/Sp(4).$$

*In further details,*

(1) *The orbit through* $(\xi_1, \xi_2, \xi_3)$ (*where* $\xi_1 > \xi_2 > |\xi_3|$) *is* $Sp(4)/(Sp(1) \times Sp(1) \times Sp(1) \times Sp(1))$.
(2) *The orbit through* $(\xi_1, \xi_2, \xi_3)$ (*where* $\xi_1 > \xi_2 = |\xi_3| \neq 0$ *or* $\xi_1 = \xi_2 > |\xi_3|$) *is* $Sp(4)/(Sp(1) \times Sp(1) \times Sp(2))$.
(3) *The orbit through* $(\xi_1, \xi_2, \xi_3)$ (*where* $\xi_1 = \xi_2 = |\xi_3| \neq 0$) *is* $Sp(4)/(Sp(1) \times Sp(3))$.
(4) *The orbit through* $(\xi_1, 0, 0)$ (*where* $\xi_1 > 0$) *is* $Sp(4)/(Sp(2) \times Sp(2))$.
(5) *The orbit through* $(0,0,0)$ *is* $Sp(4)/Sp(4)$.

*Proof.* We first observe

$$(E_{6(6)})_K = \{\alpha \in E_{6(6)} \,|\, \alpha X \times \alpha Y = \gamma\alpha\gamma(X \times Y)\},$$

where $\gamma \in E_{6(6)}, \gamma X(\xi_k, a_k + b_k e') = X(\xi_k, a_k - b_k e')$(c.f.[3]). We shall now determine the isotropy subgroup $(Sp(4))_{(\xi_1,\xi_2,\xi_3)}$ of $Sp(4)$ at the point $(\xi_1, \xi_2, \xi_3)$.

(1) 1° Case $\xi_1 > \xi_2 > |\xi_3| \neq 0$. Let $A \in (Sp(4))_{(\xi_1,\xi_2,\xi_3)}$. Then $\alpha = \psi(A) \in ((E_{6(6)})_K)_{(\xi_1,\xi_2,\xi_3)}$, that is,

$$\alpha(\xi_1 E_1 + \xi_2 E_2 + \xi_3 E_3) = \xi_1 E_1 + \xi_2 E_2 + \xi_3 E_3, \tag{4.1}$$

where $E_1 = (1,0,0), E_2 = (0,1,0), E_3 = (0,0,1)$. From $\gamma\alpha\gamma((\xi_1 E_1 + \xi_2 E_2 + \xi_3 E_3)^{\times 2}) = (\alpha(\xi_1 E_1 + \xi_2 E_2 + \xi_3 E_3))^{\times 2} = (\xi_1 E_1 + \xi_2 E_2 + \xi_3 E_3)^{\times 2}$, we have $\gamma\alpha\gamma(\xi_2\xi_3 E_1 + \xi_3\xi_1 E_2 + \xi_1\xi_2 E_3) = \xi_2\xi_3 E_1 + \xi_3\xi_1 E_2 + \xi_1\xi_2 E_3$, that is,

$$\alpha(\xi_2\xi_3 E_1 + \xi_3\xi_1 E_2 + \xi_1\xi_2 E_3) = \xi_2\xi_3 E_1 + \xi_3\xi_1 E_2 + \xi_1\xi_2 E_3. \tag{4.2}$$

Then, by ${\xi_1}^{-1} \times (4.1) - (\xi_2\xi_3)^{-1} \times (4.2)$, we get

$$\alpha(\eta_2 E_2 + \eta_3 E_3) = \eta_2 E_2 + \eta_3 E_3, \tag{4.3}$$

where $\eta_2 = {\xi_1}^{-1}\xi_2 - {\xi_2}^{-1}\xi_1 \neq 0, \eta_3 = {\xi_1}^{-1}\xi_3 - {\xi_3}^{-1}\xi_1 \neq 0$. Hence, from $\gamma\alpha\gamma((\eta_2 E_2 + \eta_3 E_3)^{\times 2}) = (\alpha(\eta_2 E_2 + \eta_3 E_3))^{\times 2} = (\eta_2 E_2 + \eta_3 E_3)^{\times 2}$, we have $\alpha(\eta_2\eta_3 E_1) = \eta_2\eta_3 E_1$, that is, $\alpha E_1 = E_1$. Similarly we have $\alpha E_2 = E_2$, $\alpha E_3 = E_3$. The converse: $\alpha = \psi(A) \in ((E_{6(6)})_K)_{(\xi_1,\xi_2,\xi_3)}$, that is, $A \in (Sp(4))_{(\xi_1,\xi_2,\xi_3)}$ is clear. Therefore, since $\psi(A)E_k = E_k$ $(k = 1,2,3)$, i.e., $A(gE_k) = (gE_k)A$ $(k = 1,2,3)$, that is, $A$ commutes with

$$gE_1 = \frac{1}{2}\mathrm{diag}(1,1,-1,-1), \quad gE_2 = \frac{1}{2}\mathrm{diag}(1,-1,1,-1),$$
$$gE_3 = \frac{1}{2}\mathrm{diag}(1,-1,-1,1),$$

it follows that $A$ belongs to $Sp(1) \times Sp(1) \times Sp(1) \times Sp(1)$. Thus we obtain that the orbit through $(\xi_1, \xi_2, \xi_3)$ is $Sp(4)/(Sp(1) \times Sp(1) \times Sp(1) \times Sp(1))$.

2° Case $\xi_1 > \xi_2 > \xi_3 = 0$. Let $A \in (Sp(4))_{(\xi_1,\xi_2,0)}$. Then $\alpha = \psi(A) \in ((E_{6(6)})_K)_{(\xi_1,\xi_2,0)}$, that is,

$$\alpha(\xi_1 E_1 + \xi_2 E_2) = \xi_1 E_1 + \xi_2 E_2. \tag{4.4}$$

From $\gamma\alpha\gamma((\xi_1 E_1 + \xi_2 E_2)^{\times 2}) = (\alpha(\xi_1 E_1 + \xi_2 E_2))^{\times 2} = (\xi_1 E_1 + \xi_2 E_2)^{\times 2}$, we have $\alpha(\xi_1\xi_2 E_3) = \xi_1\xi_2 E_3$, that is, $\alpha E_3 = E_3$. Hence, from $\gamma\alpha\gamma((\xi_1 E_1 + \xi_2 E_2) \times E_3) = \alpha(\xi_1 E_1 + \xi_2 E_2) \times \alpha E_3 = (\xi_1 E_1 + \xi_2 E_2) \times E_3$, we have

$$\alpha(\xi_2 E_1 + \xi_1 E_2) = \xi_2 E_1 + \xi_1 E_2. \tag{4.5}$$

Then, by $\xi_2 \times (4.4) - \xi_1 \times (4.5)$, we get $\alpha((\xi_2{}^2 - \xi_1{}^2)E_1) = (\xi_2{}^2 - \xi_1{}^2)E_1$, that is, $\alpha E_2 = E_2$. Similarly, by $\xi_1 \times (4.4) - \xi_2 \times (4.5)$, we have $\alpha E_1 = E_1$. Thus we obtain that the orbit through $(\xi_1, \xi_2, \xi_3)$ has the same orbit type of Case (1) 1°.

(2) 1° Case $\xi_1 > \xi_2 = \xi_3 > 0$. We may assume $\xi_1 > \xi_2 = \xi_3 = 1$. Let $A \in (Sp(4))_{(\xi_1,1,1)}$. Then $\alpha = \psi(A) \in ((E_{6(6)})_K)_{(\xi_1,1,1)}$, that is,

$$\alpha(\xi_1 E_1 + E_2 + E_3) = \xi_1 E_1 + E_2 + E_3. \tag{4.6}$$

From $\gamma\alpha\gamma((\xi_1 E_1 + E_2 + E_3)^{\times 2}) = (\alpha(\xi_1 E_1 + E_2 + E_3))^{\times 2} = (\xi_1 E_1 + E_2 + E_3)^{\times 2}$, we have $\gamma\alpha\gamma(E_1 + \xi_1 E_2 + \xi_1 E_3) = E_1 + \xi_1 E_2 + \xi_1 E_3$, that is,

$$\alpha(E_1 + \xi_1 E_2 + \xi_1 E_3) = E_1 + \xi_1 E_2 + \xi_1 E_3. \tag{4.7}$$

Then, by $\xi_1 \times (4.6) - (4.7)$, we get $\alpha((\xi_1{}^2 - 1)E_1) = (\xi_1{}^2 - 1)E_1$, that is, $\alpha E_1 = E_1$. Hence, we have $\alpha(E_2 + E_3) = \alpha((\xi_1 E_1 + E_2 + E_3) - \xi_1 E_1) = (\xi_1 E_1 + E_2 + E_3) - \xi_1 E_1 = E_2 + E_3$. Therefore, since $\psi(A)E_1 = E_1$ and $\psi(A)(E_2 + E_3) = E_2 + E_3$, i.e., $A(gE_1) = (gE_1)A$ and $A(g(E_2 + E_3)) = (g(E_2 + E_3))A$, it follows that $A$ belongs to $Sp(1) \times Sp(1) \times Sp(2)$. Thus we obtain that the orbit through $(\xi_1, 1, 1)$ is $Sp(4)/(Sp(1) \times Sp(1) \times Sp(2))$.

2° Case $\xi_1 > \xi_2 = -\xi_3 > 0$. We may assume $\xi_1 > \xi_2 = -\xi_3 = 1$. Let $A \in (Sp(4))_{(\xi_1,1-1)}$. Then, as is similar to Case (2) 1°, we get $\alpha E_1 = E_1$ and $\alpha(E_2 - E_3) = E_2 - E_3$ for $\alpha = \psi(A) \in ((E_{6(6)})_K)_{(\xi_1,1,-1)}$. Therefore, since $A(gE_1) = (gE_1)A$ and $A(g(E_2 - E_3)) = (g(E_2 - E_3))A$, it follows that $A$ belongs to $Sp(2) \times Sp(1) \times Sp(1)$. In addition, the subgroup $Sp(2) \times Sp(1) \times Sp(1)$ is conjugate to $Sp(1) \times Sp(1) \times Sp(2)$ by $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ in $Sp(4)$. Thus we obtain that the orbit through $(\xi_1, 1, -1)$ has the same orbit type of Case (2) 1°.

3° Case $\xi_1 = \xi_2 > |\xi_3|$. We may assume $\xi_1 = \xi_2 = 1 > |\xi_3|$. Let $A \in (Sp(4))_{(1,1,\xi_3)}$. Then, as is similar to Case (2) 1°, we get $\alpha E_3 = E_3$ and $\alpha(E_1 +$

$E_2) = E_1 + E_2$ for $\alpha = \psi(A) \in ((E_{6(6)})_K)_{(1,1,\xi_3)}$. Therefore, since $A(gE_3) = (gE_3)A$ and $A(g(E_1+E_2)) = (g(E_1+E_2))A$, it follows that $A$ belongs to $Sp(1) \times Sp(2) \times Sp(1)$. In addition, the subgroup $Sp(1) \times Sp(2) \times Sp(1)$ is conjugate to $Sp(1) \times Sp(1) \times Sp(2)$ by $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ in $Sp(4)$. Thus we obtain that the orbit through $(1,1,\xi_3)$ has the same orbit type of Case (2) 1°.

(3) 1° Case $\xi_1 = \xi_2 = \xi_3 > 0$. We may assume $\xi_1 = \xi_2 = \xi_3 = 1$. Let $A \in (Sp(4))_{(1,1,1)}$. Then we can easily get $\alpha E = E$ for $\alpha = \psi(A) \in ((E_{6(6)})_K)_{(1,1,1)}$. Therefore, since $\psi(A)E = E$, i.e., $A(gE) = (gE)A$, it follows that $A$ belongs to $Sp(1) \times Sp(3)$. Thus we obtain that the orbit through $(1,1,1)$ is $Sp(4)/(Sp(1) \times Sp(3))$.

2° Case $\xi_1 = \xi_2 = -\xi_3 > 0$. We may assume $\xi_1 = \xi_2 = -\xi_3 = 1$. Let $A \in (Sp(4))_{(1,1,-1)}$. Then we easily get $\alpha(E_1 + E_2 - E_3) = E_1 + E_2 - E_3$ for $\alpha = \psi(A) \in ((E_{6(6)})_K)_{(1,1,-1)}$. Therefore, since $\psi(A)(E_1 + E_2 - E_3) = E_1 + E_2 - E_3$, i.e., $A(g(E_1 + E_2 - E_3)) = (g(E_1 + E_2 - E_3))A$, it follows that $A$ belongs to $Sp(3) \times Sp(1)$. In addition, the subgroup $Sp(3) \times Sp(1)$ is conjugate to $Sp(1) \times Sp(3)$ by $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ in $Sp(4)$. Thus we obtain that the orbit through $(1,1,-1)$ has the same orbit type of Case (3) 1°.

(4) Let $A \in (Sp(4))_{(\xi_1,0,0)}$ (where $\xi_1 > 0$). Then we easily get $\alpha E_1 = E_1$ for $\alpha = \psi(A) \in ((E_{6(6)})_K)_{(\xi_1,0,0)}$. Therefore, since $\psi(A)E_1 = E_1$ i.e., $A(gE_1) = (gE_1)A$, it follows that $A$ belongs to $Sp(2) \times Sp(2)$. Thus we obtain that the orbit through $(\xi_1, 0, 0)$ is $Sp(4)/(Sp(2) \times Sp(2))$.

(5) The isotropy subgroup $(Sp(4))_{(0,0,0)}$ is obviously $Sp(4)$. Thus we obtain the orbit through $(0,0,0)$ is $Sp(4)/Sp(4)$.

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