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DIAGONAL FLIPS IN GRAPHS ON CLOSED SURFACES WITH SPECIFIED FACE SIZE DISTRIBUTIONS

By

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Abstract. We shall show that two simple graphs embedded on a closed surface with the same face size distribution can be transformed into each other, up to homeomorphism, by a sequence of diagonal flips if they have the sufficiently large number of triangular faces.

1. Introduction

A "diagonal flip" has been defined as a local deformation of triangulations on surfaces and there is a big stream [3, 4, 7, 8, 17, 19, 20, 21, 22, 23, 26, 27] of studies on the equivalence over triangulations on surfaces by diagonal flips in topological graph theory. This began with Wagner's classical work [26] on spherical triangulations and Negami's work [19] opened the way to a general theory of diagonal flips. His result states that any two triangulations on a closed surface with the same and sufficiently large number of vertices can be transformed into each other, up to homeomorphism, by a sequence of diagonal flips, and its many variations have been proved. Furthermore, Nakamoto has developed a theory of "diagonal flips" in quadrangulations on surfaces [12, 13, 14, 15, 16, 18], some part of which goes in parallel to that for triangulations.

A natural question arises; what can we say about simple graphs with faces some of which are triangular and others are quadrangular? In this paper, we shall answer this question under more general situations, as follows.

Let G be a simple graph 2-cell embedded on a closed surface F^2 and let f_i denote the number of faces of size i. Then we have $\sum_{i\geq 3} if_i = 2|E(G)|$. We call the sequence $(f_3, f_4, f_5, \ldots)$ the *face size distribution* of G. We denote the size of a face A by $|A|$. Let A_1 and A_2 be two faces adjacent along an edge v_0v_m and let $v_0v_1\cdots v_m\cdots v_{|A_1|+|A_2|-3}$ be the boundary walk of the region obtained as $A_1\cup A_2$, which is divided into A_1 and A_2 by v_0v_m. A *diagonal flip* at

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v_0v_m is to replace v_0v_m with v_1v_{m+1} (or $v_{-1}v_{m-1}$ with subscripts taken modulo $|A_1| + |A_2| - 2$). Any diagonal flip does not change the face size distribution of G. Since we have to stay in the category of simple graphs, a diagonal flip is forbidden if it produces multiple edges or loops.

A graph G embedded on a closed surface F^2 is said to be *closed 2-cell embedded* or *strongly embedded* on F^2 if each face of G is bounded by a cycle of length at least 3. The following theorem is our answer to the above question:

THEOREM 1. *Given a closed surface F^2 and a finite sequence $\varphi = (f_4, f_5, \dots)$ of non-negative integers, there exists a natural number $R = R(F^2, \varphi)$ such that two simple graphs closed 2-cell embedded on F^2 with the same face size distribution $(f_3, f_4, f_5, \dots)$ can be transformed into each other, up to homeomorphism, by a sequence of diagonal flips if $f_3 \geq R$.*

Note that triangular faces are actually necessary. For examaple, if $f_i = 0$ for all odd number i, there arises an obstruction related to an algebraic invariant called "the cycle parity". Any diagonal flip in such graphs preserves their cycle parities. Thus, they cannot be transformed into each other by a sequence of diagonal flips if they have different cycle parities. See [14, 18] for the details on cycle parities.

The property of being closed 2-cell embedded is actually necessary in our proof of the theorem to make a situation where we can use some results on triangulations. However, another problem would arise when we neglected this property. For example, how should we consider "a diagonal flip" of a cut edge?

Finally, we shall show the dual form of Theorem 1. A graph G embedded on a closed surface F^2 is said to be *semi-polyhedral* if each face of G is bounded by a cycle, possibly of length 2, and if any two faces share at most one edge. Two faces may share two or more vertices in a semi-polyhedral graph and each vertex must have degree at least 3. It is easy to see that G is semi-polyhedral if and only if its dual G^* is simple and closed 2-cell embedded on F^2.

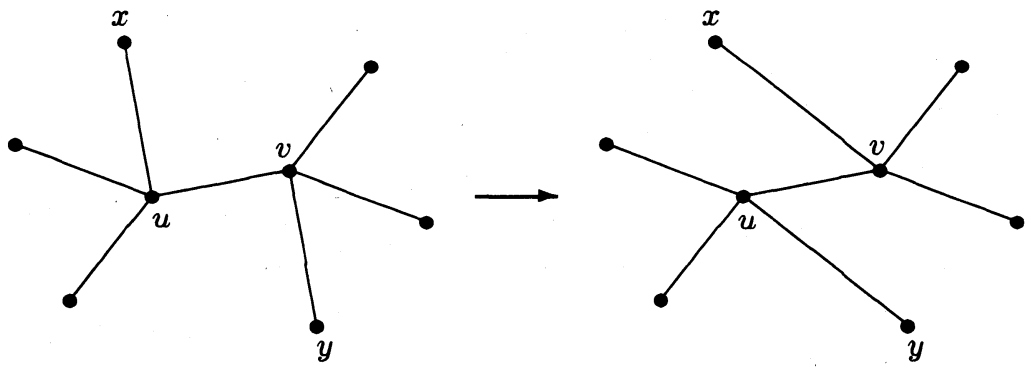

Figure 1 An edge slide along uv

Let $xuvy$ be a path of length 3 in a semi-polyhedral graph G such that xuv and uvy form two corners of distinct faces of G incident to the edge uv. An *edge slide* along uv is to replace two edges xu and vy with xv and uy, respectively, as shown in Figure 1. We do not carry out an edge slide if it makes a graph not semi-polyhedral. It is clear that an edge slide in G corresponds to a diagonal flip in G^*. Thus, the following corollary is just an immediate consequence of Theorem 1:

COROLLARY 2. *Two semi-polyhedral graphs on a closed surface with the same degree sequence can be transformed into each other, up to homeomorphism, by a sequence of edge sildes if they have sufficiently many vertices of degree* 3.

2. Proof of the theorem

A surface F^2 is called a *punctured surface* if it has a boundary. Suppose that each boundary component is assigned a non-negative integer more than or equal to 3, as its *length*. A *triangulation* on such a punctured surface F^2 is a simple graph embedded on F^2 so that a cycle of the specified length d in G is placed along each boundary component of length d and that each face is triangular. Diagonal flips in those triangulations can be defined as in the previous, but we do not flip edges on the boundary cycles.

Under this situation, Negami's arguments in [19] work well only with small changes and conclude the following theorem, which can be found in [21]. We shall prove our main theorem, using this theorem.

THEOREM 3. *Given a punctured surface F^2 with boundary components of given lengths, there exists a natural number D such that two triangulations on F^2 can be transformed into each other, up to homeomorphism, by a sequence of diagonal flips if they have the same number of vertices more than D.*

Let G be a simple graph closed 2-cell embedded on a closed surface F^2 with face size distribution $(f_3, f_4, f_5, \ldots)$ and put $\varphi = (f_4, f_5, \ldots)$. A graph H on F^2 is said to be a *φ-graph* if H is simple, closed 2-cell embedded on F^2 and has exactly f_i i-gonal faces for each $i \geq 4$. Contraction of an edge ac is to shrink ac into a point $a = c$. In particular, if ac is incident to a triangular face abc, then we have to replace a digonal face bounded by multiple edges ab and cb with a single edge. Let G/ac denote the graph obtained from G by the contraction of ac. An edge ac of G is said to be *φ-contractible* if G/ac is also a φ-graph. Thus, any edge incident to a face of size more than 3 is not φ-contractible while both sides of any φ-contractible edge must be triangular faces.

A φ-graph G on a closed surface F^2 is said to be *φ-irreducible* if G has no φ-contractible edge. We call a φ-irreducible φ-graph simply a *φ-irreducible graph.* If $\varphi = (0, 0, \ldots)$, then any φ-irreducible graph of F^2 is an irreducible triangulation of F^2 in the usual sense.

The finiteness of irreducible triangulations of F^2 in number implies that of its φ-irreducible graphs, as in the following lemma. The former has been proved in many papers [2, 5, 6, 11] and the complete lists of irreducible triangulations of the sphere, the projective plane, the torus and the Klein bottle have been already determined in [25, 1, 9, 10], respectively.

LEMMA 4. *Given a closed surface F^2 and a finite sequence $\varphi = (f_4, f_5, \ldots)$ of non-negative integers, there exist only finitely many φ-irreducible graphs of F^2, up to homeomorphism.*

Proof. Let G be any φ-irreducible graph on F^2. Call an edge "a red edge" if it is incident to a face of size more than 3 and "a black edge" otherwise. Add an extra vertex to each face of size more than 3 and join it to all vertices lying along the boundary of the face with "red edges". Let $\hat{G}$ be the resulting triangulation on F^2. The property of G being closed 2-cell embedded guarantees that $\hat{G}$ is a simple graph.

Since G is φ-irreducible, all black edges are not contractible in $\hat{G}$ while red edges might be contractible. Contract edges, red or black, until we obtain an irreducible triangulation T_0 of F^2. If a black edge and a red edge are identified in the process, then the resulting edge should be red. We shall show that each contracion decreases the number of red edges by at least one even if it contracts a black edge.

Let ac be an edge in $\hat{G}$, which is contracted in T_0. Let G' be the triangulation in which we contract the edge ac in the contracting process. We may assume that ac is still black in G' as well as in $\hat{G}$. We shall show that one of the facial cycles sharing ac in G' contains two red edges.

Since ac is not contractible in $\hat{G}$, there is a non-facial cycle axc of length 3 in $\hat{G}$. If axc were essential, then it would survive in G'. Thus, axc bounds a 2-cell region D^2. Let H be the triangulation of D^2 with boundary axc, which is a subgraph of $\hat{G}$. Since ac can be contracted in G', the region D^2 shrinks into either the edge ac or one of the faces sharing ac, say abc in G'.

First, suppose that D^2 shrinks into ac. Let A and C be the sets of vertices in H which are identified with a and c, respectively. Obviously, $A \cup C$ is a partition of $V(H)$. Since ac is black in G', any edge a_1c_1 joining A and C, called an A–C edge, is black in $\hat{G}$ and there is a non-facial cycle $a_1x_1c_1$ of length 3 in $\hat{G}$. It is clear that $a_1x_1c_1$ cannot go outside D^2 and hence it bounds a 2-cell region D_1^2 inside D^2.

Choose an A–C edge a_1c_1 so that D_1^2 is the innermost among those regions and let $a_1b_1c_1$ be the face incident to a_1c_1 in D_1^2. We may assume that $b_1 \in C$. Then the edge a_1b_1 is an A–C edge and must be contained in a non-facial cycle of length 3 bounding a 2-cell region inside D_1^2, which is contrary to the minimality of D_1^2. Thus, this case does not happen.

Suppose that D^2 shrinks into a face abc. Let A, B and C be the sets of vertices in H which are identified with a, b and c, respectively. Assume that bc is black in G'. Then all A–C edges and all B–C edges are black in $\hat{G}$ and each of them is contained in a non-facial cycle of length 3 bounding a 2-cell region inside D^2.

As in the previous case, choose an innermost one among those regions, say D_1^2. Let a_1c_1 be an A–C or B–C edge on its boundary, which is black, and let $a_1b_1c_1$ be the face incident to a_1c_1 in D_1^2. Then we may assume that $a_1 \in A \cup B$ and $c_1 \in C$. However, it is easy to see that either a_1b_1 or b_1c_1 is black in any case, which is contrary to our choice of D_1^2, again. Thus, bc must be red in G' and hence so is ab by symmetry. Since these two red edges become one, contracting ac decreases the number of red edges, as claimed.

Therefore, what we have just proved implies that the number of edges contracted to obtain T_0, say n, does not exceed the number of red edges in $\hat{G}$, which is bounded by $2\sum_{i\geq 4} if_i$. Thus we have $|V(\hat{G})| = |V(T_0)| + n$ and $n \leq 2\sum_{i\geq 4} if_i$. These imply that:

$$\begin{aligned} |V(G)| = |V(\hat{G})| - \sum_{i\geq 4} f_i \;&=\; |V(T_0)| + n - \sum_{i\geq 4} f_i \\ \leq |V(T_0)| + 2\sum_{i\geq 4} if_i - \sum_{i\geq 4} f_i \;&=\; |V(T_0)| + \sum_{i\geq 4}(2i-1)f_i \end{aligned}$$

Since $|V(T_0)|$ is bounded by a constant depending only on F^2, $|V(G)|$ is bounded by a constant depending on F^2 and φ. ■

Lemma 5. *Let G be a graph closed 2-cell embedded on a closed surface F^2, and let v be a vertex of G with neighbors $v_0, \ldots, v_{m-1}$ lying around it in this cyclic order. Add a vertex x of degree 3 to G so that x is adjacent to v, v_i and v_{i+1}. Then, the vertex x can be moved by diagonal flips so that x is adjacent to v, v_{i+1} and v_{i+2}. (The subscripts are taken modulo m.)*

Proof. By diagonal flips, replace vv_{i+1} with xv_{i+2}, and then replace xv_i with vv_{i+1}. ■

In Lemma 5, the vertex x of degree 3 added can be moved through any triangular faces. However, if no face of G incident to v is triangular, then x can only be rotated around v.

Let Δ be a 2-cell region bounded by a cycle of length 3 on a closed surface F^2 and suppose that it is subdivided into triangular faces. Since the subdivision of Δ can be regarded as a plane triangulation, it can be transformed into the "standard form" that can be obtained by adding vertices of degree 3 one by one to the empty region, in the same way as in the proof of Wagner's theorem given in [24]. Using Lemma 5 repeatedly, we can move all vertices in Δ to the outside of Δ by diagonal flips after deforming it into the standard form. So we call each of those vertices a *movable vertex* here. That is, a vertex v in a graph G on a closed surface F^2 is movable if and only if there is a cycle of length 3 in G which bounds a triangulated 2-cell region on F^2 containing v.

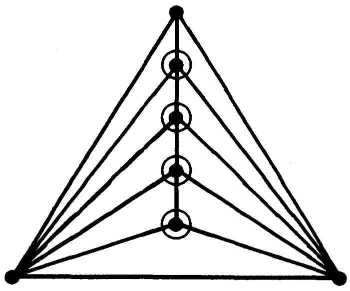

Figure 2 The "standard form" and movable vertices

Lemma 6. *Let C be a cycle of length m bounding a 2-cell region D^2 which is subdivided into triangular faces and one m-gonal face F and suppose that there is no chord of C in D^2. If the inside of C contains at least m vertices, then we can transform it by diagonal flips so that:*

(i) *There is a cycle C' of length m disjoint from C and bounding an m-gonal face.*
(ii) *Each vertex not lying on $C \cup C'$ is movable.*
(iii) *There is no chord of C in D^2.*

Proof. Let C' be the boundary cycle of F and R the region bounded by $C \cup C'$ in D^2 which is triangulated. In this initial stage, C' might not be disjoint from C. We paint movable vertices by *red* to distinguish them from other vertices, which are still *black*.

Suppose that there is a black vertex v in $R - C \cup C'$ and let $u_1 \cdots u_k$ be its *link* in the black triangulation of R, that is, the cycle of its black neighbors surrounding v. If $k = 3$, then v is movable and should have been colored by red. Thus, we have $\deg v = k \geq 4$. Then we can find consecutive four neighbors of v, say u_1 to u_4, so that u_4 does not lie on C. For, otherwise, one of edges on the link $u_1 \cdots u_k$ of v would be a chord of C. Furthermore, we may assume that each of triangular faces vu_1u_2, vu_2u_3 and vu_3u_4 does not contain any red vertex, after moving it, if any, to other triangular faces.

If u_1 and u_3 are not adjacent and do not lie on C together, we flip vu_2 to u_1u_3, so that the degree of v will decrease by one and that the new edge u_1u_3 is not a chord of C. Otherwise, u_2 and u_4 are not adjacent by the planarity and at least u_4 does not lie on C by our choice of u_1 to u_4. Flip vu_3 to u_2u_4, which will not be a chord of C, in this case. Repeat this procedure as far as possible, that is, until we have $\deg v = 3$, and color the new movable vertex v by red.

By the previous arguments, we can establish Conditions (ii) and (iii). If C' is disjoint from C, then the lemma follows. Suppose not. Since $|V(C') - V(C)| < m$, R must contain at least one red vertex. Then we can find a black triangle abc in R with $a \in V(C) \cap V(C')$, $ac \in E(C') - E(C)$ and $ab \notin E(C')$ such that we can move a red vertex x into this triangle. We may assume that x is the only vertex contained in the triangular region bounded by abc. Let d be the neighbor of a on C' other than c. Flip ac to xd and repaint x and xa, xb and xc by black. Then the new C' does not contain a. Repearting this as far as possible, we can eliminate the intersection of C and C' and obtain the final form satisfying (i), (ii) and (iii). ■

Now we shall prove Theorem 1.

Proof of Theorem 1. Let F^2 be a closed surface and let $\varphi = (f_4, f_5, \ldots)$ be any sequence of non-negative integers. By Lemma 4, there exist only finitely many φ-irreducible graphs on F^2. Thus, we let $I(F^2, \varphi)$ be the maximum number of vertices of them. Put

$$\tilde{R} = \tilde{R}(F^2, \varphi) = \max \left\{ I(F^2, \varphi) + \sum_{i \geq 4} i f_i, \ D \right\},$$

where D is the integer for which Theorem 3 is guaranteed to hold.

Let G_1 and G_2 be two φ-graphs on F^2 such that

(i) G_1 and G_2 have the same face size distribution $(f_3, f_4, f_5, \ldots)$, and
(ii) $|V(G_1)| = |V(G_2)| \geq \tilde{R}$.

Note that the second condition (ii) implies that f_3 is sufficiently large since f_i with $i \geq 4$ are fixed. Actually, we have the following bound for f_3, which can be easily derived from Euler's formula:

$$f_3 \geq R = 2\tilde{R} - 2\chi(F^2) - \sum_{i \geq 4} (i - 2) f_i$$

If all faces of size more than 3 in G_i are mutually disjoint, then G_i can be regarded as a triangulation of a punctured surface obtained from F^2 by removing the interior of those faces. In such a case, since $\tilde{R} \geq D$, Theorem 3 implies that G_1 and G_2 can be transformed into each other, up to homeomorphism, by a sequence of diagonal flips. Thus, it suffices to show that G_1 can be deformed by diagonal flips so that all faces of size more than 3 are mutually disjoint.

Since G_1 is not φ-irreducible, we let vu_1 be a φ-contractible edge in G_1. Suppose that $u_1, u_2, \ldots, u_m$ are the neighbors of v lying around it in this cyclic order. There are two triangular faces vu_1u_2 and vu_1u_m incident to vu_1 and we can flip $vu_2, vu_3, \ldots, vu_{m-2}$ to $u_1u_3, u_1u_4, \ldots, u_1u_{m-1}$ in order since the resulting graph G_i' is isomorphic to G_1/vu_1 with a vertex v of degree 3 added to one face. The added vertex v is incident to three faces, at least two of which are triangular and the other might be of size more than 3. Color the vertex of degree 3 to be *red*. Other vertices are supposed to be colored *black*.

Repeat the previous procedure as far as possible. (If a red vertex is adjacent to both ends of a φ-contractible edge e, then move the red vertex to a neighboring face before contracting e.) Then we obtain a (black) φ-irreducible graph G_0 with extra red vertices added to faces of G_0. We denote this graph by $\tilde{G}_0$. Since $|V(G_0)| \leq I(F^2, \varphi)$ and $|V(G_1)| \geq I(F^2, \varphi) + \sum_{i \geq 4} if_i$, $\tilde{G}_0$ has at least $\sum_{i \geq 4} if_i$ red vertices.

We first move these red vertices so that any triangular face of G_0 contain no red vertex. Then we can find a face F of G_0 with $|F| \geq 4$ which contains at least $|F|$ red vertices. Since each of those red vertices has been added to F as a vertex of degree 3 incident to at least two triangular faces, F contains only one $|F|$-gonal face of $\tilde{G}_0$ and others are all triangular. By Lemma 6, we can deform the inside of F so that it contains a cycle C' of length $|F|$ disjoint from the boundary cycle C of F, as desribed in the lemma. Color all vertices on C' by *blue*. They are not movable now. On the other hand, all movable vertices, which are red, can run freely through the annular region bounded by $C \cup C'$ and can go to any face adjacent to F by Lemma 5. Sweep out all of them to other faces of G_0 which have never had the structure given in Lemma 6 yet. We can continue the same deformation as in this paragraph if such faces remain.

Therefore, all non-triangular faces in the final form will be bounded by cycles consisting of only blue vertices, which are mutually disjoint. Thus, the theorem follows. ■

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