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ON EHRESMANN'S THEOREM OF QUATERNION HOLOMORPHY

By

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Abstract. Ehresmann's theorem of quaternion holomorphy on quaternionic functions is generalized by considering quaternion differentiability equation and complex matrix representation of quaternions.

1. Introduction

Let $\boldsymbol{H}=\boldsymbol{R}^{4}$ be the division algebra of quaternions over real numbers $\boldsymbol{R}$ with the product defined by W.R. Hamilton as follows (see [5]):

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}\begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix}:=\begin{pmatrix} x_0y_0-x_1y_1-x_2y_2-x_3y_3 \\ x_0y_1+x_1y_0+x_2y_3-x_3y_2 \\ x_0y_2+x_2y_0+x_3y_1-x_1y_3 \\ x_0y_3+x_3y_0+x_1y_2-x_2y_1 \end{pmatrix}.$$

For $\boldsymbol{x}\in\boldsymbol{H}$, the conjugate $\overline{\boldsymbol{x}}$, the norm $|\boldsymbol{x}|$ and the inverse element $\boldsymbol{x}^{-1}$ of $\boldsymbol{x}(\neq 0)$ are given as $\overline{\boldsymbol{x}}:=\mathrm{diag}(1,-1,-1,-1)\boldsymbol{x}$, $|\boldsymbol{x}|:=\sqrt{\boldsymbol{x}\overline{\boldsymbol{x}}}$, $\boldsymbol{x}^{-1}=\overline{\boldsymbol{x}}/|\boldsymbol{x}|^{2}$. And the standard basis of $\boldsymbol{H}$ is denoted as follows:

$$1:=\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},\ \boldsymbol{i}:=\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},\ \boldsymbol{j}:=\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},\ \boldsymbol{k}:=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\in\boldsymbol{H}.$$

Then $\boldsymbol{H}=\boldsymbol{R}1+\boldsymbol{R}\boldsymbol{i}+\boldsymbol{R}\boldsymbol{j}+\boldsymbol{R}\boldsymbol{k}$ as a direct sum of real vector subspaces of $\boldsymbol{H}$ such that $\boldsymbol{ij}=-\boldsymbol{ji}=\boldsymbol{k}$, $\boldsymbol{jk}=-\boldsymbol{kj}=\boldsymbol{i}$, $\boldsymbol{ki}=-\boldsymbol{ik}=\boldsymbol{j}, \boldsymbol{i}^{2}=\boldsymbol{j}^{2}=\boldsymbol{k}^{2}=-1$ with the unit element 1 as a real algebra. For fixed positive integers m,n, let $M(m,n;\boldsymbol{H})$ be the set of all $m\times n$-matrices with the coefficients in $\boldsymbol{H}$, on which the matrix product is considered. By abbreviation, the following notations are used: $M(n,\boldsymbol{H}):=M(n,n;\boldsymbol{H})$, $\boldsymbol{H}^{n}:=M(n,1;\boldsymbol{H})\cong\boldsymbol{R}^{4n}$ and

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$|X| := \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}|X_{ij}|^2} \in \boldsymbol{R}$ for $X := (X_{ij}) \in M(m,n;\boldsymbol{H})$. By definition, a mapping $f : \boldsymbol{H}^n \to \boldsymbol{H}^m$ is said to be *quaternion affine* if there exist $A \in M(m,n;\boldsymbol{H})$ and $B \in \boldsymbol{H}^m$ such that $f(X) = AX + B$ for all $X \in \boldsymbol{H}^n$. For a mapping $f : U \to \boldsymbol{H}^m$ defined on an open subset U of $\boldsymbol{H}^n$ and a fixed point $X \in U$, f is said to be *quaternion differentiable at* X (abbrev. Q at X) if

(Q1) f is differentiable at X, that is (cf. [8]), there exists a real linear mapping $f_{*X} : \boldsymbol{H}^n \to \boldsymbol{H}^m; Y \mapsto f_{*X}(Y)$ such that

$$\lim_{|Y|\to 0} \frac{|f(X+Y) - f(X) - f_{*X}(Y)|}{|Y|} = 0;$$

and

(Q2) there exists $A \in M(m,n;\boldsymbol{H})$ such that $f_{*X}(Y) = AY$ $(Y \in \boldsymbol{H}^n)$.

Note that A does not depend on Y or the way that Y tends to 0. However, A can depend on X in U. In the case when $m = n = 1$, $f : U \to \boldsymbol{H}$ is quaternion differentiable at $\boldsymbol{x}$ if and only if there exists $\boldsymbol{a} \in \boldsymbol{H}$ such that $f'(\boldsymbol{x}) := \lim_{|\boldsymbol{y}|\to 0}(f(\boldsymbol{x}+\boldsymbol{y}) - f(\boldsymbol{y}))\boldsymbol{y}^{-1} = \boldsymbol{a}$. By W.R. Hamilton [4, Book III, Chapter II, §3, No.324], a quadratic function $f : \boldsymbol{H} \to \boldsymbol{H}; \boldsymbol{x} \mapsto \boldsymbol{x}^2$ is not quaternion differentiable at $\boldsymbol{x} \in \boldsymbol{H}$ if $\boldsymbol{x} \notin \boldsymbol{R}$ (cf. [2, §3]). In fact, if $\boldsymbol{x} \notin \boldsymbol{R}$, then $f'(\boldsymbol{x}) = \lim_{|\boldsymbol{y}|\to 0}(x_0 + \boldsymbol{y}(x_1\boldsymbol{i} + x_2\boldsymbol{j} + x_3\boldsymbol{k})\boldsymbol{y}^{-1})$ does not tend to a definite element in $\boldsymbol{H}$, so that f is not quaternion differentiable at $\boldsymbol{x} \notin \boldsymbol{R}$, as required. On the other hand, any quaternion affine mapping is quaternion differentiable at any point in an open domain. Conversely, the following result is well-known as a theorem of Ehresmann [3] (This is stated in Besse [1, p.410, 14.58 Theorem] without proof):

THEOREM I (C. Ehresmann). *Let $f : U \longrightarrow \boldsymbol{H}^n$ be a differentiable mapping defined on an open subset U of $\boldsymbol{H}^n$. Assume that the differential f_{*X} of f at any X in U is bijective. If f is quaternion differentiable at any X in U, then f is quaternion affine.*

In Theorem I, the bijective assumption for f_{*X} $(X \in U)$ can be omitted. In fact, the following result was obtained by Sommese [7, Proposition I]:

THEOREM II (A.J. Sommese). *Let m, n be arbitrary positive integers, and $f : U \longrightarrow \boldsymbol{H}^m$ be a differentiable mapping defined on an open subset U of $\boldsymbol{H}^n$. Then f is quaternion affine if and only if f is quaternion differentiable at any X in U.*

For a proof of Theorem II, Sommese [7, Lemma III] used the fact that the zeros of a complex holomorphic function of 2-variables admit no isolated point and have a structure of a complex manifold if it is not empty (see [6, p.31, Cor.2], for example). And Sudbery [9, Theorem 1] gave Theorem II in the special case when $m = n = 1$, by the use of Hartogs's theorem (see [6, Chap.3], for example). This article gives a new proof of Theorem II, by means of quaternion differentiability equation and complex matrix representation of quaternions. It appears that Theorem II can be proved in a very elementary way. After writing up the manuscript, the author knew the previous works of Sommese [7] and Sudbery [9] (see Acknowledgement).

2. Quarternion Differentiability Equation.

For the proof, it is crucially important to translate the quaternion differentiability to a differential equation such as the complex differentiability equation of Cauchy-Riemann. For $i \in \{1, \cdots, m\}$, the i-th canonical projection is denoted as follows:

$$\pi_i : \boldsymbol{H}^m \longrightarrow \boldsymbol{H}; X = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \mapsto x_i.$$

For $j \in \{1, \cdots, n\}$ and $\ell \in \{0, 1, 2, 3\}$, put

$$x_{j\ell} \begin{pmatrix} x_{10} + x_{11}\boldsymbol{i} + x_{12}\boldsymbol{j} + x_{13}\boldsymbol{k} \\ \vdots \\ x_{n0} + x_{n1}\boldsymbol{i} + x_{n2}\boldsymbol{j} + x_{n3}\boldsymbol{k} \end{pmatrix} := x_{j\ell},$$

which gives the canonical coordinates $x_{j\ell} : \boldsymbol{H}^n \to \boldsymbol{R}; X \mapsto x_{j\ell}(X)$ on $\boldsymbol{H}^n$. And an element $F_{j\ell} \in \boldsymbol{H}^n$ is defined as $x_{j'\ell'}(F_{j\ell}) = \delta_{jj'}\delta_{\ell\ell'}$ with respect to the Kronecker's delta: $\delta_{jj'} = 1$ (or 0) if $j = j'$ (resp. $j \neq j'$).

LEMMA. *Let $f : U \to \boldsymbol{H}^m$ be a mapping defined on some open subset U of $\boldsymbol{H}^n$, and fix a point $X \in U$. Then f is quaternion differentiable at X if and only if f is differentiable at X and the differential f_{*X} of f at X commutes with $R_{\boldsymbol{x}}^{(n)} : \boldsymbol{H}^n \longrightarrow \boldsymbol{H}^n = M(n, 1; \boldsymbol{H}); Y \mapsto Y\boldsymbol{x}$, for any $\boldsymbol{x} \in \boldsymbol{H} = M(1, \boldsymbol{H})$:*

$$R_{\boldsymbol{x}}^{(m)} \circ f_{*X} = f_{*X} \circ R_{\boldsymbol{x}}^{(n)} : \boldsymbol{H}^n \longrightarrow \boldsymbol{H}^m. \tag{1}$$

Proof. Put $A := (L(F_{10}), \cdots, L(F_{n0})) \in M(m, n; \boldsymbol{H})$ for a real linear mapping $L : \boldsymbol{H}^n \to \boldsymbol{H}^m$. Then $L \circ R_{\boldsymbol{x}}^{(n)} = R_{\boldsymbol{x}}^{(m)} \circ L$ for all $\boldsymbol{x} \in \boldsymbol{H}$ if and only if $L(Y) = AY$ for all $Y \in \boldsymbol{H}^n$. Taking $L = f_{*X}$, one has then the result. ∎

PROPOSITION 1. *Let $f : U \to \boldsymbol{H}^m$ be a mapping defined on some open subset U of $\boldsymbol{H}^n$, and fix a point $X \in U$. Assume that f is differentiable at X. Then f is quaternion differentiable at X if and only if $f_i := \pi_i \circ f : U \to \boldsymbol{H}$ satisfies the* *following equations at X for $i \in \{1, \cdots, m\}$, $j \in \{1, \cdots, n\}$:*

$$\frac{\partial f_i}{\partial x_{j0}} = -\frac{\partial f_i}{\partial x_{j1}}\boldsymbol{i} = -\frac{\partial f_i}{\partial x_{j2}}\boldsymbol{j} = -\frac{\partial f_i}{\partial x_{j3}}\boldsymbol{k}. \tag{2}$$

Proof. By Lemma, f is Q at X if and only if $f_{*X}(Y\boldsymbol{x}) = (f_{*X}Y)\boldsymbol{x}$ for all $Y \in \boldsymbol{H}^n$ and any $\boldsymbol{x} \in \boldsymbol{H}$. Since f is differentiable at X, each f_i is differentiable at X. And $f_{*X}(Y) = {}^t(f_{1*X}(Y), \cdots, f_{m*X}(Y))$ for all $Y \in \boldsymbol{H}^n$. Hence, f is Q at X if and only if $f_{i*X}(Y\boldsymbol{x}) = (f_{i*X}Y)\boldsymbol{x}$ for all $Y \in \boldsymbol{H}^n$, $\boldsymbol{x} \in \boldsymbol{H}$ and $i \in \{1, \cdots, m\}$. Note that $F_{j0} = -F_{j1}\boldsymbol{i} = -F_{j2}\boldsymbol{j} = -F_{j3}\boldsymbol{k}$. Taking $Y = F_{j\ell}$, one has that f is Q at X if and only if

$$f_{i*X}(F_{j0}) = -f_{i*X}(F_{j1})\boldsymbol{i} = -f_{i*X}(F_{j2})\boldsymbol{j} = -f_{i*X}(F_{j3})\boldsymbol{k} \tag{3}$$

for $j \in \{1, \cdots, n\}$ and $i \in \{1, \cdots, m\}$. With respect to the identification: $\boldsymbol{H}^n = T_X\boldsymbol{H}^n$, the equation (3) is equivalent to the equation (2) at X. ■

3. Complex Matrix Representation of Quaternions

Note that $\boldsymbol{H}$ contains a real subalgebra of complex numbers as $\boldsymbol{C} := \boldsymbol{R}1+\boldsymbol{R}\boldsymbol{i}$ such that the complex conjugate $\overline{x}$ of $x \in \boldsymbol{C}$ is same with the conjugate in $\boldsymbol{H}$. Then $\boldsymbol{H} = \boldsymbol{C}+\boldsymbol{j}\boldsymbol{C}$ is a direct sum of vector subspaces such that $(a+\boldsymbol{j}b)(c+\boldsymbol{j}d) = (ac-\overline{b}d)+\boldsymbol{j}(\overline{a}d+bc)$ for $a, b, c, d \in \boldsymbol{C}$. And quaternions can be realized by complex matrixes as follows:

$$\kappa : \boldsymbol{H} = \boldsymbol{C} + \boldsymbol{j}\boldsymbol{C} \longrightarrow \boldsymbol{C}^2; a + \boldsymbol{j}b \mapsto \begin{pmatrix} a \\ b \end{pmatrix} =: \begin{pmatrix} \kappa_1(a+\boldsymbol{j}b) \\ \kappa_2(a+\boldsymbol{j}b) \end{pmatrix};$$
$$\tilde{\kappa} : \boldsymbol{H} = \boldsymbol{C} + \boldsymbol{j}\boldsymbol{C} \longrightarrow M(2, \boldsymbol{C}); a + \boldsymbol{j}b \mapsto \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix}.$$

Then $\kappa \circ R_z^{(1)} = R_z^{(2)}|_{\boldsymbol{C}^2} \circ \kappa$ holds for all $z \in \boldsymbol{C}$. For $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{H}$, one has that

$$|\kappa(\boldsymbol{x})| = |\boldsymbol{x}|, \ \kappa(\boldsymbol{x}\boldsymbol{y}) = \tilde{\kappa}(\boldsymbol{x})\kappa(\boldsymbol{y}) \text{ and } \tilde{\kappa}(\boldsymbol{x}\boldsymbol{y}) = \tilde{\kappa}(\boldsymbol{x})\tilde{\kappa}(\boldsymbol{y}).$$

According to [11, p.463 (1)], a mapping $\tilde{\kappa}(m, n) : M(m, n; \boldsymbol{H}) \to M(2m, 2n; \boldsymbol{C})$ is defined as follows:

$$\tilde{\kappa}(m, n)(A) := (\tilde{\kappa}(\boldsymbol{a}_{ij}))_{i\in\{1,\cdots,m\}, j\in\{1,\cdots,n\}},$$

where $A = (\boldsymbol{a}_{ij})_{i\in\{1,\cdots,m\}, j\in\{1,\cdots,n\}}$. Then

$$\tilde{\kappa}(m, k)(AB) = \tilde{\kappa}(m, n)(A)\tilde{\kappa}(n, k)(B)$$

for $A \in M(m,n;\boldsymbol{H})$ and $B \in M(n,k;\boldsymbol{H})$. Moreover, a real linear isomorphism from $\boldsymbol{H}^n$ onto $\boldsymbol{C}^{2n}$ is defined as follows:

$$\kappa^{(n)} : \boldsymbol{H}^n \longrightarrow \boldsymbol{C}^{2n}; X = \begin{pmatrix} \boldsymbol{x}_1 \\ \vdots \\ \boldsymbol{x}_n \end{pmatrix} \mapsto \kappa^{(n)}(X) := \begin{pmatrix} \kappa(\boldsymbol{x}_1) \\ \vdots \\ \kappa(\boldsymbol{x}_n) \end{pmatrix}.$$

Then $\kappa^{(n)} \circ R_z^{(n)} = R_z^{(2n)}|_{\boldsymbol{C}^{2n}} \circ \kappa^{(n)}$ for any $z \in \boldsymbol{C}$. Hence,

$$\kappa^{(n)} : (\boldsymbol{H}^n, R_z^{(n)}\ (z \in \boldsymbol{C})) \longrightarrow (\boldsymbol{C}^{2n}, R_z^{(2n)}|_{\boldsymbol{C}^{2n}}\ (z \in \boldsymbol{C}))$$

is a complex linear isomorphism such as

$$\kappa^{(n)} \begin{pmatrix} x_{10} + x_{11}\boldsymbol{i} + x_{12}\boldsymbol{j} + x_{13}\boldsymbol{k} \\ \vdots \\ x_{n0} + x_{n1}\boldsymbol{i} + x_{n2}\boldsymbol{j} + x_{n3}\boldsymbol{k} \end{pmatrix} = \begin{pmatrix} x_{10} + x_{11}\boldsymbol{i} \\ x_{12} - x_{13}\boldsymbol{i} \\ \vdots \\ x_{n0} + x_{n1}\boldsymbol{i} \\ x_{n2} - x_{n3}\boldsymbol{i} \end{pmatrix} =: \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{2n-1} \\ z_{2n} \end{pmatrix}.$$

Then $\kappa^{(n)}$ gives the canonical complex coordinates on $\boldsymbol{H}^n$ such that

$$|\kappa^{(n)}(Y)| = |Y|, \quad \kappa^{(m)}(AY) = \tilde{\kappa}(m,n)(A)\ \kappa^{(n)}(Y) \tag{4}$$

for $Y \in \boldsymbol{H}^n$ and $A \in M(m,n;\boldsymbol{H})$. And $z_{2j-1} = \kappa_1(\boldsymbol{x}_j)$, $z_{2j} = \kappa_2(\boldsymbol{x}_j)$.

PROPOSITION 2. *Let $f : U \longrightarrow \boldsymbol{H}^m$ be a mapping defined on some open subset U of $\boldsymbol{H}^n$. For a fixed $X \in U$, assume that f is quaternion differentiable at X. Then, for any $i \in \{1, \cdots, m\}$,*

$$\kappa \circ f_i : U \longrightarrow \boldsymbol{C}^2; X' \mapsto \begin{pmatrix} (\kappa_1 \circ f_i)(X') \\ (\kappa_2 \circ f_i)(X') \end{pmatrix}$$

is complex differentiable at X with respect to the complex structure of U given by the canonical complex coordinates $(z_1, \cdots, z_{2n})$ on $\boldsymbol{H}^n$.

Proof. Fix any $i \in \{1, \cdots, m\}$. By Proposition 1, $f_i = \pi_i \circ f$ is quaternion differentiable at X. Hence, there exists $A \in M(1,n;\boldsymbol{H})$ such that

$$\lim_{|Y| \to 0} \frac{|f_i(X+Y) - f_i(X) - AY|}{|Y|} = 0.$$

Operating κ or $\kappa^{(n)}$ in the above formula, by the equation (4), one has that

$$\lim_{|\kappa^{(n)}(Y)| \to 0} \frac{|(\kappa \circ f_i)(X+Y) - (\kappa \circ f_i)(X) - \tilde{\kappa}(1,n)(A)\kappa^{(n)}(Y)|}{|\kappa^{(n)}(Y)|} = 0.$$

Since the mapping : $\boldsymbol{C}^{2n} \to \boldsymbol{C}^2; \kappa^{(n)}(Y) \mapsto \tilde{\kappa}(1,n)(A)\kappa^{(n)}(Y)$ is complex linear, $\kappa \circ f_i$ is complex differentiable at X, as required. ∎

Remark. This representation of quaternions is related to *Pauli's spin matrixes* as $\sigma_z := -i\tilde{\kappa}(\boldsymbol{i}), \sigma_y := i\tilde{\kappa}(\boldsymbol{j}), \sigma_x := i\tilde{\kappa}(\boldsymbol{k})$ (cf. [10, (15.9), (15.10)]).

4. Proof of Theorem II

Let $f : U \longrightarrow \boldsymbol{H}^m$ be a quaternion holomorphic mapping defined on an open subset U of $\boldsymbol{H}^n$. For $i \in \{1, \cdots, m\}$ and $\ell \in \{1, 2\}$, put

$$f_{i\ell} := \kappa_\ell \circ \pi_i \circ f : U \to \boldsymbol{C}.$$

Then, by the quaternion differentiability equation (2), one has that

$$\frac{\partial(f_{i1} + \boldsymbol{j} f_{i2})}{\partial x_{j0}} = -\frac{\partial(f_{i1} + \boldsymbol{j} f_{i2})}{\partial x_{j2}}\boldsymbol{j}$$

on U for $j \in \{1, \cdots, n\}$. Note that $z\boldsymbol{j} = \boldsymbol{j}\overline{z}$ for $z \in \boldsymbol{C}$. Hence,

$$\frac{\partial f_{i1}}{\partial x_{j0}} = \frac{\partial \overline{f}_{i2}}{\partial x_{j2}} \quad \text{and} \quad \frac{\partial f_{i2}}{\partial x_{j0}} = -\frac{\partial \overline{f}_{i1}}{\partial x_{j2}} \quad \text{on} \quad U. \tag{5}$$

By Proposition 2, f_{i1} and f_{i2} are holomorphic on U with respect to the complex coordinates $(z_1, \cdots, z_{2n})$. In particular, $\partial f_{i1}/\partial x_{j0} = \partial f_{i1}/\partial z_{2j-1}$ and $\partial f_{i2}/\partial x_{j0} = \partial f_{i2}/\partial z_{2j-1}$ are holomorphic, and that $\partial \overline{f}_{i2}/\partial x_{j2} = \partial \overline{f}_{i2}/\partial z_{2j}$ and $\partial \overline{f}_{i1}/\partial x_{j2} = \partial \overline{f}_{i1}/\partial z_{2j}$ are anti-holomorphic. By the equation (5), they are both holomorphic and anti-holomorphic on U, which are constants, that is, for certain $a_{i\ 2j},\ a_{i\ 2j-1} \in \boldsymbol{C}$ $(j \in \{1, \cdots, n\})$,

$$\frac{\partial f_{i1}}{\partial z_{2j-1}} = \frac{\partial f_{i1}}{\partial x_{j0}} = a_{i,2j-1}, \quad \frac{\partial f_{i1}}{\partial z_{2j}} = \frac{\partial f_{i1}}{\partial x_{j2}} = -\overline{a}_{i,2j};$$
$$\frac{\partial f_{i2}}{\partial z_{2j-1}} = \frac{\partial f_{i2}}{\partial x_{j0}} = a_{i,2j}, \quad \frac{\partial f_{i2}}{\partial z_{2j}} = \frac{\partial f_{i2}}{\partial x_{j2}} = \overline{a}_{i,2j-1}$$

on U. Then there exist constants $b_{i1}, b_{i2} \in \boldsymbol{C}$ such that

$$\begin{cases} f_{i1}(X) = \sum_{j=1}^n (a_{i,2j-1} z_{2j-1} - \overline{a}_{i,2j} z_{2j}) + b_{i1} \\ f_{i2}(X) = \sum_{j=1}^n (a_{i,2j} z_{2j-1} + \overline{a}_{i,2j-1} z_{2j}) + b_{i2} \end{cases}$$

for $X = {}^t(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n)$ with $\boldsymbol{x}_j = z_{2j-1} + \boldsymbol{j} z_{2j}$. By the equation (4), one has

$$f_i(X) = f_{i1}(X) + \boldsymbol{j} f_{i2}(X) = \sum_{j=1}^n \boldsymbol{a}_{ij} \boldsymbol{x}_j + \boldsymbol{b}_i,$$

where $\boldsymbol{a}_{ij} := a_{i,2j-1} + ja_{i,2j}$ and $\boldsymbol{b}_i := b_{i1} + jb_{i2}$. Hence,

$$f(X) = \begin{pmatrix} f_1(X) \\ \vdots \\ f_m(X) \end{pmatrix} = \begin{pmatrix} \boldsymbol{a}_{11} & \cdots & \boldsymbol{a}_{1n} \\ \vdots & \vdots & \vdots \\ \boldsymbol{a}_{m1} & \cdots & \boldsymbol{a}_{mn} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_1 \\ \vdots \\ \boldsymbol{x}_n \end{pmatrix} + \begin{pmatrix} \boldsymbol{b}_1 \\ \vdots \\ \boldsymbol{b}_n \end{pmatrix}.$$

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